# Generalized SIP-modules

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## Abstract

We say an $R$-module $M$ has the *generalized summand intersection property* (briefly $GSIP$), if the intersection of any two direct summands is isomorphic to a direct summand. This is a generalization of SIP modules. In this note, the characterization of this property over rings and modules is investigated and some useful propositions obtained in SIP modules are generalized to GSIP modules.

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## 1. Introduction

Throughout this paper all rings are associative with unity and $R$ always denotes such a ring. Modules are unital and for an Abelian group $M$, we use $M_R$ to denote a right $R$-module. For any terminology, the definition of which is not given please see [2, 6, 13]. An $R$-module $M$ has the *summand intersection property* (briefly $SIP$) if the intersection of any two direct summands is again a direct summand. The definition of SIP was first given by Wilson [17] and this definition together with its generalizations were later studied by many authors [1, 3, 12]. $M$ is said to have the *strong summand intersection property* (briefly $SSIP$) if the intersection of any number of direct summands is again a direct summand. A ring $R$ is called a right *V-ring* if every simple right $R$-module is injective. For any $R$-module $M$, $E(M_R)$, $End(M_R)$ and $r(X)$ (resp. $r(x)$) will denote the injective hull of $M$, the ring of $R$-endomorphisms of $M$, and the right annihilator of a subset $X$ (resp. single element) of $M$ in $R$, respectively. For any nonempty subset $N$ of $M$, $N \leq M$, $N \leq_e M$ and $N \leq_d M$ will denote $N$ is a submodule of $M$, $N$ is an essential submodule of $M$ and $N$ is a direct summand of $M$, respectively. $M \cong N$ means that $M$ is isomorphic to $N$. Recall that in a commutative ring $R$, the ideal $I$ is *prime* if $ab \in I$ implies $a \in I$ or $b \in I$ [11, p.1]. A module $M$ is called a $CS$ (or $(C_1)$) module if every submodule of $M$ is essential in a direct summand of $M$. Recall that a module $M$ satisfies $(C_{11})$ if every submodule of $M$ has a complement which is a direct summand [15]. A submodule $N$ of $M$ is *fully invariant*, if for every $\varphi \in End_R(M)$, $\varphi(N) \subseteq N$. Recall that a module $M$ is called a (*weak*) *duo* module provided every (direct summand) submodule of $M$ is fully

invariant. Fuchs [7] defines a module M to have the *absolute direct summand property* (briefly *ADS*), if for every decomposition $M = A \oplus B$ of $M$ and every relative complement $C$ of $A$ in $M$ we have $M = A \oplus C$. A module $M$ satisfies $(C_2)$ if a submodule $A$ of $M$ is isomorphic to a direct summand of $M$, then $A$ is the direct summand of $M$. A module $M$ satisfies $(C_3)$ if for any direct summands $A$ and $B$ of $M$ with $A \cap B = 0$, $A \oplus B$ is a direct summand of $M$. A module $M$ is called a *quasi-continuous* module if $M$ satisfies both $(C_1)$ and $(C_3)$, and a *continuous* module if $M$ satisfies both $(C_1)$ and $(C_2)$.

