

## Optimal investment strategy and liability ratio for insurer with Lévy risk process

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### Abstract

We investigate an insurer's optimal investment and liability problem by maximizing the expected terminal wealth under different utility functions. The insurer's aggregate claim payments are modeled by a Lévy risk process. We assume that the financial market consists of a riskless and a risky assets. It is also assumed that the insurer's liability is negatively correlated with the return of the risky asset. The closed-form solution for the optimal investment and liability ratio is obtained using Pontryagin's Maximum Principle. Moreover, the solutions of the optimal control problems are examined and compared to the findings where the jump sizes are assumed to be constant.

**Keywords:** Lévy risk process, Control theory, Pontryagin's maximum principle, Optimal investment strategy, Optimal liability ratio

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## 1. Introduction

The insurer is expected to invest in the financial market to compensate the claim payments for hedging purposes. However, the liability risk can not be eliminated by only investing in the financial market, and thus the insurer needs to consider insurance risk as well as the financial risk. The control problems can be defined as an optimization problem under the certain criterion function from insurer's standpoint. In terms of the actuarial literature, the objective is either to maximize the terminal wealth and to minimize the ruin probability or both.

Cramer-Lundberg model that assumes the aggregate claim payments follow compound Poisson distribution, was introduced by Lundberg in 1903 and republished by Cramer in 1930s. This model has been the main model for insurance problems for a long time. In recent years, the insurer risk process is modeled by diffusion processes since it offers insightful solutions to the actuarial control problems. Browne [2] makes use of diffusion process to obtain an optimal investment strategy by both maximizing the terminal wealth under the exponential utility function and minimizing the ruin probability. Promislow and Young [12] focus on the implication of diffusion processes to obtain both the optimal investment and the reinsurance strategy. The optimal reinsurance is obtained by Schmidli [13] for minimizing the ruin probability when the risk process is controlled by both the Cramer-Lundberg model and diffusion model. In addition to minimizing the ruin probability of insurer, optimal reinsurance policy is achieved by maximizing the survival function of the insurer in the literature. (See Castañer and Claramunt [3]) Hipp and Plum [8] investigate the optimal investment strategy by minimizing the ruin probability for compound Poisson process. Jump structure has also naturally been introduced to insurer's problem as the compound Poisson process approximates to the jump-diffusion process at the limit [15]. Yang and Zhang [18] assume that the risk process is driven by jump-diffusion and the risky asset follows a geometric Brownian motion. Then, the closed-form solution is obtained by maximizing expected terminal wealth under the exponential utility function. Wang et al. [17] show that the optimal investment strategy is obtained via utility maximization for the risk process modeled by a Lévy process.

In this study, we investigate the Pontryagin's Maximum Principle to solve the stochastic control problems. There are two approaches in solving the insurance control problems in the literature: Martingale approach and HJB equation. For the use of martingale approach in financial mathematics to solve optimal investment and consumption problem, see Harrison and Kreps [7], Cadenillas and Karatzas [9]). The optimal investment problems for insurance are also solved by employing the Martingale method (See Zou and Cadenillas [19], Wang et al. [16]). The HJB equation is a dynamic programming and also applied to stochastic control problems based on verification theorem. For theoretical detail, see Fleming and Soner [5]. In insurance market, the HJB equation is applied to stochastic control problems for the different objective functions, for instance, the optimal investment is obtained by maximizing expected terminal utility in Zou and Cadenillas [19] and minimizing the ruin probability in Hipp and Plum [8]. However the Pontryagin's Maximum Principle alternatively suggested instead of HJB for dynamic programming, see Chapter 3 in Øksendal and Sulem [11]. The investment and consumption problems related to finance can also be solved by employing Maximum Principle method, which relaxes the Markovian assumption see Framstad et al. [6].

In the recent work by Zou and Cadenillas [19], the dynamics of risky asset is supposed to follow geometric Brownian motion and the insurer's risk process evolves over time governed by a pure jump-diffusion process. They also impose negative correlation between the liability of insurer and return of the risky asset (see Stein [14]). Then, the optimal investment strategy and liability ratio is obtained by maximizing of the expected

terminal wealth for the logarithmic, power and exponential utility functions using the martingale method.

In insurance market, the claim payments are generally constant and determined at contract date for the life insurance product, whereas the claim payments are randomly emerged for the non-life insurance products. Motivated by that, we extend the model developed by Zou and Cadenillas [19] to the case where sizes of jumps are random and the optimal investment strategy and liability ratio are obtained for insurer's risk process, which follows Lévy process (See Kyprianou [10], Biffis and Kyprianou [1] discussions on Lévy risk process). Then, we assume that the number of policies controls to the insurer's risk.

The main objective of this study is to obtain a closed-form solution for the optimal investment strategy and the liability ratio by using Pontryagin's Maximum Principle, when the insurer's risk process is controlled by Lévy process. This provides an important extension to the current set of the model where it is assumed that the risk process follows jump-diffusion model.

The paper is organized as follows. In Section 2, detailed information of price dynamics is given for the insurer's risk process based on Lévy process. Then, we arrange the insurer's wealth process and define the control problem for insurer. In Section 3, the sufficient conditions for Maximum Principle are explained and closed-form solutions are obtained under exponential, power and logarithmic utilities for control variables. Based on closed-form solution for exponential utility, the optimal controls are analytically obtained for exponential and gamma distributed jump sizes. In section 4, numerical work and the related results are presented.

## 2. The Model

In the financial market, it is assumed that there are two investment opportunities: one riskless and one risky assets. The riskless asset is considered as the banking account whose return is known risk-free interest rate while risky asset is assumed to follow a geometric Brownian motion. The price dynamics for riskless ( $P_0$ ) and risky ( $P_1$ ) assets are given by the following SDEs;

$$(2.1) \quad dP_0(t) = r(t)P_0(t)dt,$$

$$(2.2) \quad dP_1(t) = P_1(t) \left( \mu(t)dt + \sigma(t)dW^{(1)}(t) \right),$$

where  $\mu$ ,  $\sigma$  and  $r$  are positive bounded deterministic function and  $W^{(1)}(t)$  is the standard Brownian motion.

