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Research Article

On locally ϕ-semisymmetric Sasakian manifolds

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Abstract

Generalizing the notion of local ϕ-symmetry of Takahashi [Sasakian ϕ-symmetric spaces, Tohoku Math. J., 1977], in the present paper, we introduce the notion of *local ϕ-semisymmetry* of a Sasakian manifold along with its proper existence and characterization. We also study the notion of local Ricci (resp., projective, conformal) ϕ-semisymmetry of a Sasakian manifold and obtain its characterization. It is shown that the local ϕ-semisymmetry, local projective ϕ-semisymmetry and local concircular ϕ-semisymmetry are equivalent. It is also shown that local conformal ϕ-semisymmetry and local conharmonical ϕ-semisymmetry are equivalent.

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1. Introduction

Let M be an n-dimensional, $n \geq 3$, connected smooth Riemannian manifold endowed with the Riemannian metric g. Let ∇, R, S and r be the Levi-Civita connection, curvature tensor, Ricci tensor and the scalar curvature of M respectively. The manifold M is called locally symmetric due to Cartan ([2, 3]) if the local geodesic symmetry at $p \in M$ is an isometry, which is equivalent to the fact that $\nabla R = 0$. Generalizing the concept of local symmetry, the notion of semisymmetry was introduced by Cartan [4] and fully classified by Szabó ([17–19]). The manifold M is said to be semisymmetric if

$$(R(U, V).R)(X, Y)Z = 0$$

for all vector fields X, Y, Z, U, V on M, where $R(U, V)$ is considered as the derivation of the tensor algebra at each point of M. Every locally symmetric manifold is semisymmetric but the converse is not true, in general. However, the converse is true only for $n = 3$. As a weaker version of local symmetry, in 1977 Takahashi [20] introduced the notion of local ϕ-symmetry on a Sasakian manifold. A Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X, Y)Z) = 0$$

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for all horizontal vector fields X, Y, Z, W on M, where ϕ is the structure tensor of the manifold M. The concept of local ϕ-symmetry on various structures and their generalizations or extensions are studied in [6, 8–15]. By extending the notion of semisymmetry and generalizing the concept of local ϕ-symmetry of Takahashi [20], in the present paper, we introduce the notion of *local ϕ-semisymmetry* on a Sasakian manifold. A Sasakian manifold M, $n \geq 3$, is said to be *locally ϕ-semisymmetric* if

$$\phi^2((R(U,V).R)(X,Y)Z) \quad = \quad 0$$

for all horizontal vector fields X, Y, Z, U, V on M. We note that every locally ϕ-symmetric as well as semisymmetric Sasakian manifold is locally ϕ-semisymmetric but not conversely.

The object of the present paper is to study the geometric properties of a locally ϕ-semisymmetric Sasakian manifold along with its proper existence and characterization. The paper is organized as follows. Section 2 deals with the rudiments of Sasakian manifolds. By extending the definition of local ϕ-symmetry, in Section 3, we derive the defining condition of a locally ϕ-semisymmetric Sasakian manifold and proved that a Sasakian manifold is locally ϕ-semisymmetric if and only if each Kählerian manifold, which is a base space of a local fibering, is Hermitian locally semisymmetric. We cite an example of a locally ϕ-semisymmetric Sasakian manifold which is not locally ϕ-symmetric. We also obtain a characterization of locally ϕ-semisymmetric Sasakian manifold by considering the horizontal vector fields. Section 4 is devoted to the characterization of locally ϕ-semisymmetric Sasakian manifold for arbitrary vector fields. As a generalization of Ricci (resp., projectively, conformally) semisymmetric Sasakian manifold, in the last section, we introduce the notion of *locally Ricci (resp., projectively, conformally) ϕ-semisymmetric Sasakian manifold* and obtain the characterization of such notions. Recently Shaikh and Kundu [16] defined a generalized curvature tensor, called B-tensor, by the linear combination of R, S and g which includes various curvature tensors as particular cases. We study the characterization of locally B-ϕ-semisymmetric Sasakian manifolds. It is shown that local ϕ-semisymmetry, local projective ϕ-semisymmetry and local concircular ϕ-semisymmetry are equivalent and hence they are of the same characterization. Also it is proved that local conformal ϕ-semisymmetry and local conharmonical ϕ-semisymmetry are equivalent. Finally, we conclude that the study of local ϕ-semisymmetry and local conformal ϕ-semisymmetry are meaningful as they are not equivalent. However, the study of local ϕ-semisymmetry with any other generalized curvature tensor of type (1,3) (which are the linear combination of R, S and g) is either meaningless or redundant due to their equivalency.

2. Sasakian manifolds

An $n(= 2m+1, m \geq 1)$-dimensional C^∞ manifold M is said to be a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on M. Given a contact form η, it is well-known that there exists a unique vector field ξ, called the characterstic vector field of η, satisfying $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for any vector field X on M. A Riemannian metric g is said to be an associated metric if there exists a tensor field ϕ of type (1,1) such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\cdot) \quad = \quad g(\cdot, \xi), \quad d\eta(\cdot, \cdot) = g(\cdot, \phi\cdot). \tag{2.1}$$

Then the structure (ϕ, ξ, η, g) on M is called a contact metric stucture and the manifold M equipped with such a stucture is called a contact metric manifold [1].
From (2.1) it is easy to check that the following holds:

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi\cdot, \cdot) = -g(\cdot, \phi\cdot), \tag{2.2}$$

$$g(\phi\cdot, \phi\cdot) = g(\cdot, \cdot) - \eta \otimes \eta. \tag{2.3}$$

Given a contact metric manifold M there is an (1,1) tensor field h given by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the operator of Lie differentiation. Then h is symmetric. The vector field ξ is a Killing vector field with respect to g if and only if $h = 0$. A contact metric manifold M for which ξ is a Killing vector is said to be a K-contact manifold. A contact structure on M gives rise to an almost complex structure J on the product $M \times \mathbb{R}$ defined by

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\phi X - f\xi, \eta(X)\frac{d}{dt}\Big),$$

where f is a real valued function, is integrable, then the structure is said to be normal and the manifold M is a Sasakian manifold. Equivalently, a contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.4}$$

holds for all X, Y on M.