In this paper, we say a module $M$ has the *generalized summand intersection property* (briefly *GSIP*), if the intersection of any two direct summands is isomorphic to a direct summand. A characterization of GSIP modules are provided in Theorem 2.2 which state that an $R$-module $M$ has the GSIP if and only if for every pair of direct summands $K$ and $L$ with $\pi : M \longrightarrow K$, the projection map, the kernel of the restricted map $\pi|_L$ is isomorphic to a direct summand. Another characterization of GSIP modules is given in Theorem 2.3 which state that an $R$-module $M$ has the GSIP if and only if for every decomposition $M = A \oplus B$ and every $R$-homomorphism $\phi$ from $A$ to $B$, the kernel of $\phi$ is isomorphic to a direct summand. Example 2.4 demonstrates that there is a $\mathbb{Z}$-module which has the GSIP but does not have the SIP. Example 2.5 shows that there is a module family which has the GSIP but does not have the SIP. Two necessary conditions for the equivalence of SIP and GSIP conditions are given in Proposition 2.7 and Proposition 2.15 that if a module $M$ is quasi-continuous or satisfies $(C_2)$, then $M$ has the SIP if and only if $M$ has the GSIP. Naturally it is of interest to examine whether or not an algebraic notion is inherited by direct summands and direct sums. It is shown in Example 2.16 that a direct sum of two modules having the GSIP, may not have the GSIP. As an answer to this question, it is proved in Proposition 2.18 that if $A$ and $B$ being two $R$-modules having the GSIP with the property $r(A) + r(B) = R$, $A \oplus B$ has the GSIP. Also, it is proved that assuming $M = \oplus_{i \in I} M_i$ is a direct sum of fully invariant submodules $M_i$ of $M$, then each $M_i$ has the GSIP if and only if the module $M$ has the GSIP. It is proved in Proposition 2.15 that if $M$ is a quasi-continuous module, then $M$ has the GSIP if and only if $E(M)$ has the GSIP. A characterization of semisimple rings is given in Corollary 2.9 that a ring $R$ is semisimple if and only if all $R$-modules have the GSIP if and only if all injective $R$-modules have the GSIP. Furthermore, a characterization of V-ring is provided in Theorem 2.21 that a ring $R$ is a right V-ring if and only if every finitely cogenerated $R$-module has the GSIP if and only if every finitely copresented $R$-module has the GSIP. It is proved in Corollary 2.12 that over a commutative Noetherian domain $R$, an injective $R$-module $M$ is torsion-free if and only if $M \oplus M$ has the GSIP. At the end of the paper, in Example 2.24, it is shown that the GSIP is not Morita invariant.

## 2. GSIP modules

**Definition 2.1.** An $R$-module $M$ has the *generalized summand intersection property* (briefly *GSIP*) if the intersection of every pair of direct summands of $M$ is isomorphic to a direct summand of $M$. We say a ring $R$ has the right *GSIP* if the module $R_R$ has the *GSIP* i.e., for every pair of idempotents $c$, $d$ in $R$ there exists $e^2 = e \in R$ such that $cR \cap dR \cong eR$.

Clearly semisimple, indecomposable and uniform modules have the SIP. It is well known that a module is weak duo if and only if its endomorphism ring is Abelian. Moreover any module with an Abelian endomorphism has the SIP. Thus (weak) duo modules have the SIP. Recall that, if a module M is CS and a polyform module, then $M$ has the SIP by [1, Lemma 11], and if $M$ is injective and a prime $R$-module, then $M$ has the SIP from [8, Proposition 2.1], and if $M$ is a projective module over a hereditary ring, then $M$ has the SIP by [17, Proposition 3(a)], and if $F$ is a free module over a principle ideal domain (PID), then $F$ has the SIP by [10]. It is obvious from the definitions of SIP and GSIP

that if a module has the SIP, then it has the GSIP. Thus, the above mentioned module families have the GSIP.

**Theorem 2.2.** *An R-module M has the GSIP if and only if for every pair of direct summands K and L with $\pi : M \longrightarrow K$ the projection map, the kernel of the restricted map $\pi|_L$ is isomorphic to a direct summand of M.*

**Proof.** Assume $M$ is an $R$-module with the GSIP. Let $\pi : M \longrightarrow K$ be the projection map. For $K' = Ker\ \pi$, we get $M = K \oplus K'$. Thus, $Ker\ \pi|_L = L \cap K'$ is isomorphic to a direct summand. Conversely, suppose that $M$ has the stated property. Let $K$, $L$ be the direct summands of $M$. Then there exists a submodule $K'$ of $M$ such that $M = K \oplus K'$, and let $\rho : M \longrightarrow K'$ be the projection map. It follows that $K \cap L = Ker\ \rho|_L$ is isomorphic to a direct summand. $\square$

**Theorem 2.3.** *An R-module M has the GSIP if and only if for every decomposition $M = A \oplus B$ and every R-homomorphism $\phi$ from A to B, the kernel of $\phi$ is isomorphic to a direct summand of M.*