Under the Lévy process, the surplus for the insurer is given by

$$(2.3) \quad dS(t) = pdt - dR(t)$$

where  $p$  is the premium rate at time  $t$  and

$$dR(t) = \bar{a}dt + b d\bar{W}(t) + \int_{\varepsilon \leq \gamma(t,z) < 1} \gamma(t,z) \tilde{N}(dt, dz) + \int_{\gamma(t,z) \geq 1} \gamma(t,z) \tilde{N}(dt, dz)$$

represent to the individual claim process at time  $t$  where  $\bar{a} = a + \int_{\gamma(t,z) \geq 1} \gamma(t,z) \nu(dz)$  and  $\nu(dz)$  is the Lévy measure on  $\mathbb{R}$ .  $a$  and  $b$  are assumed to be positive constants. The drift parameter  $a$  is considered as a consumption per policy and  $b$  is considered as a partition parameter. In the risk process,  $\bar{W}(t)$  is the standard Brownian motion. After the financial crises of 2007-2008, Stein [14] shows that wealth process model has to

assume negative correlation between insurer's liabilities and financial returns. According to idea of Stein [14],  $\bar{W}(t)$  is recently defined as the following form

$$(2.4) \quad \bar{W}(t) = \rho W^{(1)}(t) + \sqrt{1 - \rho^2} W^{(2)}(t)$$

where  $\rho \in [-1, 0]$  is a correlation coefficient,  $W^{(2)}(t)$  is another standard Brownian motion which is independent of  $W^{(1)}(t)$ . Then, the risk process can be written as follows

$$(2.5) \quad dR(t) = \bar{a}dt + b d\bar{W}(t) + \int_{\mathbb{R}} \gamma(t, z) \tilde{N}(dt, dz).$$

(For further discussions on risk process see Appendix A.)

Let  $\tilde{\pi}(t)$  be the amount of risky asset in portfolio and  $L(t)$  be the total amount of liability at time  $t$ . The amount of risky asset and the total amount of liability are defined as control variables as  $\tilde{u}(t) = (\tilde{\pi}(t), L(t))$ . In the literature, premium is calculated by various methods and approximations. We assume that the premium is a constant ratio of insurer's liability as  $p$  to simplify the analysis. Then, the premium income is calculated as  $pL(t)$  at time  $t$ . The wealth process is composed of the cash flow realized as a result of insurance operations and defined as

$$\text{The Wealth} = \text{Initial Wealth} + \text{Financial Gain} + \text{Surplus}.$$

Therefore, the wealth process treats the control variable  $\tilde{u} = (\tilde{\pi}(t), L(t))$  in the following equation

$$(2.6) \quad \begin{aligned} dX^{\tilde{u}}(t) = & \left[ r(t)X^{\tilde{u}}(t) + (\mu(t) - r(t))\tilde{\pi}(t) + (p - \bar{a})L(t) \right] dt \\ & + (\sigma(t)\tilde{\pi}(t) - b\rho L(t))dW^{(1)}(t) - bL(t)\sqrt{1 - \rho^2}dW^{(2)}(t) \\ & - \int_{\mathbb{R}} L(t)\gamma(t, z)\tilde{N}(dt, dz) \end{aligned}$$

It is defined that the liability ratio over wealth at time  $t$  as  $\kappa(t) = \frac{L(t)}{X^{\tilde{u}}(t)}$ . The investment strategy is denoted by  $\pi(t)$  and described as the proportion of wealth allocated to risk investment opportunity at time  $t$ . Then, the wealth process is rearranged for the control variables  $u(t) = (\pi(t), \kappa(t))$  as the following equation

$$(2.7) \quad \begin{aligned} \frac{dX^{\tilde{u}}(t)}{X^{\tilde{u}}(t)} = & [r(t) + (\mu(t) - r(t))\pi(t) + (p - \bar{a})\kappa(t)] dt - \int_{\mathbb{R}} \kappa(t)\gamma(t, z)\tilde{N}(dt, dz) \\ & + (\sigma(t)\pi(t) - b\rho\kappa(t))dW^{(1)}(t) - b\kappa(t)\sqrt{1 - \rho^2}dW^{(2)}(t) \end{aligned}$$

**2.1. Remark.** In financial market, it is expected that the risk is compensated by an extra return, hence we assume the following condition for the drift parameters of riskless and risky assets

$$\mu(t) \geq r(t).$$

The insurer must achieve a balance between expected premium income, expected losses and her expenses. Expected losses and expenses are compensated by expected premium income, hence the premium has a lower bound as

$$p \geq \bar{a} = a + \int_{\gamma(t, z) \geq 1} \gamma(t, z)\nu(dz).$$

The price process of the riskless and risky assets, insurer's risk process and consequently the wealth process are defined on a filtered complete probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  where  $\mathcal{F}_t$  is the collection of information in the market until time  $t$ . The control variables  $u(t) = (\pi(t), \kappa(t))$  (or  $\tilde{u}(t) = (\tilde{\pi}(t), L(t))$ ) are admissible control, then they are predictable with respect to adaptive filtration  $\mathcal{F}_t$  and  $u(t) \in \mathcal{A}$ , where  $\mathcal{A}$  is

set of all admissible strategy with initial condition  $X^u(t) = x$ . The admissible controls satisfy the following conditions

$$\int_0^t \tilde{\pi}(s) ds < \infty \quad \text{and} \quad \int_0^t L(s) ds < \infty.$$

In this study, we aim to obtain the closed-form solutions of optimal controls  $(u(t) = (\pi(t), \kappa(t)))$  for maximizing expected utility of terminal wealth. Hence the objective function is defined as

$$(2.8) \quad J(t, x; u) = E \left[ U(X^{\tilde{u}}(T) | X^{\tilde{u}}(t) = x) \right]$$

where  $T > 0$  is terminal time and  $E$  is expectation operator under probability measure  $\mathbb{P}$ . The utility function is supposed to be strictly increasing and concave function which means that the insurer is risk averse. The optimal investment strategy and liability ratio are represented  $u^*(t) = (\pi^*(t), \kappa^*(t))$  and satisfy the following condition

$$(2.9) \quad J(t, x; u^*) = \sup_{u \in \mathcal{A}} J(t, x; u).$$

**2.2. Problem.** For constant  $(t, x)$ , the insurer's wealth process is defined in equation (2.7) and the control variables  $u(t) = (\pi(t), \kappa(t))$  are admissible. We select the control variables  $u^* = (\pi^*, \kappa^*)$ , which maximize the expected utility of terminal wealth in equation (2.9).

### 3. The Maximum Principle

In this section, we introduce the maximum principle approach with some basic definition and sufficient conditions for insurer's wealth process and optimization problem. We also give the verification theorem for our problem.

For convenience, we suppose that the wealth process of controlled Lévy process on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  is given as

$$(3.1) \quad \begin{aligned} dX^u(t) &= b(t, X, u)dt + \sigma_1(t, X, u)dW^{(1)}(t) \\ &+ \sigma_2(t, X, u)dW^{(2)}(t) + \int_{\mathbb{R}} \gamma(t, X, u)\tilde{N}(dt, dz) \end{aligned}$$

where,  $b : [0, T] \times \mathbb{R} \times U \rightarrow \mathbb{R}$ ,  $\sigma_1 : [0, T] \times \mathbb{R} \times U \rightarrow \mathbb{R}$ ,  $\sigma_2 : [0, T] \times \mathbb{R} \times U \rightarrow \mathbb{R}$  and  $\gamma : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \rightarrow \mathbb{R}$  are continuous function,  $W^{(1)}(t)$  and  $W^{(2)}(t)$  are standard Brownian motion processes and  $\tilde{N}(dt, dz)$  is compensated Lévy process with intensity measure  $\int_{\mathbb{R}} \gamma(t, z)\nu(dz)dt$ .