In an n-dimensional Sasakian manifold M the following relations hold ([1], [22]):

$$R(\xi, X)Y = (\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X = -R(X,\xi)Y, \tag{2.5}$$

$$\nabla_X \xi = -\phi X, \quad (\nabla_X \eta)(Y) = g(X, \phi Y), \tag{2.6}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.7}$$

$$(\nabla_W R)(X,Y)\xi = g(W,\phi Y)X - g(W,\phi X)Y + R(X,Y)\phi W, \tag{2.8}$$

$$(\nabla_W R)(X,\xi)Z = g(X,Z)\phi W - g(Z,\phi W)X + R(X,\phi W)Z, \tag{2.9}$$

$$S(X,\xi) = (n-1)\eta(X), \quad S(\xi,\xi) = (n-1) \tag{2.10}$$

for all vector fields X, Y, Z and W on M. In a Sasakian manifold, for any X, Y, Z on M, we also have [21]

$$\begin{aligned} R(X,Y)\phi W =& g(W,\phi X)Y - g(W,Y)\phi X \\ & - g(W,\phi Y)X + g(W,X)\phi Y + \phi R(X,Y)W. \end{aligned} \tag{2.11}$$

From (2.8) and (2.11), it follows that

$$(\nabla_W R)(X,Y)\xi = g(W,X)\phi Y - g(W,Y)\phi X + \phi R(X,Y)W. \tag{2.12}$$

3. Locally ϕ-semisymmetric Sasakian manifolds

Let M be an $n(= 2m+1, m \geq 1)$-dimensional Sasakian manifold endowed with the structure (ϕ, ξ, η, g). Let $\tilde{U}$ be an open neighbourhood of $x \in M$ such that the induced Sasakian structure on $\tilde{U}$, denoted by the same letters, is regular. Let $\pi : \tilde{U} \to N = \tilde{U}/\xi$ be a (local) fibering and let $(J, \bar{g})$ be the induced Kählerian structure on N [7]. Let R and $\bar{R}$ be the curvature tensors constructed by g and $\bar{g}$ respectively. For a vector field $\bar{X}$ on N, we denote its horizontal lift (with respect to the connection form η) by $\bar{X}^*$. Then we have, for any vector fields $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ on N,

$$(\bar{\nabla}_{\bar{X}} \bar{Y})^* = \nabla_{\bar{X}^*} \bar{Y}^* - \eta(\nabla_{\bar{X}^*} \bar{Y}^*)\xi, \tag{3.1}$$

$$(\bar{R}(\bar{X},\bar{Y})\bar{Z})^* = R(\bar{X}^*,\bar{Y}^*)\bar{Z}^* + g(\phi\bar{Y}^*,\bar{Z}^*)\phi\bar{X}^* - g(\phi\bar{X}^*,\bar{Z}^*)\phi\bar{Y}^* - 2g(\phi\bar{X}^*,\bar{Y}^*)\phi\bar{Z}^*,$$

$$((\bar{\nabla}_{\bar{V}} \bar{R})(\bar{X},\bar{Y})\bar{Z})^* = -\phi^2[(\nabla_{\bar{V}^*} R)(\bar{X}^*,\bar{Y}^*)\bar{Z}^*] \tag{3.2}$$

where $\bar{\nabla}$ is the Levi-Civita connection for $\bar{g}$. The relations (3.1) and (3.2) are due to Ogiue [7] and the relation (3.2) is due to Takahashi [20].

Making use of (2.1), (2.4)-(2.11) and (3.1)-(3.2), we get by straightforward calculation

$$((\bar{R}(\bar{U},\bar{V}) \cdot \bar{R})(\bar{X},\bar{Y})\bar{Z})^* = -\phi^2[(R(\bar{U}^*,\bar{V}^*) \cdot R)(\bar{X}^*,\bar{Y}^*)\bar{Z}^*] \tag{3.3}$$

for any vector fields $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}$ and $\bar{V}$ on N, where $R(U, V)$ is considered as the derivation of the tensor algebra at each point of N. Hence from (3.3) it is natural to define the following:

Definition 3.1. A Sasakian manifold is said to be a locally ϕ-semisymmetric if

$$\phi^2[(R(U,V) \cdot R)(X,Y)Z] = 0 \tag{3.4}$$

for any horizontal vector fields X, Y, Z, U and V on M, where a horizontal vector is a vector which is horizontal with respect to the connection form η of the local fibering, that is, orthogonal to ξ.

Thus from (3.3) and (3.4), we can state the following:

Theorem 3.2. *A Sasakian manifold is locally ϕ-semisymmetric if and only if each Kählerian manifold, which is a base space of a local fibering, is a Hermitian locally semisymmetric space.*

If we consider a Sasakian manifold of non-constant ϕ-sectional curvature, then the Kählerian base manifold is not of constant sectional curvature. Suppose that $R(X, \phi X, Y, \phi Y) = f \in C^\infty(M)$. Then $(\nabla_V R)(X, \phi X, Y, \phi Y) = (Vf) \neq 0$, i.e. the Kählerian manifold is not Hermitian locally symmetric and therefore the Sasakian manifold is not locally ϕ-symmetric. Now $(\nabla_U \nabla_V R)(X, \phi X, Y, \phi Y) = U(Vf) \neq 0$, which implies that $(R(U, V) \cdot R)(X, \phi X, Y, \phi Y) = 0$, i.e. the Kählerian manifold is Hermitian locally semisymmetric. Hence the Sasakian manifold is locally ϕ-semisymmetric but not locally ϕ-symmetric.

First we suppose that M is a Sasakian manifold such that

$$\phi^2[(R(U,V) \cdot R)(X,Y)\xi] = 0 \tag{3.5}$$

for any horizontal vector fields X, Y, U and V on M.