**Proof.** Assume $M$ is an $R$-module with the GSIP. Let $M = A \oplus B$ and $\phi$ be an $R$-homomorphism from $A$ to $B$. Let $C = \{a + \phi(a) \mid a \in A\}$. We want to show that $M = C \oplus B$. Let $x \in M$, then $x = a + b$ where $a \in A$ and $b \in B$. Now, $x = a + \phi(a) - \phi(a) + b$. But $a + \phi(a) \in C$ and $-\phi(a) + b \in B$. So, $M = C + B$. Let us choose $x \in C \cap B$. We can write $x = a + \phi(a)$ where $a \in A$ and hence $a = x - \phi(a) \in A \cap B = 0$. Therefore $\phi(a) = 0$ which gives $x = 0$. So, $M = C \oplus B$. Since $M$ has the GSIP, $C \cap A$ is isomorphic to a direct summand of $M$. It is a straightforward matter to show that $C \cap A = Ker\ \phi$. Hence, $Ker\ \phi$ is isomorphic to a direct summand of $M$. To prove the converse, suppose that $M = A \oplus B$ has the stated property. Let $M = X \oplus X_1$, $M = Y \oplus Y_1$ and let $\pi_{X_1} : M \longrightarrow X_1$ and $\pi_Y : M \longrightarrow Y$ be the natural epimorphisms. Define $\sigma = (\pi_{X_1} o \pi_Y)|_X$. Notice that $\sigma$ acts from $X$ to $X_1$. Thus, there is a direct summand $P$ of $M$ such that $Ker\ \sigma \cong P$. It is easy to check that $Ker\ \sigma = (X \cap Y) \oplus (X \cap Y_1)$. Since $X \cap Y$ is a direct summand of $Ker\ \sigma$, and $Ker\ \sigma \cong P$, $X \cap Y$ is isomorphic to a direct summand of $P$. Then, $X \cap Y$ is isomorphic to a direct summand of $M$. Thus, $M$ has the GSIP. $\square$

The following two examples illustrate modules having the GSIP but not the SIP.

**Example 2.4.** Let $p$ be a prime number. Consider $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ as a $\mathbb{Z}$-module. It is clear that $0$, $M$, $\langle(\bar{0}, \bar{1})\rangle$, $\langle(\bar{1}, \bar{1})\rangle$, $\langle(\bar{1}, \bar{0})\rangle$ and $\langle(\bar{1}, \bar{p})\rangle$ are direct summands of $M$. Only one of the intersections, namely $\langle(\bar{0}, \bar{1})\rangle \cap \langle(\bar{1}, \bar{1})\rangle = \langle(\bar{0}, \bar{p})\rangle$ is not a direct summand of $M$. Thus, $M_{\mathbb{Z}}$ does not have the SIP. But this intersection is isomorphic to direct summand $\langle(\bar{1}, \bar{0})\rangle$ of $M$. So, $M$ has the GSIP.

**Example 2.5.** Let $A$ be a commutative PID and $R = \begin{bmatrix} A & A/xA \\ 0 & A \end{bmatrix}$. Then the following are true:

   (i) if $x = 0$ then $R_R$ has the SIP.
   (ii) if $x \neq 0$ and $x \neq 1$ then $R_R$ has the GSIP but not the SIP.

If $x = 0$ then $R = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$. Routine calculations show that idempotents are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & r \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & r \\ 0 & 0 \end{bmatrix}$, for all $r \in A$. Therefore, direct summands are $R$, $[0]$, $\begin{bmatrix} A & A \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & rA \\ 0 & A \end{bmatrix} = \left\{ \begin{bmatrix} 0 & ry \\ 0 & y \end{bmatrix} : y \in A \right\}$. It can easily be checked that $R_R$ has the SIP.

Now, if $x \neq 0$ and $x \neq 1$ then idempotents are $\begin{bmatrix} 0 & \bar{0} \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & \bar{0} \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & \bar{r} \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & \bar{r} \\ 0 & 0 \end{bmatrix}$,

for all $\overline{r} \in A/xA$. Thus, the direct summands are $R$, $[0]$, $\begin{bmatrix} A & A/xA \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \overline{r}A \\ 0 & A \end{bmatrix}$.