The wealth process  $(X^u(t))$  which is controlled by a Lévy process in equation (3.1), satisfies linear growth and Lipschitz conditions. The control variable  $(u(t))$  is admissible, while the wealth process  $(X^u(t))$  has an existence and unique solution with the initial condition  $X(t) = x$ .

$$E[|X^u(t)|^2 | X^u(t) = x] < \infty \quad \text{for all } t \in [0, T]$$

(See Oksendal and Sulem [11], Theorem 1.19).

The control process is denoted by  $u(t)$ , predictable and càd-làg process. The objective function is defined for maximizing the expected utility of terminal wealth at equation (2.8).

$$J(t, x; u) = E[U(X^u(T)) | X(t) = x], \quad u \in \mathcal{A}$$

If the control variables are admissible, then the following condition is satisfied

$$E[\max_{u \in \mathcal{A}} U(X^u(T))] < \infty.$$

Indeed the optimal control variables are denoted by  $u^*(t)$  which maximizes the objective function as

$$J(t, x; u^*) = \sup_{u \in \mathcal{A}} J(t, x; u), \quad u \in \mathcal{A}.$$

**3.1. Definition.** The Hamiltonian function for maximizing expected utility of terminal wealth is defined as

$$(3.2) \quad \begin{aligned} H(t, x, u, q_1, q_2, q_3, q_4) = & b(t, x, u)q_1 + \sigma_1(t, x, u)q_2 + \sigma_2(t, x, u)q_3 \\ & + \int_{\mathbb{R}} \gamma(t, x, u, z)q_4(t, z)\nu(dz) \end{aligned}$$

where  $H : [0, T] \times \mathbb{R} \times \mathcal{A} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous function.

It is assumed that the Hamiltonian function is differentiable with respect to  $t$  and  $x$ . Then there exists an adjoint equation corresponding to the admissible pair  $(u, X^u(t))$  which follows the backward stochastic differential equation (BSDE) for unknown adapted processes (adjoint processes)  $(q_1(t) \in \mathbb{R}, q_2(t) \in \mathbb{R}, q_3(t) \in \mathbb{R}, q_4(t, z) \in ([0, T], \mathbb{R}))$ .

**3.2. Definition.** The adjoint equation is defined as

$$(3.3) \quad \begin{aligned} dq_1(t) = & -\frac{\partial}{\partial x} H(t, X(t), u(t), q_1(t), q_2(t), q_3(t), q_4(t, z)) + q_2(t)dW^{(1)}(t) \\ & + q_3(t)dW^{(2)}(t) + \int_{\mathbb{R}} q_4(t, z)\tilde{N}(dt, dz) \end{aligned}$$

with terminal condition

$$(3.4) \quad q_1(T) = \frac{\partial}{\partial x} U(X(T)).$$

The sufficient conditions are given for the maximum principle and the verification theorem for the optimality. Reader can see Framstad et al. [6] with the financial examples.

**3.1. Theorem.** Let  $(u^*(t), X^{u^*}(t))$  be an admissible pair with corresponding solutions  $X^{u^*}(t), q_1^*(t), q_2^*(t), q_3^*(t), q_4^*(t, z)$  process is defined as in equation (3.1), equation (3.3) and equation (3.4). Assume  $u^*$  is an optimal control when the following conditions are satisfied

- (1)  $U$  is linear and concave function of  $x$ .
- (2) The growth condition is satisfied;

$$E \left[ \int_0^T \left( q_2^*(t)^2 + q_3^*(t)^2 + \int_{\mathbb{R}} q_4^*(t, z)^2 \nu(dz) \right) dt \right] < \infty.$$

- (3) Moreover, suppose that

$$(3.5) \quad \begin{aligned} & H^*(t, X(t), u^*(t), q_1^*(t), q_2^*(t), q_3^*(t), q_4^*(t, z)) \\ & = \sup_{u \in \mathcal{A}} H(t, X(t), u(t), q_1^*(t), q_2^*(t), q_3^*(t), q_4^*(t, z)) \end{aligned}$$

for  $\forall t$ .

*Proof.* See also Øksendal and Sulem [11] and Framstad et al. [6]. □

**3.1. Optimal Controls for Utility Functions.** In this section, we aim to obtain optimal controls  $(u^*(t) = \pi^*(t), \kappa^*(t))$  in Problem 2.2 which makes the expected utility of terminal wealth for logarithmic, power and exponential utility function maximum. The wealth process is controlled by Lévy process as in the equation (2.6) with  $X^{\bar{u}}(t) = x > 0$ .

**3.1. Proposition.** Let us assume that the wealth process  $X^{\bar{u}}(t)$  and the objective function are defined as in equation (2.6) and equation (2.8), respectively. Suppose further that the exponential utility function is given by

$$U(x) = -\frac{1}{\alpha}e^{-\alpha x}, \alpha > 0.$$

Then, the optimal investment strategy is obtained as

$$(3.6) \quad \pi_t^* = e^{-r(T-t)} \frac{\mu_t - r_t}{\alpha x \sigma_t^2} + \frac{\rho b}{\sigma_t} \kappa_t^*.$$

Then, the optimal liability ratio satisfies the following equation

$$(3.7) \quad \begin{aligned} \Lambda(\kappa_t^*) &= -(p - \bar{a}) + (-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*) \alpha x e^{r(T-t)} \\ &+ \int_{\mathbb{R}} \gamma(t, z) \left[ \exp(\alpha e^{r(T-t)} \gamma(t, z) \kappa_t^* x) - 1 \right] \nu(dz). \end{aligned}$$

*Proof.* We give the verification theorem for general case in the previous section. Then, we get the Hamiltonian by using the wealth process dynamics defined in equation (2.6)

$$(3.8) \quad \begin{aligned} H(t, X^{\bar{u}}(t) = x, \tilde{\pi}_t, L(t), q_1, q_2, q_3, q_4) &= [xr_t + (\mu_t - r_t)\tilde{\pi}(t) + (p - \bar{a})L(t)] q_1 \\ &+ (\sigma_t \tilde{\pi}(t) - b\rho L(t)) q_2 \\ &- bL(t)\sqrt{1 - \rho^2} q_3 \\ &- \int_{\mathbb{R}} \gamma(t, z) L(t) q_4(t^-, z) \nu(dz) \end{aligned}$$

and the adjoint equation

$$(3.9) \quad dq_1(t) = -r_t q_1(t) + q_2(t) dW^{(1)}(t) + q_3(t) dW^{(2)}(t) + \int_{\mathbb{R}} q_4(t^-, z) \tilde{N}(dt, dz)$$

with terminal condition  $q_1(T) = \exp(-\alpha X(T))$ . In order to obtain other parameters which are defined in equation (3.9), we assume that  $q_1^*(t)$  has the following equation

$$(3.10) \quad q_1^*(t) = \phi(t) \exp(-\alpha e^{r(T-t)} X(t))$$

where  $\phi(t) \in \mathcal{C}^1$  is deterministic function with terminal conditions  $\phi(T) = 1$ .