Differentiating (2.12) covariantly with respect to a horizontal vector field U we get

$$\begin{aligned}(\nabla_U \nabla_V R)(X,Y)\xi = \{&g(Y,U)g(X,V) - g(X,U)g(Y,V) \\ &-R(X,Y,U,V)\}\xi + \phi((\nabla_U R)(X,Y)V).\end{aligned} \tag{3.6}$$

Alternating U and V on (3.6) we get

$$\begin{aligned}(\nabla_V \nabla_U R)(X,Y)\xi = \{&g(Y,V)g(X,U) - g(X,V)g(Y,U) \\ &-R(X,Y,V,U)\}\xi + \phi((\nabla_V R)(X,Y)U).\end{aligned} \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$\begin{aligned}(R(U,V) \cdot R)(X,Y)\xi = 2\{&g(Y,U)g(X,V) - g(X,U)g(Y,V) - R(X,Y,U,V)\}\xi \\ &+\phi\{(\nabla_U R)(X,Y)V - (\nabla_V R)(X,Y)U\}.\end{aligned} \tag{3.8}$$

Again from (3.5) we have

$$(R(U,V) \cdot R)(X,Y)\xi = 0. \tag{3.9}$$

From (3.8) and (3.9) we have

$$\begin{aligned}&2\{g(Y,U)g(X,V) - g(X,U)g(Y,V) - R(X,Y,U,V)\}\xi \\ &\quad +\phi\{(\nabla_U R)(X,Y)V - (\nabla_V R)(X,Y)U\} = 0.\end{aligned} \tag{3.10}$$

Applying ϕ on (3.10) and using (2.11), (2.12) and (2.2) we get

$$(\nabla_U R)(X,Y)V - (\nabla_V R)(X,Y)U = 0. \tag{3.11}$$

In view of (3.11), (3.10) yields

$$R(X,Y,U,V) = g(Y,U)g(X,V) - g(X,U)g(Y,V) \tag{3.12}$$

for any horizontal vector fields X, Y, U and V on M. Hence M is of constant ϕ-holomorphic sectional curvature 1 and hence of constant curvature 1. This leads to the following:

Theorem 3.3. *If a Sasakian manifold M satisfies the condition $\phi^2[(R(U,V)\cdot R)(X,Y)\xi] = 0$ for all horizontal vector fields X, Y, Z, U and V on M, then it is a manifold of constant curvature 1.*

Now we consider a locally ϕ-semisymmetric Sasakian manifold. Then from (3.4) we have

$$(R(U,V)\cdot R)(X,Y)Z = g((R(U,V)\cdot R)(X,Y)Z,\xi)\xi,$$

from which we get

$$(R(U,V)\cdot R)(X,Y)Z = -g((R(U,V)\cdot R)(X,Y)\xi,Z)\xi \tag{3.13}$$

for all horizontal vector fields X, Y, Z, U and V on M.
In view of (3.8), it follows from (3.13) that

$$(R(U,V)\cdot R)(X,Y)Z = [(\nabla_U R)(X,Y,V,\phi Z) - (\nabla_V R)(X,Y,U,\phi Z)]\xi. \tag{3.14}$$

Now differentiating (2.11) covariantly with respect to a horizontal vector field V, we obtain

$$\begin{aligned}(\nabla_V R)(X,Y)\phi Z = [R(X,Y,Z,V) - \{g(Y,Z)g(X,V)\\ +g(X,Z)g(Y,V)]\xi + \phi((\nabla_V R)(X,Y)Z).\end{aligned} \tag{3.15}$$

Taking inner product of (3.15) with a horizontal vector field U, we obtain

$$g((\nabla_V R)(X,Y)\phi Z,U) = -g((\nabla_V R)(X,Y)Z,\phi U). \tag{3.16}$$

Using (3.16) in (3.14) we get

$$(R(U,V)\cdot R)(X,Y)Z = [(\nabla_U R)(X,Y,Z,\phi V) - (\nabla_V R)(X,Y,Z,\phi U)]\xi \tag{3.17}$$

for any horizontal vector fields X, Y, Z, U and V on M. Hence we can state the following:

Theorem 3.4. *A necessary and sufficient condition for a Sasakian manifold M to be a locally ϕ-semisymmetric is that it satisfies the relation* (3.17) *for all horizontal vector fields on M.*

4. Characterization of a locally ϕ-semisymmetric Sasakian manifold

In the section we investigate the condition of local ϕ-semisymmetry of a Sasakian manifold for arbitrary vector fields on M. To find the condition we need the following lemmas.

Lemma 4.1 ([20])**.** *For any horizontal vector fields X, Y and Z on M, we get*

$$(\nabla_\xi R)(X,Y)Z = 0. \tag{4.1}$$

Now Lemma 4.1, (2.9) and (2.12) together imply the following:

Lemma 4.2 ([20])**.** *For any vector fields X, Y, Z, V on M, we get*

$$\begin{aligned}(\nabla_{\phi^2 V} R)(\phi^2 X,\phi^2 Y)\phi^2 Z = (\nabla_V R)(X,Y)Z + \eta(X)\{g(Y,Z)\phi V\\ -g(\phi V,Z)Y + R(Y,\phi V)Z\} - \eta(Y)\{g(X,Z)\phi V - g(\phi V,Z)X\\ +R(X,\phi V)Z\} - \eta(Z)\{g(X,V)\phi Y - g(Y,V)\phi X + \phi R(X,Y)V\}.\end{aligned} \tag{4.2}$$

Now let X, Y, Z, U, V be arbitrary vector fields on M. We now compute the term $(R(\phi^2 U,\phi^2 V)\cdot R)(\phi^2 X,\phi^2 Y)\phi^2 Z$ in two different ways. Firstly, from (3.17), (2.1) and (4.2) we get

$$\begin{aligned}(R(\phi^2 U,\phi^2 V)\cdot R)(\phi^2 X,\phi^2 Y)\phi^2 Z = [\{(\nabla_V R)(X,Y,Z,\phi U)\\ -(\nabla_U R)(X,Y,Z,\phi V)\} + \eta(X)\{g(Y,\phi V)g(\phi U,Z)\\ -g(Y,\phi U)g(\phi V,Z) - R(Y,Z,\phi U,\phi V)\}\\ -\eta(Y)\{g(X,\phi V)g(\phi U,Z) - g(X,\phi U)g(\phi V,Z)\\ -R(X,Z,\phi U,\phi V)\} - 2\eta(Z)\{g(X,V)g(U,Y)\\ -g(X,U)g(V,Y) - R(X,Y,U,V)\}]\xi.\end{aligned} \tag{4.3}$$