Let $\overline{r_1}$, $\overline{r_2} \in A/xA$ and $\overline{r_1} \neq \overline{r_2}$. Only one of the intersections, namely $\begin{bmatrix} 0 & \overline{r_1}A \\ 0 & A \end{bmatrix} \cap$

$\begin{bmatrix} 0 & \overline{r_2}A \\ 0 & A \end{bmatrix} = \begin{bmatrix} 0 & \overline{0} \\ 0 & rA \end{bmatrix}$ is not a direct summand but it is isomorphic to direct summand

$\begin{bmatrix} 0 & \overline{0} \\ 0 & 1 \end{bmatrix} .R = \begin{bmatrix} 0 & \overline{0} \\ 0 & A \end{bmatrix}$. Hence $R_R$ has the GSIP but not the SIP.

The next proposition is a consequence of [14, Lemma 5], but we will present a proof below.

**Proposition 2.6.** *A module $M$ having the GSIP has the SIP if and only if for any two direct summands $A_1$ and $A_2$ of $M$, with an $R$-isomorphism $\sigma : A_1 \cap A_2 \longrightarrow P$ such that $P$ be a direct summand of $M$, then $\sigma$ extends to some $\theta \in End(M)$.*

**Proof.** Assume that a module $M$ having the GSIP has the SIP. Let $A_1$ and $A_2$ be two direct summands of $M$ and $\sigma : A_1 \cap A_2 \longrightarrow P$ be an $R$-isomorphism where $P$ be a direct summand of $M$. By assumption, $M$ has the SIP. So, $A_1 \cap A_2$ is a direct summand of $M$. Then there exists a submodule $B$ of $M$ such that $M = (A_1 \cap A_2) \oplus B$. Now, define $\theta : M \longrightarrow M$ by $\theta(a + b) = \sigma(a)$ where $a \in A_1 \cap A_2$ and $b \in B$. Clearly, $\theta(a) = \sigma(a)$ for all $a \in A_1 \cap A_2$. So, $\theta|_{A_1 \cap A_2} = \sigma$. Conversely, $M$ has the stated property. Let $A_1$ and $A_2$ be direct summands of $M$. Since $M$ has the GSIP, there exist a monomorphism $\sigma : A_1 \cap A_2 \longrightarrow M$ such that $\sigma(A_1 \cap A_2)$ is a direct summand of $M$. By hypothesis, $\sigma$ can be lifted to a homomorphism $\theta : M \longrightarrow M$. Let $\pi : M \longrightarrow \sigma(A_1 \cap A_2)$ denote the canonical projection. Then $\psi = \pi\theta : M \longrightarrow \sigma(A_1 \cap A_2)$ is a homomorphism. Note that $\psi(x) = \pi\theta(x) = \pi\sigma(x) = \sigma(x)$ for all $x \in A_1 \cap A_2$. Let $m \in M$. Then $\psi(m) = \sigma(n) = \psi(n)$ for some $n \in A_1 \cap A_2$ since $\sigma$ is an isomorphism of $A_1 \cap A_2$ onto $M$. Then $m - n \in Ker\ \psi$. It follows that $M = Ker\ \psi + (A_1 \cap A_2)$. Now, let $z \in Ker\ \psi \cap (A_1 \cap A_2)$. Then $z \in A_1 \cap A_2$ and $\sigma(z) = \psi(z) = 0$, so $z = 0$. Hence $M = Ker\ \psi \oplus (A_1 \cap A_2)$. Thus, $A_1 \cap A_2$ is a direct summand of $M$. So, $M$ has the SIP. $\square$

**Proposition 2.7.** *If an $R$-module $M$ satisfies $(C_2)$, then $M$ has the SIP if and only if $M$ has the GSIP.*

**Proof.** The necessity is clear. The inverse follows immediately from the definition of the $(C_2)$ property. $\square$

**Corollary 2.8.** *Let $M$ be an $R$-module. Then the following are equivalent:*

(a) *$E(M)$ has the GSIP.*
(b) *For submodules $A$ and $B$ of $M$, the following equality holds:*
$E(A \cap B) = E(A) \cap E(B)$.