Using equation (2.6) with  $X^{\bar{u}}(t) = x$ , the differentiation of the unknown adapted process  $(q_1^*(t))$ , which are defined equation (3.10), yields

$$(3.11) \quad \begin{aligned} dq_1^*(t) &= \phi'(t) \exp\{-\alpha e^{r(T-t)} X^{\bar{u}}(t)\} dt + \phi(t) \alpha r e^{r(T-t)} x \exp\{-\alpha e^{r(T-t)} x\} \\ &- \phi(t) \alpha e^{r(T-t)} \exp\{-\alpha e^{r(T-t)} X^{\bar{u}}(t)\} [r_t x + (\mu_t - r_t) \tilde{\pi}_t + (p - \bar{a}) L_t] dt \\ &- \phi(t) \alpha e^{r(T-t)} \exp\{-\alpha e^{r(T-t)} X^{\bar{u}}(t)\} \left[ (\sigma_t \tilde{\pi}_t - b\rho L_t) dW^{(1)}(t) - bL_t \sqrt{1 - \rho^2} dW^{(2)}(t) \right] \\ &+ \phi(t) \alpha^2 e^{2r(T-t)} \exp\{-\alpha e^{r(T-t)} X^{\bar{u}}(t)\} \frac{1}{2} [\sigma_t^2 \tilde{\pi}_t - 2b\rho \sigma_t \tilde{\pi}_t L_t + b^2 L_t^2] dt \\ &+ \int_{\mathbb{R}} \phi(t) \left[ \exp\{-\alpha e^{r(T-t)} (X^{\bar{u}}(t) - L(t) \gamma(t, z))\} - \exp\{-\alpha e^{r(T-t)} X^{\bar{u}}(t)\} \right. \\ &\quad \left. + \alpha e^{r(T-t)} \exp\{-\alpha e^{r(T-t)} X^{\bar{u}}(t)\} \gamma(t, z) \mathbf{1}_{\varepsilon \leq \gamma(t, z) < 1} \right] \nu(dz) dt \\ &+ \int_{\mathbb{R}} \phi(t) \left[ \exp\{-\alpha e^{r(T-t)} (X^{\bar{u}}(t) - L(t) \gamma(t, z))\} - \exp\{-\alpha e^{r(T-t)} X^{\bar{u}}(t)\} \right] d\tilde{N}(dt, dz) \end{aligned}$$

Comparing with equation (3.9) and equation (3.11), we get the following equations for the unknown adapted processes  $(q_2^*(t), q_3^*(t)$  and  $q_4^*(t^-, z))$

(3.12a)

$$\begin{aligned} q_2^*(t) &= -\phi(t)\alpha e^{r(T-t)} \exp\{-\alpha e^{r(T-t)}x\}(\sigma_t\tilde{\pi}_t - b\rho L_t) \\ &= -\phi(t)\alpha x e^{r(T-t)} \exp\{-\alpha e^{r(T-t)}x\}(\sigma_t\pi_t - b\rho\kappa_t), \end{aligned}$$

(3.12b)

$$\begin{aligned} q_3^*(t) &= -\phi(t)\alpha e^{r(T-t)} \exp\{-\alpha e^{r(T-t)}x\}(-bL_t\sqrt{1-\rho^2}) \\ &= -\phi(t)\alpha x e^{r(T-t)} \exp\{-\alpha e^{r(T-t)}x\}(-b\kappa_t\sqrt{1-\rho^2}) \end{aligned}$$

(3.12c)

$$\begin{aligned} q_4^*(t^-, z) &= \phi(t) \left[ \exp\{-\alpha e^{r(T-t)}(x - L_t\gamma(t, z))\} - \exp\{-\alpha e^{r(T-t)}x\} \right] \\ &= \phi(t) \exp\{-\alpha e^{r(T-t)}x\} \left[ \exp\{\alpha e^{r(T-t)}x\kappa_t\gamma(t, z)\} - 1 \right] \end{aligned}$$

where  $\pi = \tilde{\pi}/x$  and  $\kappa = L/x$  are called investment strategy and liability ratio defined in Section 2.

Finally, we get the linear system to obtain optimal control variable  $(u^*(t) = (\pi^*(t), \kappa^*(t)))$ ;

$$\begin{aligned} H^*(t, x, \tilde{\pi}_t, L_t, q_1^*(t), q_2^*(t), q_3^*(t), q_4^*(t^-, z)) &= [r_t x + (\mu_t - r_t)\tilde{\pi}_t + (p - \bar{a})L_t] q_1^*(t) \\ &+ (\sigma_t\tilde{\pi}_t - b\rho L_t) q_2^*(t) - bL_t\sqrt{1-\rho^2} q_3^*(t) - \int_{\mathbb{R}} L_t\gamma(t, z) q_4^*(t^-, z) \nu(dz). \end{aligned}$$

The critical points refer to the maximum level of the linear system. Hence, the optimal investment amount  $\tilde{\pi}_t^*$  satisfies the following equation

$$(3.13) \quad \frac{\partial H^*}{\partial \tilde{\pi}_t} = (\mu_t - r_t) q_1^*(t) + \sigma_t q_2^*(t) = 0.$$

We get the following equation by substituting the unknown adapted processes  $q_1^*(t)$  and  $q_2^*(t)$  in equation (3.10) and equation (3.12a) into equation (3.13),

$$\begin{aligned} \frac{\partial H^*}{\partial \tilde{\pi}_t} &= (\mu_t - r_t)\phi(t) \exp\{-\alpha e^{r(T-t)}x\} \\ &+ \sigma_t\phi(t) \left( -\alpha e^{r(T-t)}x \right) \exp\{-\alpha e^{r(T-t)}x\}(\sigma_t\pi_t - b\rho\kappa_t^*) = 0. \end{aligned}$$

Hence, the optimal investment strategy is obtained as following equation

$$\pi_t^* = e^{-r(T-t)} \frac{\mu_t - r_t}{x\alpha\sigma_t^2} + \frac{\rho b}{\sigma_t} \kappa_t^*.$$