Again using (2.11) in (4.3), we obtain

$$(R(\phi^2U,\phi^2V)\cdot R)(\phi^2X,\phi^2Y)\phi^2Z = [\{(\nabla_V R)(X,Y,Z,\phi U) - (\nabla_U R)(X,Y,Z,\phi V)\} - \eta(X)H(Y,Z,U,V) + \eta(Y)H(X,Z,U,V) + 2\eta(Z)H(X,Y,U,V)]\xi \tag{4.4}$$

where $H(X,Y,Z,U) = g(\mathcal{H}(X,Y)Z,U)$ and the tensor field $\mathcal{H}$ of type (1,3) is given by

$$\mathcal{H}(X,Y)Z = R(X,Y)Z - g(Y,Z)X + g(X,Z)Y \tag{4.5}$$

for all vector fields X,Y,Z on M. Secondly, we have

$$(R(\phi^2U,\phi^2V)\cdot R)(\phi^2X,\phi^2Y)\phi^2Z = R(\phi^2U,\phi^2V)R(\phi^2X,\phi^2Y)\phi^2Z - R(R(\phi^2U,\phi^2V)\phi^2X,\phi^2Y)\phi^2Z - R(\phi^2X,R(\phi^2U,\phi^2V)\phi^2Y)\phi^2Z - R(\phi^2X,\phi^2Y)R(\phi^2U,\phi^2V)\phi^2Z. \tag{4.6}$$

By straightforward calculation, from (4.6) we get

$$(R(\phi^2U,\phi^2V)\cdot R)(\phi^2X,\phi^2Y)\phi^2Z = -(R(U,V)\cdot R)(X,Y)Z + \eta(U)[H(X,Y,Z,V)\xi + \eta(X)\mathcal{H}(V,Y)Z + \eta(Y)\mathcal{H}(X,V)Z + \eta(Z)\mathcal{H}(X,Y)V] - \eta(V)[H(X,Y,Z,U)\xi + \eta(X)\mathcal{H}(U,Y)Z + \eta(Y)\mathcal{H}(X,U)Z + \eta(Z)\mathcal{H}(X,Y)U]. \tag{4.7}$$

From (4.4) and (4.7) it follows that

$$(R(U,V)\cdot R)(X,Y)Z = [\{(\nabla_U R)(X,Y,Z,\phi V) - (\nabla_V R)(X,Y,Z,\phi U)\} + \eta(X)H(Y,Z,U,V) - \eta(Y)H(X,Z,U,V) - 2\eta(Z)H(X,Y,U,V) + \eta(U)H(X,Y,Z,V) - \eta(V)H(X,Y,Z,U)]\xi + \eta(U)[\eta(X)\mathcal{H}(V,Y)Z + \eta(Y)\mathcal{H}(X,V)Z + \eta(Z)\mathcal{H}(X,Y)V] - \eta(V)[\eta(X)\mathcal{H}(U,Y)Z + \eta(Y)\mathcal{H}(X,U)Z + \eta(Z)\mathcal{H}(X,Y)U]. \tag{4.8}$$

Thus in a locally ϕ-semisymmetric Sasakian manifold, the relation (4.8) holds for any arbitrary vector fields X,Y,Z,U and V on M. Next, if the relation (4.8) holds in a Sasakian manifold, then for any horizontal vector fields X,Y,Z,U and V on M, we get the relation (3.17) and hence the manifold is locally ϕ-semisymmetric. Thus we can state the following:

Theorem 4.3. *A Sasakian manifold M is locally ϕ-semisymmetric if and only if the relation* (4.8) *holds for any arbitrary vector fields X,Y,Z,U and V on M.*

Corollary 4.4 ([21])**.** *A semisymmetric Sasakian manifold is a manifold of constant curvature 1.*

5. Locally Ricci (resp., projectively, conformally) ϕ-semisymmetric Sasakian manifolds

Definition 5.1. Let M be a Sasakian manifold. Then M is said to be a locally Ricci ϕ-semisymmetric if the relation

$$\phi^2[(R(U,V)\cdot Q)(X)] = 0 \tag{5.1}$$

holds for all horizontal vector fields X,Y,Z,U and V on M, Q being the Ricci operator of the manifold.

We know that

$$(R(U,V)\cdot Q)(X) = R(U,V)QX - QR(U,V)X. \tag{5.2}$$

Applying ϕ^2 on both sides of (5.2) we get

$$\phi^2[(R(U,V)\cdot Q)(X)] = -(R(U,V)\cdot Q)(X) \tag{5.3}$$

for all horizontal vector fields U, V and X on M. This leads to the following:

Theorem 5.2. *A Sasakian manifold M is locally Ricci ϕ-semisymmetric if and only if $(R(U,V)\cdot Q)(X) = 0$ for all horizontal vector fields U, V and X on M.*

Now let M be a locally ϕ-semisymmetric Sasakian manifold. Then the relation (3.17) holds on M. Taking inner product of (3.17) with a horizontal vector field W and then contracting over X and W, we get $(R(U,V)\cdot S)(Y,Z) = 0$ from which it follows that $(R(U,V)\cdot Q)(Y) = 0$ for all horizontal vector fields U, V and Y on M. Thus in view of the Theorem 5.2, we can state the following:

Theorem 5.3. *A locally ϕ-semisymmetric Sasakian manifold M is locally Ricci ϕ-semisymmetric.*

Now let U, V and X are arbitrary vector fields on a Sasakian manifold M. Then in view of (2.1), (2.4), (2.5) and (2.10), (5.2) yields

$$\begin{aligned}(R(\phi^2U,\phi^2V)\cdot Q)(\phi^2X) = -(R(U,V)\cdot Q)(X) + \{E(X,V)\eta(U) \\ -E(X,U)\eta(V)\}\xi - \eta(X)\{\eta(V)\mathcal{E}U - \eta(U)\mathcal{E}V\},\end{aligned} \tag{5.4}$$

where $g(\mathcal{E}X,Y) = E(X,Y)$ and E is given by

$$E(X,Y) = S(X,Y) - (n-1)g(X,Y). \tag{5.5}$$

Since ϕ^2U, ϕ^2V and ϕ^2X are orthogonal to ξ, in a locally Ricci ϕ-semisymmetric Sasakian manifold M, from (5.4) we have

$$\begin{aligned}(R(U,V)\cdot Q)(X) = \{E(X,V)\eta(U) - E(X,U)\eta(V)\}\xi \\ -\eta(X)\{\eta(V)\mathcal{E}U - \eta(U)\mathcal{E}V\}.\end{aligned} \tag{5.6}$$

Thus in a locally Ricci ϕ-semisymmetric Sasakian manifold M the relation (5.6) holds for any arbitrary vector fields U, V and X on M. Next, if the relation (5.6) holds in a Sasakian manifold M, then for all horizontal vector fields U, V and X, we have $(R(U,V)\cdot Q)(X) = 0$ and hence M is locally Ricci ϕ-semisymmetric. Thus we can state the following:

Theorem 5.4. *A Sasakian manifold M is locally Ricci ϕ-semisymmetric if and only if the relation* (5.6) *holds for any arbitrary vector fields U, V and X on M.*

Corollary 5.5 ([21])**.** *A Ricci semisymmetric Sasakian manifold is an Einstein manifold.*

Definition 5.6. A Sasakian manifold M is said to be a locally projectively (resp. conformally) ϕ-semisymmetric if the relation

$$\phi^2[(R(U,V)\cdot P)(X,Y)Z] \text{ (resp. } \phi^2[(R(U,V)\cdot C)(X,Y)Z]) = 0 \tag{5.7}$$

holds for all horizontal vector fields X, Y, Z, U and V on M, P (resp. C) being the projective (resp. conformal) curvature tensor of the manifold.