**Proof.** Immediate by Proposition 2.7 and [16, Theorem 4.13]. $\square$

**Corollary 2.9.** *The following are equivalent for a ring $R$:*

(a) *$R$ is semisimple.*
(b) *All $R$-modules have the GSIP.*
(c) *All injective $R$-modules have the GSIP.*

**Proof.** Immediate by Proposition 2.7 and [17, Propoisition 3(b)]. $\square$

**Corollary 2.10.** *Let $R$ be a commutative Noetherian ring. Then an injective $R$-module $M$ has the SSIP if and only if $M$ has the GSIP.*

**Proof.** Immediate by Proposition 2.7 and [17, Proposition 4]. $\square$

**Corollary 2.11.** *Let $R$ be a commutative Noetherian domain. Then the following are equivalent for an injective $R$-module $M$:*

(a) *$M$ has the SSIP.*
(b) *$M$ has the SIP.*
(c) *$M$ has the GSIP.*
(d) *either* (*i*) *$M$ is torsion-free.*
  *or* (*ii*) *$M$ is torsion and for any two distinct, indecomposable direct summands $A$ and $B$ of $M$, $Hom(A, B) = 0$.*

**Proof.** Equivalence of (a), (b) and (d) are proved in [17, Proposition 6]. $(b) \Rightarrow (c)$ is clear. $(c) \Rightarrow (b)$ is a direct consequence of Proposition 2.7 since injective modules satisfy $(C_2)$. □

**Corollary 2.12.** *Let $R$ be a commutative Noetherian domain and $M$ be an injective $R$-module. Then the following are equivalent:*

(a) *$M$ is torsion-free.*
(b) *$M \oplus M$ has the GSIP.*
(c) *$\bigoplus_{\Lambda} M$ has the GSIP for any index set $\Lambda$.*

**Proof.** Immediate by Proposition 2.7 and [8, Theorem 3.8]. □

**Corollary 2.13.** *Let $M$ be a module with $S = End(M)$.*
*If $M$ satisfies $(C_2)$, then $M \oplus M$ has the GSIP if and only if $S$ is a regular ring.*

**Proof.** Immediate by Proposition 2.7 and [1, Theorem 29]. □

The following proposition generalizes [17, Lemma 2].

**Proposition 2.14.** *Let $R$ be a commutative Noetherian ring and $M = M_1 \oplus M_2$ with indecomposable submodules $M_1$ and $M_2$. Assume that $M$ has the GSIP and satisfies $(C_2)$, then either*

(a) *$Hom(M_1, M_2) = 0$ or*
(b) *$M_1$ is isomorphic to $M_2$ and there is some prime ideal $A$ of $R$ with $r(x) = A$ for every nonzero $x \in M_1$.*

**Proof.** Assume that $M$ has the stated property and let $0 \neq \alpha \in Hom(M_1, M_2)$. Since $M$ has the GSIP and satisfies $(C_2)$, $M$ has the SIP from Proposition 2.7. Then by [9, Proposition 1.4], $Ker\ \alpha$ is a direct summand of $M_1$. Since $M_1$ is an indecomposable submodule and $\alpha \neq 0$, we have $Ker\ \alpha = 0$. So, $\alpha$ is a monomorphism. By [1, Lemma 19(1)], $M = M_1 \oplus M_2$ has the $SSP$. Thus, $\alpha(M_1)$ is a direct summand of $M_2$ from [1, Theorem 8]. Since $M_2$ is an indecomposable submodule, $\alpha$ is onto and therefore $M_1$ is isomorphic to $M_2$.

It remains to show the condition on annihilators. Let $x, y \in M$ be nonzero and suppose that there is some $a \in r(x)$ with $a \notin r(y)$. Define $\beta : M_1 \longrightarrow M_2$ by $\beta(m) = \alpha(am)$ for $m \in M_1$. We see that $x \in Ker\ \beta$, so $\beta$ is not a monomorphism. Also $y \notin Ker\ \beta$, so $\beta \neq 0$. Since $M_1 \oplus M_2$ has the GSIP and satisfies $(C_2)$, $M_1 \oplus M_2$ has the SIP from Proposition 2.7. Thus, $Ker\ \beta$ is not a direct summand, contradicting [9, Proposition 1.4]. Thus, $r(x) = r(y)$ for all nonzero $x, y \in M_1$. Then $r(x)$ is a prime ideal follows immediately from [11, Theorem 6]. □