By the similar argument, the optimal liability amount  $L_t^*$  satisfies the following equation at the critical point.

$$(3.14) \quad \frac{\partial H^*}{\partial L_t^*} = (p - \bar{a}) q_1^*(t) - b\rho q_2^*(t) - b\sqrt{1-\rho^2} q_3^*(t) - \int_{\mathbb{R}} \gamma(t, z) q_4^*(t^-, z) \nu(dz) = 0$$

The unknown adapted processes  $(q_1^*(t), q_2^*(t), q_3^*(t)$  and  $q_4^*(t^-, z))$  are given in equation (3.10) and equation (3.12), hence the right hand side of equation (3.14)

is rearranged

$$\begin{aligned} \frac{\partial H^*}{\partial L_t^*} &= [p - \bar{a}] \phi(t) \exp(-\alpha e^{r(T-t)} x) \\ &+ (-\rho b) \left[ -\phi(t) \alpha x e^{r(T-t)} \exp\{-\alpha e^{r(T-t)} x\} (\sigma_t \pi_t - b \rho \kappa_t) \right] \\ &+ ((-b\sqrt{1-\rho^2})) \left[ -\phi(t) \alpha x e^{r(T-t)} \exp\{-\alpha e^{r(T-t)} x\} (-b\kappa_t \sqrt{1-\rho^2}) \right] \\ &- \phi(t) \exp(-\alpha e^{r(T-t)} x) \int_{\mathbb{R}} \gamma(t, z) \left[ \exp(\alpha e^{r(T-t)} \gamma(t, z) \kappa_t^* x) - 1 \right] \nu(dz) = 0. \end{aligned}$$

Then the optimal liability ratio satisfies the following equation as

$$\begin{aligned} \Lambda(\kappa_t^*) &= -(p - \bar{a}) + (-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*) \alpha x e^{r(T-t)} \\ &+ \int_{\mathbb{R}} \gamma(t, z) \left[ \exp(\alpha e^{r(T-t)} \gamma(t, z) \kappa_t^* x) - 1 \right] \nu(dz) = 0. \end{aligned}$$

□

**3.2. Proposition.** Let us assume that the wealth process  $X^{\bar{u}}(t)$  and the objective function are defined as in equation (2.6) and equation (2.8), respectively. Suppose further that the power utility function is given by

$$U(x) = \frac{1}{\alpha} x^\alpha, \quad \alpha < 1, \quad \alpha \neq 0.$$

The optimal investment strategy satisfies the following equation as

$$\pi_t^* = -\frac{\mu_t - r_t}{\sigma_t^2(\alpha - 1)} + \frac{\rho b}{\sigma_t} \kappa_t^*.$$

The optimal liability ratio satisfies the following equation as

$$\begin{aligned} \Lambda(\kappa_t^*) &= (p - \bar{a}) + [-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*](\alpha - 1) \\ &- \int_{\mathbb{R}} \gamma(t, z) [(1 - \gamma(t, z) \kappa_t^*)^{\alpha-1} - 1] \nu(dz) = 0. \end{aligned}$$

*Proof.* The proof can be obtained in a similar way to that in proof of Proposition 3.1. The Hamiltonian function and adjoint equation are defined in equation (3.8) and equation (3.9) for the Problem 2.2, respectively. However, the terminal condition of adjoint equation is  $q_1(T) = (X^{\bar{u}}(T))^{\alpha-1}$  under the power utility function. Hence, it is assumed that the unknown adapted process  $(q_1^*(t))$  follows the process as

$$(3.15) \quad q_1^*(t) = \phi(t) X^{\bar{u}}(t)^{\alpha-1}$$

where  $\phi(t) \in \mathcal{C}^1$  is deterministic function with terminal conditions  $\phi(T) = 1$ . Based on the wealth process is defined in equation (2.6), the differentiation of the unknown

adapted process  $(q_1^*(t))$  yields the following equation for the power utility function.

$$\begin{aligned}
 dq_1^*(t) &= \phi'(X^{\bar{u}}(t))^{\alpha-1} \\
 &+ \phi(t)(\alpha-1)(X^{\bar{u}}(t))^{\alpha-2} \left[ \{r_t X^{\bar{u}}(t) + (\mu_t - r_t)\tilde{\pi}_t + (p - \bar{a})L_t\} dt \right. \\
 &\quad \left. + (\sigma_t \tilde{\pi}_t - b\rho L_t) dW^{(1)}(t) - bL_t \sqrt{1 - \rho^2} dW^{(2)}(t) \right] \\
 (3.16) \quad &+ \phi(t)(\alpha-1)(\alpha-2)(X^{\bar{u}}(t))^{\alpha-3} \frac{1}{2} [\sigma_t^2 \tilde{\pi}_t^2 - 2b\rho\sigma_t \tilde{\pi}_t L_t + b^2 L_t^2] dt \\
 &+ \int_{\mathbb{R}} \phi(t) \left[ (X^{\bar{u}}(t) - L_t \gamma(t, z))^{\alpha-1} - (X^{\bar{u}}(t))^{\alpha-1} - (\alpha-1)(X^{\bar{u}}(t))^{\alpha-2} \gamma(t, z) \right] \nu(dz) \\
 &+ \int_{\mathbb{R}} \phi(t) \left[ (X^{\bar{u}}(t) - L_t \gamma(t, z))^{\alpha-1} - (X^{\bar{u}}(t))^{\alpha-1} \right] \tilde{N}(dt, dz)
 \end{aligned}$$

Comparing with the equation (3.9) and equation (3.16), we attain the unknown adapted processes  $(q_2^*(t), q_3^*(t)$  and  $q_4^*(t^-, z))$  as the following equations:

$$\begin{aligned}
 (3.17a) \quad q_2^*(t) &= \phi(t)(\alpha-1)X^{\bar{u}}(t)^{\alpha-2}(\sigma_t \tilde{\pi}_t - \rho b L_t) \\
 &= \phi(t)(\alpha-1)X^{\bar{u}}(t)^{\alpha-1}(\sigma_t \pi_t - \rho b \kappa_t),
 \end{aligned}$$

$$\begin{aligned}
 (3.17b) \quad q_3^*(t) &= \phi(t)(\alpha-1)X^{\bar{u}}(t)^{\alpha-2}(-b\sqrt{1-\rho^2})L_t \\
 &= \phi(t)(\alpha-1)X^{\bar{u}}(t)^{\alpha-1}(-b\sqrt{1-\rho^2}\kappa_t),
 \end{aligned}$$

$$\begin{aligned}
 (3.17c) \quad q_4^*(t^-, z) &= \phi(t)[(X^{\bar{u}}(t) - \gamma(t, z)L_t)^{\alpha-1} - X^{\bar{u}}(t)^{\alpha-1}] \\
 &= \phi(t)X^{\bar{u}}(t)^{\alpha-1}[(1 - \gamma(t, z)\kappa_t^*)^{\alpha-1} - 1]
 \end{aligned}$$

where  $\pi = \tilde{\pi}/x$  and  $\kappa = L/x$  are called investment strategy and liability ratio in Section 2, respectively.