The projective transformation is such that geodesics transformed into geodesics [23] and as the invariant of such transformation the Weyl projective curvature tensor P of type (1,3) is given by [23]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{5.8}$$

The conformal transformation is an angle preserving mapping and as the invariant of such transformation the Weyl conformal curvature tensor C of type (1,3) on a Riemannian manifold M, $n>3$, is given by [23]

$$\begin{aligned} C(X,Y)Z \;=\; & R(X,Y)Z-\frac{1}{n-2}\{S(Y,Z)X-S(X,Z)Y \\ & +g(Y,Z)QX-g(X,Z)QY\} \\ & +\frac{r}{(n-1)(n-2)}\{g(Y,Z)X-g(X,Z)Y\}. \end{aligned} \tag{5.9}$$

From (5.8) we get

$$\begin{aligned} (R(U,V)\cdot P)(X,Y)Z &= (R(U,V)\cdot R)(X,Y)Z \\ & -\frac{1}{n-1}\left[(R(U,V)\cdot S)(Y,Z)X -(R(U,V)\cdot S)(X,Z)Y\right]. \end{aligned} \tag{5.10}$$

Applying ϕ^2 on both sides of (5.10) and using (3.8) we obtain

$$\begin{aligned} \phi^2[(R(U,V)\cdot P)(X,Y)Z] &= -(R(U,V)\cdot R)(X,Y)Z \\ & +[(\nabla_U R)(X,Y,Z,\phi V)-(\nabla_V R)(X,Y,Z,\phi U)]\,\xi \\ & +\frac{1}{n-1}\left[(R(U,V)\cdot S)(Y,Z)X-(R(U,V)\cdot S)(X,Z)Y\right] \end{aligned} \tag{5.11}$$

for all horizontal vector fields X,Y,Z,U and V on M.

Now we suppose that M is a locally projectively ϕ-semisymmetric Sasakian manifold. Then from (5.11) we obtain

$$\begin{aligned} (R(U,V)\cdot R)(X,Y)Z &= [(\nabla_U R)(X,Y,Z,\phi V) \\ & -(\nabla_V R)(X,Y,Z,\phi U)]\,\xi+\frac{1}{n-1}\left[(R(U,V)\cdot S)(Y,Z)X\right. \\ & \left.-(R(U,V)\cdot S)(X,Z)Y\right] \end{aligned} \tag{5.12}$$

for all horizontal vector fields X,Y,Z,U and V on M. Taking inner product of (5.12) with a horizontal vector field W and then contracting over X and Z, we get

$$(R(U,V)\cdot S)(Y,W)=0 \tag{5.13}$$

for all horizontal vetor fields U,V,Y and W on M and hence by Theorem 5.2, it follows that the manifold M is locally Ricci ϕ-semisymmetric. Using (5.13) in (5.12), it follows that the manifold M is locally ϕ-semisymmetric.

Next, we suppose that M is a locally ϕ-semisymmetric Sasakian manifold. Then the relation (3.17) holds on M. Taking inner product of (3.17) with a horizontal vector field W and then contracting over X and W, we get $(R(U,V)\cdot S)(Y,Z)=0$ for all horizontal vetor fields U,V,Y and Z on M and hence from (5.11) it follows that the manifold M is locally projectively ϕ-semisymmetric. This leads to the following:

Theorem 5.7. *A locally projectively ϕ-semisymmetric Sasakian manifold M is locally ϕ-semisymmetric and vice versa.*

Now from (5.9) we get

$$\begin{aligned} (R(U,V)\cdot C)(X,Y)Z &= (R(U,V)\cdot R)(X,Y)Z \\ & -\frac{1}{n-2}[(R(U,V)\cdot S)(Y,Z)X-(R(U,V)\cdot S)(X,Z)Y \\ & +g(Y,Z)(R(U,V)\cdot Q)(X)-g(X,Z)(R(U,V)\cdot Q)(Y)]. \end{aligned} \tag{5.14}$$

Applying ϕ^2 on both sides of (5.14) and using (3.8) and (5.3) we obtain

$$\begin{aligned}\phi^2[(R(U,V)\cdot C)(X,Y)Z] = &-(R(U,V)\cdot R)(X,Y)Z \\ &+[(\nabla_U R)(X,Y,Z,\phi V)-(\nabla_V R)(X,Y,Z,\phi U)]\xi \\ &+\frac{1}{n-2}[(R(U,V)\cdot S)(Y,Z)X-(R(U,V)\cdot S)(X,Z)Y \\ &+g(Y,Z)(R(U,V)\cdot Q)(X)-g(X,Z)(R(U,V)\cdot Q)(Y)]\end{aligned} \tag{5.15}$$

for all horizontal vector fields X, Y, Z, U and V on M. This leads to the following:

Theorem 5.8. *A Sasakian manifold M is locally conformally ϕ-semisymmetric if and only if the relation*

$$\begin{aligned}(R(U,V)\cdot R)(X,Y)Z = &[(\nabla_U R)(X,Y,Z,\phi V) \\ &-(\nabla_V R)(X,Y,Z,\phi U)]\,\xi+\frac{1}{n-2}[(R(U,V)\cdot S)(Y,Z)X \\ &-(R(U,V)\cdot S)(X,Z)Y+g(Y,Z)(R(U,V)\cdot Q)(X) \\ &-g(X,Z)(R(U,V)\cdot Q)(Y)]\end{aligned} \tag{5.16}$$

holds for all horizontal vector fields X, Y, Z, U and V on M.

Let M be a locally ϕ-semisymmetric Sasakian manifold. Then M is locally Ricci ϕ-semisymmetric and thus in view of (3.17), it follows from (5.15) that $\phi^2[(R(U,V)\cdot C)(X,Y)Z]=0$ for all horizontal vector fields X, Y, Z, U and V on M. Hence the manifold M is locally conformally ϕ-semisymmetric.