**Proposition 2.15.** *Let $M$ be a quasi-continuous module (or satisfies $(C_{11})$ and ADS property). Then the following are equivalent:*

(a) *$M$ has the SIP.*
(b) *$E(M)$ has the SIP.*
(c) *$M$ has the SSIP.*

(d) $E(M)$ *has the SSIP.*
(e) $M$ *has the GSIP.*
(f) $E(M)$ *has the GSIP.*
(g) $M$ *is UC-module.*
(h) $E(M)$ *is UC-module.*

**Proof.** Equivalence of $(a)$, $(b)$, $(c)$ and $(d)$ are proved in [1, Proposition 18]. $(a) \Leftrightarrow (g)$ and $(b) \Leftrightarrow (h)$ are a result of [1, Lemma 17]. $(a) \Rightarrow (e)$ and $(b) \Rightarrow (f)$ are clear. $(f) \Rightarrow (b)$ is a consequence of Proposition 2.7 since injective modules satisfy $(C_2)$.

$(e) \Rightarrow (b)$ Assume $M$ has the GSIP and let $A$ and $B$ be direct summands of $E(M)$. Then $E(M) = A \oplus A'$, $E(M) = B \oplus B'$ for some $A', B' \leq E(M)$ and $A = E(A)$ and $B = E(B)$. By [13, Theorem 2.8], $A \cap M$ and $B \cap M$ are direct summands of $M$. By assumption $A \cap B \cap M$ is isomorphic to a direct summand $T$ of $M$. By [13, Corollary 2.32], isomorphic submodules have isomorphic closures. Closure of $T$ is $T$ and let the closure of $A \cap B \cap M$ be $N$ such that $N \cong T$. Since $A \cap B \cap M \leq_e N$, $E(A \cap B \cap M) = E(N)$. Since $M$ satisfies $(C_1)$, then $N$ is direct summand of $M$. Hence there exist an $L$ submodule of $M$ such that $M = N \oplus L$. Therefore $E(M) = E(N) \oplus E(L) = E(A \cap B \cap M) \oplus E(L)$. Since $A \cap M \leq_e A$ and $B \cap M \leq_e B$, $A \cap B \cap M \leq_e A \cap B$. Hence $E(M) = E(A \cap B) \oplus E(L)$. Since $A = E(A \cap B) \oplus (E(L) \cap A)$ and $B = E(A \cap B) \oplus (E(L) \cap B)$, then $E(A \cap B) \leq A \cap B \leq E(A \cap B)$ implies $A \cap B = E(A \cap B)$ is a direct summand of $E(M)$. So, $E(M)$ has the SIP.

Proof of the proposition is again routine in the case that $M$ satisfies $(C_{11})$ and *ADS* property. Instead of giving a long proof of the proposition, we state that the proof follows straightforwardly from [4, Proposition 1.3(2)]. $\square$

The following example shows that direct sum of two modules having the GSIP, may not have the GSIP.

**Example 2.16.** Consider $\mathbb{Z}_4$ as a $\mathbb{Z}$-module. It is clear that $\mathbb{Z}_4$ is indecomposable and hence, has the GSIP. Define $f : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4$ by $f(\bar{x}) = 2\bar{x}$. Then $Ker\ f = 2\mathbb{Z}_4$ is not isomorphic to a direct summand. Therefore by Theorem 2.3, $(\mathbb{Z}_4 \oplus \mathbb{Z}_4)_{\mathbb{Z}}$ does not have the GSIP.

**Proposition 2.17.** *[8, Proposition 3.9] Let $M = M_1 \oplus M_2$ be an R-module. If $r(M_1) + r(M_2) = R$, then every submodule $N$ of $M$ can be written as $N = N_1 \oplus N_2$, where $N_1 \leq M_1$ and $N_2 \leq M_2$.*

Proposition 2.18 presents the condition under which the direct sum of modules having the GSIP has the GSIP.