Finally, we get linear system to obtain optimal control variables  $(u^*(t) = (\pi^*(t), \kappa^*(t)))$  under the power utility function. The critical points refer to the maximum level of the linear system. Hence, the optimal investment strategy is obtained as

$$\pi_t^* = -\frac{\mu_t - r_t}{(\alpha-1)\sigma_t^2} + \frac{\rho b}{\sigma_t} \kappa_t^*.$$

In a similar manner, the optimal liability ratio satisfies the following equation

$$\begin{aligned}
 A(\kappa_t^*) &= (p - \bar{a}) + [-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*](\alpha - 1) \\
 &\quad - \int_{\mathbb{R}} \gamma(t, z) [(1 - \gamma(t, z)\kappa_t^*)^{\alpha-1} - 1] \nu(dz) = 0.
 \end{aligned}$$

□

**3.3. Proposition.** Let us assume that the wealth process  $X^{\bar{u}}(t)$  and the objective function are defined as in equation (2.6) and equation (2.8), respectively. Suppose further that the logarithmic utility function is given by

$$U(x) = \ln(x), \quad x > 0.$$

Then, the optimal investment strategy is obtained

$$\pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{\rho b}{\sigma_t} \kappa_t^*$$

and the optimal liability ratio satisfies the following equation

$$(3.18) \quad A(\kappa_t^*) = (p - \bar{a}) - [-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*] - \int_{\mathbb{R}} \left[ \frac{\gamma(t, z)}{1 + \kappa_t^* \gamma(t, z)} - 1 \right] \nu(dz) = 0.$$

*Proof.* Note that under the logarithmic utility function similar in spirit of the last two proofs. Therefore, the Hamiltonian function and adjoint equation are defined in equation (3.8) and equation (3.9) with terminal condition  $\frac{1}{X(T)}$  for logarithmic utility function, respectively. We assume that the unknown adapted process  $q_1^*(t)$  has the following form

$$(3.19) \quad q_1^*(t) = \frac{\phi(t)}{X(t)}$$

where  $\phi(t) \in \mathcal{C}^1$  is deterministic function with terminal conditions  $\phi(T) = 1$ . In that case, we achieve the unknown adapted processes ( $q_2^*(t)$ ,  $q_3^*(t)$  and  $q_4^*(t^-, z)$ ) by comparing the adjoint equation and the differentiation of the unknown adapted process ( $q_1^*(t)$ ) which is defined equation (3.19). The critical points refer to the maximum of the linear system. The optimal investment strategy ( $\pi_t^*$ ) is obtained

$$\pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{\rho b}{\sigma_t} \kappa_t^*$$

and the optimal liability ratio ( $\kappa_t^*$ ) satisfies the following equation

$$\Lambda(\kappa_t^*) = (p - \bar{a}) - [-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*] - \int_{\mathbb{R}} \left[ \frac{\gamma(t, z)}{1 + \kappa_t^* \gamma(t, z)} - 1 \right] \nu(dz) = 0.$$

□

**3.2. The Analysis Under The Exponential Utility Function.** The closed-form solution of optimal controls ( $\pi^*(t)$ ,  $\kappa^*(t)$ ) for the risk process in equation (2.5), which is infinite activity Lévy process, are obtained under exponential, power and logarithmic utility functions in Section 3.1. Since the jumps arrive infinitely often for such kind of Lévy processes, the distribution of the jumps size does not exist (See Cont and Tankov [4]). Hence, in order to apply our results, in this section we assume that the insurer's risk per policy follows jump-diffusion dynamics. The optimal controls ( $\pi^*(t)$ ,  $\kappa^*(t)$ ) are analytically obtained by maximizing expected terminal wealth under the exponential utility function.

$$(3.20) \quad dR(t) = \bar{a}dt + bd\bar{W}(t) + \int_{\mathbb{R}} \gamma \tilde{N}(dt, d\gamma)$$

where  $\nu(d\gamma) = \lambda F(d\gamma)$  Lévy measure and  $\bar{a} = a + \lambda \int_{\mathbb{R}} \gamma F(d\gamma)$ ,  $F$  is distribution function of jumps size (For further discussions see Appendix A). The claim size is defined as a non-negative continuous random variable. Furthermore, it is assumed that the claim sizes follow exponential family of distributions. (See Schmidli [13], Yang et al.[18]).

Based on the results of Proposition 3.1, the optimal liability ratio for the risk process controlled jump-diffusion process in equation (3.20) satisfies the following equation

$$(3.21) \quad \begin{aligned} \Lambda(\kappa_t^*) = & -(p - a) + (-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*) \alpha x e^{r(T-t)} \\ & + \int_{\mathbb{R}} z \exp(\alpha e^{r(T-t)} z \kappa_t^* x) \lambda F(dz) \end{aligned}$$

where  $F$  is a distribution function of jumps size and  $\lambda$  is Poisson intensity.

**3.2. Remark.** Zou and Cadenillas [19], assumed that the jump size in risk model is a positive constant, thus the optimal controls  $u^*(t) = (\pi^*(t), \kappa^*(t))$  are obtained by maximizing expected terminal wealth under the exponential utility function. The optimal investment strategy is obtained by

$$\pi_t^* = e^{-r(T-t)} \frac{\mu_t - r_t}{\alpha x \sigma_t^2} + \frac{\rho b}{\sigma_t} \kappa_t^*$$

and the optimal liability ratio satisfies the following equation as

$$\Lambda(\kappa_t^*) = -(p - a) + (-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*) \alpha x e^{r(T-t)} + \lambda \gamma \exp(\alpha e^{r(T-t)} \gamma \kappa_t^* x) = 0.$$

**3.4. Proposition.** The closed-form of the optimal controls  $(\pi^*(t), \kappa^*(t))$  are defined as in equation (3.6) and equation (3.21) for jump-diffusion risk process under the exponential utility function. If the jump sizes follow an exponential distribution with parameter  $(\beta)$ , then  $\kappa^*(t)$  is satisfied by

$$(3.22) \quad \Lambda(\kappa_t^*) = -(p - a) + (-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*) \alpha x e^{r(T-t)} + \lambda \frac{\beta}{(\beta - \alpha \kappa_t^* e^{r(T-t)})^2} = 0$$

with the condition  $\beta - \alpha \kappa_t^* e^{r(T-t)} \neq 0$ . As a result of equation (3.22), the optimal controls  $(\pi^*(t), \kappa^*(t))$  are implicitly obtained.