Again, we consider M as the locally conformally ϕ-semisymmetric Sasakian manifold. If M is locally ϕ-semisymmetric Sasakian manifold, then from (5.16) it follows that $(R(U,V)\cdot S)(Y,Z)=0$ which implies that $(R(U,V)\cdot Q)(Y)=0$ for all horizontal vector fields U, V and Y on M and hence by Theorem 5.2, the manifold M is locally Ricci ϕ-semisymmetric. Again, if M is locally Ricci ϕ-semisymmetric, then $(R(U,V)\cdot Q)(Y)=0$ for all horizontal vector fields U, V and Y on M and hence by Theorem 3.4, it follows from (5.16) that the manifold M is locally ϕ-semisymmetric. This leads to the following:

Theorem 5.9. *A locally ϕ-semisymmetric Sasakian manifold M is locally conformally ϕ-semisymmetric. The converse is true if and only if the manifold M is locally Ricci ϕ-semisymmetric.*

Now let X, Y, Z, U and V be any arbitrary vector fields on a Sasakian manifold M. Then using (2.1), (2.10), (4.2) and (5.16) we obtain

$$\begin{aligned}(R(\phi^2U,\phi^2V)\cdot R)(\phi^2X,\phi^2Y)\phi^2Z = &[\{(\nabla_V R)(X,Y,Z,\phi U) \\ &-(\nabla_U R)(X,Y,Z,\phi V)\}-\eta(X)H(Y,Z,U,V) \\ &+\eta(Y)H(X,Z,U,V)+2\eta(Z)H(X,Y,U,V)]\xi \\ &-\frac{1}{n-2}[\{(R(U,V)\cdot S)(Y,Z)X \\ &-(R(U,V)\cdot S)(X,Z)Y\}-\{(R(U,V)\cdot S)(Y,Z)\eta(X) \\ &-(R(U,V)\cdot S)(X,Z)\eta(Y)\}\xi \\ &-\{E(V,Z)\eta(U)-E(U,Z)\eta(V)\}\{\eta(Y)X-\eta(X)Y\} \\ &+\{E(Y,U)X-E(X,U)Y\}\eta(Z)\eta(V)\end{aligned} \tag{5.17}$$

$$-\{E(Y,V)X - E(X,V)Y\}\eta(Z)\eta(U)]$$
$$-\frac{1}{n-2}[\{g(Y,Z)(R(U,V).Q)(X) - g(X,Z)(R(U,V).Q)(Y)\}$$
$$-\{\eta(Y)(R(U,V).Q)(X) - \eta(X)(R(U,V).Q)(Y)\}\eta(Z)$$
$$+\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\{\eta(V)\mathcal{E}U - \eta(U)\mathcal{E}V\}$$
$$-\{E(X,V)\eta(U) - E(X,U)\eta(V)\}g(Y,Z)\xi$$
$$+\{E(Y,V)\eta(U) - E(Y,U)\eta(V)\}g(X,Z)\xi]$$

where $g(\mathcal{E}U,V) = E(U,V)$ and E is given by (5.5).
From (5.17) and (4.7) it follows that

$$\begin{aligned}(R(U,V)\cdot R)(X,Y)Z = &[\{(\nabla_U R)(X,Y,Z,\phi V)\\
&-(\nabla_V R)(X,Y,Z,\phi U)\} + \eta(X)H(Y,Z,U,V) - \eta(Y)H(X,Z,U,V)\\
&-2\eta(Z)H(X,Y,U,V) + \eta(U)H(X,Y,Z,V) - \eta(V)H(X,Y,Z,U)]\xi\\
&+\eta(U)[\eta(X)\mathcal{H}(V,Y)Z + \eta(Y)\mathcal{H}(X,V)Z + \eta(Z)\mathcal{H}(X,Y)V]\\
&-\eta(V)[\eta(X)\mathcal{H}(U,Y)Z + \eta(Y)\mathcal{H}(X,U)Z + \eta(Z)\mathcal{H}(X,Y)U]\\
&+\frac{1}{n-2}[\{(R(U,V)\cdot S)(Y,Z)X - (R(U,V)\cdot S)(X,Z)Y\}\\
&-\{(R(U,V)\cdot S)(Y,Z)\eta(X) - (R(U,V)\cdot S)(X,Z)\eta(Y)\}\xi\\
&-\{E(V,Z)\eta(U) - E(U,Z)\eta(V)\}\{\eta(Y)X - \eta(X)Y\}\\
&+\{E(Y,U)X - E(X,U)Y\}\eta(Z)\eta(V) - \{E(Y,V)X - E(X,V)Y\}\eta(Z)\eta(U)]\\
&+\frac{1}{n-2}[\{g(Y,Z)(R(U,V).Q)(X) - g(X,Z)(R(U,V).Q)(Y)\}\\
&-\{\eta(Y)(R(U,V).Q)(X) - \eta(X)(R(U,V).Q)(Y)\}\eta(Z)\\
&+\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\{\eta(V)\mathcal{E}U - \eta(U)\mathcal{E}V\}\\
&-\{E(X,V)\eta(U) - E(X,U)\eta(V)\}g(Y,Z)\xi\\
&+\{E(Y,V)\eta(U) - E(Y,U)\eta(V)\}g(X,Z)\xi]\end{aligned} \tag{5.18}$$

where $H(X,Y,Z,U) = g(\mathcal{H}(X,Y)Z,U)$ and $g(\mathcal{E}U,V) = E(U,V)$, $\mathcal{H}$ and E are given by (4.5) and (5.5) respectively. Thus in a locally conformally ϕ-semisymmetric Sasakian manifold M the relation (5.18) holds for any arbitrary vector fields X,Y,Z,U and V on M. Next, if the relation (5.18) holds in a Sasakian manifold M, then for all horizontal vector fields X,Y,Z,U and V on M, we have (5.16), that is, the manifold is locally conformally ϕ-semisymmetric. This leads to the following:

Theorem 5.10. *A Sasakian manifold M is locally conformally ϕ-semisymmetric if and only if the relation* (5.18) *holds for any arbitrary vector fields X,Y,Z,U and V on M.*

Corollary 5.11 ([5])**.** *A conformally semisymmetric Sasakian manifold is a manifold of constant curvature 1.*

Remark 5.12. Since the skew-symmetric operator $R(X,Y)$ and the structure tensor ϕ of the Sasakian manifold both are commutes with the contraction, it follows from Theorem 6.6(ii) of Shaikh and Kundu [16] that the same conclusion of the Theorem 5.8, Theorem 5.9 and Theorem 5.10 holds for locally conharmonically ϕ-semisymmetric Sasakian manifold.