**Proposition 2.18.** *If $A$ and $B$ are two R-modules having the GSIP such that $r(A) + r(B) = R$, then $A \oplus B$ has the GSIP.*

**Proof.** Let $X$ and $Y$ be two direct summands of $A \oplus B$. By Proposition 2.17, $X = A_1 \oplus B_1$ and $Y = A_2 \oplus B_2$, where $A_1$ and $A_2$ are submodules of $A$, $B_1$ and $B_2$ are submodules of $B$. It is easy to show that $A_1$ and $A_2$ are direct summands of $A$, $B_1$ and $B_2$ are direct summands of $B$. If $A$ and $B$ have the GSIP, then $A_1 \cap A_2$ is isomorphic to a direct summand of $A$ and $B_1 \cap B_2$ is isomorphic to a direct summand of $B$. Therefore, $(A_1 \cap A_2) \oplus (B_1 \cap B_2)$ is isomorphic to a direct summand of $A \oplus B$. Now $(A_1 \cap A_2) \oplus (B_1 \cap B_2) = (A_1 \oplus B_1) \cap (A_2 \oplus B_2) = X \cap Y$. Thus $X \cap Y$ is isomorphic to a direct summand of $A \oplus B$ and hence, $A \oplus B$ has the GSIP. $\square$

To the best of authors' knowledge, it is not yet known whether a direct summand of a module having GSIP has the GSIP. Theorem 2.19 shows that a fully invariant direct summand of a module having the GSIP inherits the property and this result gives conditions under which the direct sum of modules having the GSIP has the GSIP.

**Theorem 2.19.** *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of fully invariant submodules $M_i$ of $M$ where $I$ is an index set. Then $M_i$ ($\forall i \in I$) has the GSIP if and only if $M$ has the GSIP.*

**Proof.** Assume that $M_i$ has the GSIP. Let $S$ be any direct summand of $M$. Since $M_i$ are fully invariant submodules of $M$, thus by [5, Lemma 1.1(3)] $S = \bigoplus (S \cap M_i)$. Now let $S$, $T$ be direct summands of $M$. So $S \cap T = \bigoplus [(S \cap M_i) \cap (T \cap M_i)]$. Since each $M_i$ has the GSIP, $S \cap T$ is isomorphic to a direct summand of $M$. To prove the converse, suppose that $M$ has the GSIP. Let $A$, $B$ be direct summands of $M_i$. Then $A$ and $B$ are direct summands of $M$. Since $M$ has the GSIP, there exists a direct summand $K$ of $M$ such that $A \cap B \cong K$. If we show $K$ is a direct summand of $M_i$, then the proof is complete. Let $f : A \cap B \longrightarrow K$ and $g : K \longrightarrow A \cap B$ denote the isomorphisms. $\pi_K : M \longrightarrow K$ denote the canonical projection and let $i_K : K \longrightarrow M$ denote inclusion. Then $h = i_K \, o \, f \, o \, g \, o \, \pi_K$ is an endomorphism of $M$. Because $M_i$ is fully invariant in $M$, $h(M_i) \subseteq M_i$, so that $f(A \cap B) = K \subseteq M_i$. Now, it is easy to see that $K$ is a direct summand of $M_i$. Thus $M_i$ has the GSIP. $\square$

**Proposition 2.20.** *If $M$ has the GSIP and $f : M \longrightarrow N$ is an isomorphism, then $N$ has the GSIP.*

**Proof.** Let $K$ and $L$ be direct summands of $N$. So there exist $A$ and $B$ which are direct summands of $M$ such that $f(A) = K$, $f(B) = L$. Since $M$ has the GSIP, $A \cap B$ is isomorphic to some $T$ which is a direct summand of $M$.

$$f(T) \cong f(A \cap B) = f(A) \cap f(B) = K \cap L$$

Since $K \cap L \cong f(T)$ and $f(T)$ is a direct summand of $N$, it follows that $N$ has the GSIP. $\square$

Let $R$ be a ring. An $R$-module $M$ is *finitely cogenerated* if and only if $Soc(M)$ is finitely generated and essential in $M$ [18, 21.3(1)]. An $R$-module $X$ is called a *finitely copresented* module if (*i*) $X$ is finitely cogenerated and (*ii*) in every exact sequence $0 \to X \to L \to N \to 0$ in *Mod-R* with $L$ finitely cogenerated, $N$ is also finitely cogenerated. $R$ is a right *V-ring* if and only if every finitely cogenerated $R$-module is semisimple [18, 23.1] if and only if every finitely copresented $R$-module is injective [18, 31.7].