**3.5. Proposition.** The closed-form solution of the optimal controls  $(\pi^*(t), \kappa^*(t))$  are defined in equation (3.6) and equation (3.21) for jump-diffusion risk process under exponential utility function. If the jump sizes  $\gamma$  have a gamma distribution with parameters  $(\theta, k)$ , then  $\kappa^*(t)$  satisfies by

$$(3.23) \quad \Lambda(\kappa_t^*) = -(p - a) + (-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*) \alpha x e^{r(T-t)} + \lambda \frac{\theta k}{(1 - \theta \alpha e^{r(T-t)} \kappa_t^*)^{k+1}} = 0$$

with the condition  $(1 - \theta \alpha e^{r(T-t)} \kappa_t^*)^{k+1} \neq 0$ . As a result of equation (3.23), the optimal controls are implicitly obtained.

## 4. Numerical Example

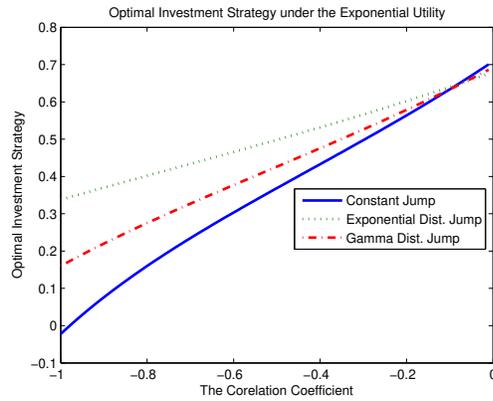
In this section, we present an example to illustrate the impact of jump sizes assumptions. There are two assumptions regarding to risk model: constant jump sizes as in Zou and Cadenillas [19] and random jump sizes as in Section 2, respectively. In this study, Exponential and Gamma distributions are used for modeling the random jump sizes. The uncertainty of the risk process is identified by the variance of risk process. We investigate the impact of the variance (volatility) of jump sizes (risk process) on the optimal control variables. For the comparison purposes, the parameter values are selected as same as the ones used in Zou and Cadenillas [19]. The following parameters shown in Table 1 are assumed to be constant with the corresponding values presented at the second row.

**Table 1.** The parameters for wealth process

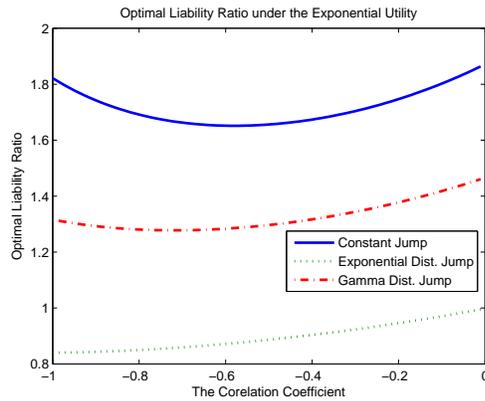
x	$\mu$	r	$\sigma$	a	b	p	$\lambda$	$\alpha$
1	0.05	0.01	0.25	0.08	0.1	0.15	0.1	1

The impact of the negative correlation between the insurance liability and financial returns on the control variables is examined by Zou and Cadenillas [19] when the jump sizes are fixed at 0.3. For comparing our numerical results, we take that the jump sizes have exponential or gamma distributions with mean 0.3. Therefore, the parameters of exponential/gamma distributions are  $(\beta = 10/3)/(\theta = 0.1, k = 3)$ . The Figs. (1a) and (1b) show that the impact of correlation coefficient on the considered variables is similar to the constant and random jump sizes assumption cases. Fig. 1(a) shows that the optimal investment strategy is an increasing function of the correlation for both assumption, this behavior can be explained in equation (3.6). It can be concluded that the optimal investment strategy  $(\pi^*(t))$  attains its maximum value when insurer's liability and financial returns are uncorrelated. We observe that the optimal liability ratio  $\kappa^*(t)$  is convex function of the correlation coefficient for both assumption. It attains minimum

value when the correlation coefficient takes values around the middle of interval  $[-1, 0]$ , since the uncertainty of risk process attains its maximum value.



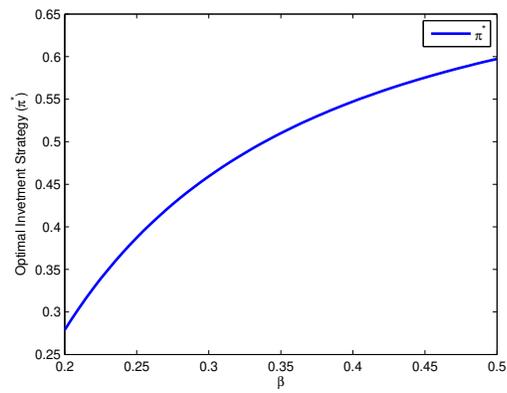
(a) The optimal investment strategy ( $\pi^*$ )



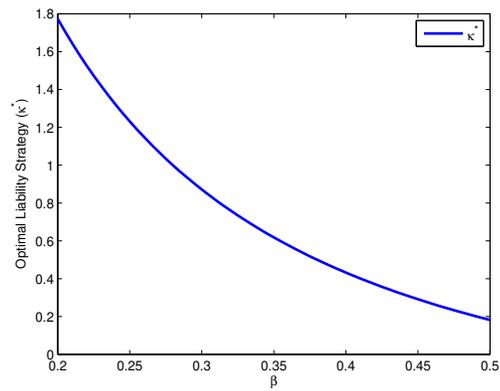
(b) The optimal liability ratio ( $\kappa^*$ )

**Figure 1.** The impact of negative correlation on the optimal controls  $(\pi^*, \kappa^*)$  for exponential utility function

It is seen that the optimal liability ratio  $(\kappa^*(t))$  for case of exponential distribution has lowest value. It can be explained that the variance of the risk process under the exponential distribution with parameter  $(\beta = 10/3)$  is higher than others and an increase in variance leads an increase in volatility of the risk process. As a result, the optimal liability decreases. On the other hand, optimal investment strategy has to have a higher value to compensate the possible increase in the volatility of the risk process.