Again, by linear combination of R, S and g, Shaikh and Kundu [16] defined a generalized curvature tensor B (see, equation (2.1) of [16]) of type (1,3), called B-tensor which includes various curvature tensors as particular cases. Then Shaikh and Kundu (see [16, Eq. (5.5)]) showed that this B-tensor turns into the following form:

$$\begin{aligned} B(X,Y)Z &= b_0R(X,Y)Z + b_1\{S(Y,Z)X - S(X,Z)Y \\ &\quad +g(Y,Z)QX - g(X,Z)QY\} \\ &\quad +b_2r\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{5.19}$$

where b_0, b_1 and b_2 are scalars. We note that if
(a) $b_0 = 1$, $b_1 = 0$ and $b_2 = -\frac{1}{n(n-1)}$;
(b) $b_0 = 1$, $b_1 = -\frac{1}{(n-2)}$ and $b_2 = \frac{1}{(n-1)(n-2)}$;
(c) $b_0 = 1$, $b_1 = -\frac{1}{(n-2)}$ and $b_2 = 0$;
and (d) $b_2 = -\frac{1}{n}(\frac{b_0}{n-1} + 2b_1)$,
then from (5.19) it follows that the B-tensor turns into the (a) concircular, (b) conformal, (c) conharmonic and (d) quasi-conformal curvature tensor respectively. For details about the B-tensor we refer the reader to see Shaikh and Kundu [16] and also references therein.

Definition 5.13. A Sasakian manifold M is said to be a locally B-ϕ-semisymmetric if the relation

$$\phi^2[(R(U,V)\cdot B)(X,Y)Z] = 0 \tag{5.20}$$

holds for all horizontal vector fields X, Y, Z, U and V on M, B being the generalized curvature tensor of the manifold.

From (5.19) we get

$$\begin{aligned} (R(U,V)\cdot B)(X,Y)Z &= b_0(R(U,V)\cdot R)(X,Y)Z \\ &\quad +b_1\left[(R(U,V)\cdot S)(Y,Z)X - (R(U,V)\cdot S)(X,Z)Y\right. \\ &\quad \left.+ g(Y,Z)(R(U,V)\cdot Q)(X) - g(X,Z)(R(U,V)\cdot Q)(Y)\right]. \end{aligned} \tag{5.21}$$

Applying ϕ^2 on both sides of (5.21) and using (3.8) and (5.3) we obtain

$$\begin{aligned} \phi^2[(R(U,V)\cdot B)(X,Y)Z] &= -b_0[(R(U,V)\cdot R)(X,Y)Z \\ &\quad - \{(\nabla_U R)(X,Y,Z,\phi V) - (\nabla_V R)(X,Y,Z,\phi U)\}\xi] \\ &\quad -b_1\left[(R(U,V)\cdot S)(Y,Z)X - (R(U,V)\cdot S)(X,Z)Y\right. \\ &\quad \left.+ g(Y,Z)(R(U,V)\cdot Q)(X) - g(X,Z)(R(U,V)\cdot Q)(Y)\right] \end{aligned} \tag{5.22}$$

for all horizontal vector fields X, Y, Z, U and V on M. This leads to the following:

Theorem 5.14. *A Sasakian manifold M is locally B-ϕ-semisymmetric if and only if*

$$\begin{aligned} (R(U,V)\cdot R)(X,Y)Z &= [(\nabla_U R)(X,Y,Z,\phi V) \\ &\quad - (\nabla_V R)(X,Y,Z,\phi U)]\,\xi - \frac{b_1}{b_0}\left[(R(U,V)\cdot S)(Y,Z)X\right. \\ &\quad - (R(U,V)\cdot S)(X,Z)Y + g(Y,Z)(R(U,V)\cdot Q)(X) \\ &\quad \left.- g(X,Z)(R(U,V)\cdot Q)(Y)\right] \end{aligned} \tag{5.23}$$

for all horizontal vector fields X, Y, Z, U and V on M, provided $b_0 \neq 0$.

Now taking inner product of (5.23) with a horizontal vector field W and then contracting over X and W, we get

$$\{b_0 + (n-2)b_1\}(R(U,V)\cdot S)(Y,Z) = 0 \tag{5.24}$$

for all horizontal vector fields X, Y, Z, U and V on M.
From (5.24) following two cases arise:
Case-I. If $b_0 + (n-2)b_1 \neq 0$, then from (5.24) we have

$$(R(U,V)\cdot S)(Y,Z) = 0 \tag{5.25}$$

for all horizontal vector fields X, Y, Z, U and V on M, from which it follows that $(R(U, V) \cdot Q)(Y) = 0$ for all horizontal vector fields U, V and Y on M. This leads to the following:

Theorem 5.15. *A locally B-ϕ-semisymmetric Sasakian manifold M is locally Ricci ϕ-semisymmetric provided that $b_0 + (n-2)b_1 \neq 0$.*

Corollary 5.16. *A locally concircularly ϕ-semisymmetric Sasakian manifold M is locally Ricci ϕ-semisymmetric.*

Corollary 5.17. *A locally quasi-conformally ϕ-semisymmetric Sasakian manifold M is locally Ricci ϕ-semisymmetric provided that $b_0 + (n-2)b_1 \neq 0$.*

Now if $b_0 + (n-2)b_1 \neq 0$, then in view of (5.25), (5.23) takes the form (3.17) for all horizontal vector fields X, Y, Z, U and V on M and hence the manifold M is locally ϕ-semisymmetric. Again, if we consider the manifold M as locally ϕ-semisymmetric, then the relation (3.17) holds on M. Taking inner product of (3.17) with a horizontal vector field W and then contracting over X and W, we get $(R(U, V) \cdot S)(Y, Z) = 0$ for all horizontal vetor fields U, V, Y and Z on M and hence from (5.22) it follows that the manifold M is locally B-ϕ-semisymmetric. Thus we can state the following:

Theorem 5.18. *In a Sasakian manifold M, local $B - \phi$-semisymmetry and local ϕ-semisymmetry are equivalent provided that $b_0 + (n-2)b_1 \neq 0$.*

Corollary 5.19. *In a Sasakian manifold M, local concircular ϕ-semisymmetry and local ϕ-semisymmetry are equivalent.*

Corollary 5.20. *In a Sasakian manifold M, local quasi-conformal ϕ-semisymmetry and local ϕ-semisymmetry are equivalent provided that $b_0 + (n-2)b_1 \neq 0$.*

Remark 5.21. Since the skew-symmetric operator $R(X, Y)$ and the structure tensor ϕ of the Sasakian manifold both are commutes with the contraction, it follows from Theorem 6.6(i) of Shaikh and Kundu [16] that the same conclusion of the Corollary 5.16 and Corollary 5.19 holds for locally projectively ϕ-semisymmetric Sasakian manifold as the contraction on projective curvature tensor gives rise the Ricci operator although projective curvature tensor is not a generalized curvature tensor.