**Theorem 2.21.** *The following are equivalent for a ring $R$:*
 (a) *$R$ is a right V-ring.*
 (b) *Every finitely cogenerated $R$-module has the GSIP.*
 (c) *Every finitely copresented $R$-module has the GSIP.*

**Proof.** $(a) \Rightarrow (b) \Rightarrow (c)$ These are clear by [18, 23.1] and the definitions.
$(c) \Rightarrow (a)$ Let $M$ be a finitely copresented $R$-module. By [18, 30.1], $E(M)$ and $E(M)/M$ are finitely cogenerated. Since $E(M)/M$ is finitely cogenerated, $E(E(M)/M)$ is finitely cogenerated. Since any finitely cogenerated injective module is finitely copresented, by (c) and [18, 21.4], $E(M) \oplus E(E(M)/M)$ has the GSIP. Let $f$ be a canonical epimorphism $E(M)$ to $E(M)/M$ and $i$ be the inclusion homomorphism from $E(M)/M$ to $E(E(M)/M)$. Since $E(M) \oplus E(E(M)/M)$ has the GSIP, then $Ker \, (iof) = Ker \, f = M$ isomorphic to a direct summand of $E(M) \oplus E(E(M)/M)$ from Theorem 2.3. Since a direct summand of an injective module is an injective module, $M$ is isomorphic to an injective module. Then $M$ is an injective module. By [18, 31.7], $R$ is a right V-ring. $\square$

**Proposition 2.22.** *If a ring $R$ satisfies the condition "Any direct sum of $R$-modules having the GSIP has the GSIP", then $R$ is a right $V$-ring.*

**Proof.** Let $M$ be a finitely cogenerated $R$-module. Then $M$ is a direct sum of indecomposable $R$-modules by [18, 21.3]. Since indecomposable modules have the GSIP, then $M$ has the GSIP by hypothesis. Then by Theorem 2.21, $R$ is a right V-ring. $\square$

It is well known that a ring $R$ is right Noetherian if and only if every direct sum of injective right $R$-modules is injective [2, Proposition 18.13].

**Proposition 2.23.** *The following are equivalent for a ring R:*

(a) *R is a right Noetherian ring with the property "Any direct sum of R-modules having the GSIP has the GSIP".*

(b) *R is semisimple.*

**Proof.** $(a) \Rightarrow (b)$ Let $M$ be an injective $R$-module. Since $R$ is a right Noetherian ring, $M$ is a direct sum of indecomposable modules. Since indecomposable modules have the *GSIP*, $M$ has the GSIP. By Corollary 2.9, $R$ is semisimple.

$(b) \Rightarrow (a)$ If $R$ is semisimple, then every $R$-module is injective by [18, 20.7]. And also every $R$-module has the GSIP by Corollary 2.9. Hence, $(a)$ holds. $\square$

The following example shows that right GSIP modules cannot be Morita invariant.

**Example 2.24.** Let us consider the ring $\mathbb{Z}_4$. Although $\mathbb{Z}_4$ has the GSIP, the ring of $2 \times 2$ matrices over $\mathbb{Z}_4$ does not have GSIP. Let $R = \begin{bmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 \end{bmatrix}$, $e = \begin{bmatrix} \bar{0} & \bar{2} \\ \bar{0} & \bar{1} \end{bmatrix}$, and $f = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}$. Then $eR \cap fR = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{2} \end{bmatrix} R$, which is a nil right ideal. Since $R$ is a quasi-Frobenius ring, it satisfies $(C_2)$. Hence $eR \cap fR$ cannot be isomorphic to an idempotent generated right ideal (i.e., a direct summand). Therefore the right $R$-module $R$ does not have the GSIP, so GSIP is not a Morita invariant.

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