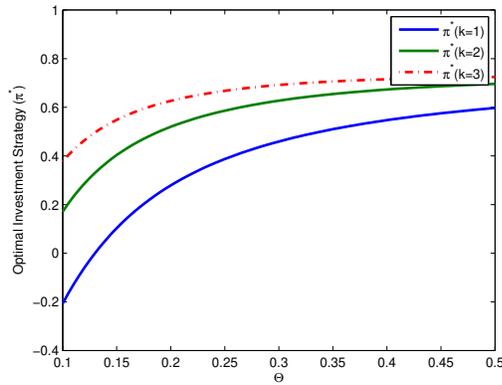


(a) The optimal investment strategy ( $\pi^*$ )

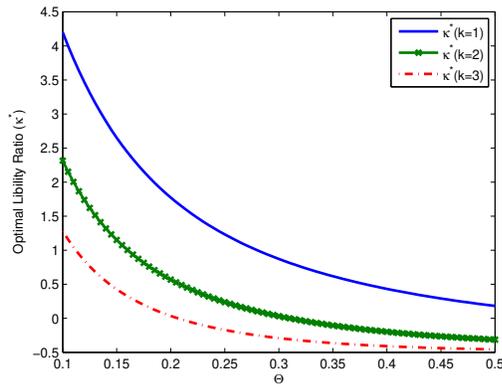


(b) The optimal liability ratio ( $\kappa^*$ )

**Figure 2.** The impact of exponential distribution parameter ( $\beta$ ) on the optimal controls ( $\pi^*$ ,  $\kappa^*$ ) for exponential utility function



(a) The optimal investment strategy ( $\pi^*$ )



(b) The optimal liability ratio ( $\kappa^*$ )

**Figure 3.** The impact of gamma distribution parameters  $(\theta, k)$  on the optimal controls  $(\pi^*, \kappa^*)$  for exponential utility function

Secondly, the impact of the parameters of exponential/gamma distributions is analyzed on the optimal control variables. In this part, the correlation coefficient between the insurance liability and financial returns is fixed at  $\rho = -0.5$ , the exponential distribution parameter  $\beta$  is assumed as between  $[0.2, 0.5]$  and the gamma distribution parameter  $\theta$  is examined in the interval  $[0.1, 0.5]$  for  $k = 1, 2, 3$ . If the variation in the parameters causes an increase in the expected value or volatility of claim payments, then the optimal liability ratio decreases and the optimal investment strategy increases to compensate the higher claim payments. From Figs. (2) and (3), it can be seen that the optimal investment strategy ( $\pi^*$ ) is an increasing function of parameters  $(\beta, \theta, k)$  for exponential and gamma distributed jump sizes model, thus the optimal liability strategy ( $\kappa^*$ ) is decreasing function of parameters  $(\beta, \theta, k)$ .

Lastly, the impact of risk-aversion parameter on the optimal control variables  $(\pi^*, \kappa^*)$  is examined in this section. If the utility function is concave, the insurer or investor has a risk-averse utility function. The exponential utility function is risk-averse according to the risk aversion parameter  $\alpha$ . Hence, insurer can tolerate the risk based on risk aversion

parameter selection. For example, the risk aversion parameter  $\alpha$  is 0.5 and 1 for low risk aversion and high risk aversion respectively. It can be seen that the optimal investment strategy and liability ratio are lower for the higher risk aversion model.

## 5. Conclusion

This paper has two contributions to the literature. Firstly, the investment strategy and the liability ratio are implicitly obtained for the random claims size model by maximizing expected terminal wealth under exponential utility. It is apparently a reasonable relaxation of the constancy assumption, as real life experience shows us that the claims sizes indeed arrive in different sizes over time. As a second contribution, we have shown that the control problems for insurer's decision can also be solved by The Pontryagin's Maximum Principle in addition to HJB or martingale methods.

The findings of numerical results indicate that the random jump sizes assumption causes an increase in the risk process volatility, hence the optimal liability ratio, as expected, is lower than the constant jump sizes model. On the other hand, the ratio of the investment in risky asset increases to compensate the possible increase in the volatility of risk process. This research is concerned with implicit solution for the optimal control variables that maximize the terminal wealth under exponential utility choice. The results have shown that closed-form solutions could only be obtained for power and logarithmic utilities.

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## Appendix A.

**A.1. Proposition.** Let  $R(t)_{0 \leq t \leq T}$  be a risk process controlled by infinite activity Lévy process on  $\mathbb{R}$ . Then by Itô-Lévy decomposition; the dynamic of the insurer's risk process satisfies

$$dR(t) = adt + bd\bar{W}(t) + \gamma_t^l + \lim_{\varepsilon \downarrow 0} \gamma_t^\varepsilon$$

where

$$\gamma_t^l = \int_{\gamma(t,z) \geq 1} \gamma(t,z) N(dt, dz)$$

and

$$\gamma_t^\varepsilon = \int_{\varepsilon \leq \gamma(t,z) < 1} \gamma(t,z) [N(dt, dz) - \nu(dz)dt].$$

$\gamma_t^l$  can be written following form

$$\gamma_t^l = \int_{\gamma(t,z) \geq 1} \gamma(t,z) [\tilde{N}(dt, dz) + \nu(dz)dt]$$

where  $N$  is a Poisson random measure on  $[0, \infty) \times \mathbb{R}$  with intensity measure  $\nu(dz)$ .

*Proof.* See Proposition 3.7 in Cont and Tankov [4]. □

**A.1. Remark.** Note that every compound Poisson process can be written in the following form

$$X(t) = \sum_{i=1}^{N(t)} \gamma_i = \int_0^t \int_{\mathbb{R}} \gamma N(dt, d\gamma).$$

Indeed this is just a special case of Lévy-Itô decomposition. (see Proposition 3.5 in Cont and Tankov [4])

Hence, if the insurer's risk per policy is assumed to be a jump-diffusion process (finite activity Lévy process), then its dynamics can be represented in the following form

$$(A.1) \quad dR(t) = adt + bd\bar{W}(t) + \int_{\mathbb{R}} \gamma N(dt, d\gamma)$$

with intensity measure  $\nu(dz) = \lambda F(dz)$  where  $F$  is distribution function of jump sizes. Or equivalently, it can be written as follows the risk process is

$$(A.2) \quad dR(t) = \bar{a}dt + bd\bar{W}(t) + \int_{\mathbb{R}} \gamma \tilde{N}(dt, d\gamma)$$

where  $\bar{a} = a + \int_{\mathbb{R}} \gamma \nu(d\gamma)$ .

The optimal liability ratio satisfies equation (3.7) for infinite Lévy process the under the exponential utility function, however, it is obtained for the assumption of jump-diffusion process equation (A.2) in the following equation

$$(A.3) \quad \Lambda(\kappa_t^*) = - \left( p - a - \int_{\mathbb{R}} \gamma \nu(d\gamma) \right) + (-\rho b \sigma_t \pi_t^* + b^2 \kappa_t^*) \alpha x e^{r(T-t)} \\ + \int_{\mathbb{R}} \gamma \left[ \exp(\alpha e^{r(T-t)} \gamma \kappa_t^* x) - 1 \right] \nu(d\gamma)$$

where  $\nu(d\gamma) = \lambda F(d\gamma)$  and  $F$  distribution function.