Case-II. If $b_0 + (n-2)b_1 = 0$, then from (5.22) we have

$$\begin{aligned}\phi^2[(R(U,V)\cdot B)(X,Y)Z] = &- b_0[(R(U,V)\cdot R)(X,Y)Z \\ &- \{(\nabla_U R)(X,Y,Z,\phi V) - (\nabla_V R)(X,Y,Z,\phi U)\}\,\xi] \\ &+ \frac{b_0}{n-2}[(R(U,V)\cdot S)(Y,Z)X - (R(U,V)\cdot S)(X,Z)Y \\ &+ g(Y,Z)(R(U,V)\cdot Q)(X) - g(X,Z)(R(U,V)\cdot Q)(Y)]\end{aligned} \tag{5.26}$$

for all horizontal vector fields X, Y, Z, U and V on M. This leads to the following:

Theorem 5.22. *A Sasakian manifold M is locally B-ϕ-semisymmetric if and only if*

$$\begin{aligned}(R(U,V)\cdot R)(X,Y)Z = &[(\nabla_U R)(X,Y,Z,\phi V) \\ &- (\nabla_V R)(X,Y,Z,\phi U)]\,\xi + \frac{1}{n-2}[(R(U,V)\cdot S)(Y,Z)X \\ &- (R(U,V)\cdot S)(X,Z)Y + g(Y,Z)(R(U,V)\cdot Q)(X) \\ &- g(X,Z)(R(U,V)\cdot Q)(Y)]\end{aligned} \tag{5.27}$$

for all horizontal vector fields X, Y, Z, U and V on M provided that $b_0 + (n-2)b_1 = 0$.

Corollary 5.23. *A Sasakian manifold M is locally conformally (resp. conharmonically) ϕ-semisymmetric if and only if the relation (5.27) holds.*

Corollary 5.24. *A Sasakian manifold M is locally quasi-conformally ϕ-semisymmetric if and only if the relation (5.27) holds provided that $b_0 + (n-2)b_1 = 0$.*

Let M be a locally ϕ-semisymmetric Sasakian manifold. Then M is locally Ricci ϕ-semisymmetric and thus in view of (3.17), it follows from (5.22) that $\phi^2[(R(U,V) \cdot B)(X,Y)Z] = 0$ for all horizontal vector fields X, Y, Z, U and V on M. Hence the manifold M is locally B-ϕ-semisymmetric.

Again, we consider M as the locally B-ϕ-semisymmetric Sasakian manifold. If $b_0+(n-2)b_1 \neq 0$, then M is locally ϕ-semisymmetric. So we suppose that $b_0+(n-2)b_1 = 0$. If M is locally ϕ-semisymmetric, then from (5.27) it follows that $(R(U,V) \cdot S)(Y,Z) = 0$, which implies that $(R(U,V) \cdot Q)(Y) = 0$ for all horizontal vector fields U, V and Y on M. Thus in view of Theorem 5.2, the manifold M is locally Ricci ϕ-semisymmetric. Again, if M is locally Ricci ϕ-semisymmetric, then $(R(U,V) \cdot Q)(Y) = 0$ for all horizontal vector fields U, V and Y on M. Thus in view of Theorem 3.4, it follows from (5.27) that the manifold M is locally ϕ-semisymmetric. This leads to the following:

Theorem 5.25. *A locally ϕ-semisymmetric Sasakian manifold M is locally B-ϕ-semisymmetric. The converse is true for $b_0 + (n-2)b_1 = 0$ if and only if the manifold M is locally Ricci ϕ-semisymmetric.*

If X, Y, Z, U and V are arbitrary vector fields on M, then proceeding similarly as in the case of conformal curvature tensor, it is easy to check that (5.18) holds for $b_0+(n-2)b_1 = 0$. Hence we can state the following:

Theorem 5.26. *A Sasakian manifold M is locally B-ϕ-semisymmetric if and only if the relation* (5.18) *holds for any arbitrary vector fields X, Y, Z, U and V on M provided that* $b_0 + (n-2)b_1 = 0$.

Corollary 5.27. *A Sasakian manifold M is locally conformally (resp. conharmonically) ϕ-semisymmetric if and only if the relation* (5.18) *holds for any arbitrary vector fields X, Y, Z, U and V on M.*

Corollary 5.28. *A Sasakian manifold M is locally quasi-conformally ϕ-semisymmetric if and only if the relation* (5.18) *holds for any arbitrary vector fields X, Y, Z, U and V on M provided that $b_0 + (n-2)b_1 = 0$.*

6. Conclusion

From the above discussion and results, we conclude that the study of local ϕ-semisymmetry is meaningful as a generalized notion of local ϕ-symmetry and semisymmetry. From Theorem 6.6 and Corollary 6.2 of Shaikh and Kundu [16] we also conclude that the same characterization of local ϕ-semisymmetry of a Sasakian manifold holds for the locally projectively ϕ-semisymmetric and locally concircularly ϕ-semisymmetric Sasakian manifolds as the contraction on projective or concircular curvature tensor gives rise the Ricci operator. And also from Theorem 6.6 and Corollary 6.2 of Shaikh and Kundu [16] we again conclude that the local conformal ϕ-semisymmetry and local conharmonical ϕ-semisymmetry on a Sasakian manifold are equivalent. However, the study of local ϕ-semisymmetry and local conformal ϕ-semisymmetry are meaningful as they are not equivalent. Finally, we conclude that the study of local ϕ-semisymmetry on a Sasakian manifold by considering any other generalized curvature tensor of type (1,3)(which are the linear combination of R, S and g) is either meaningless or redundant due to their equivalency.

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