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On the Lifts of $F^K + F = 0$, $(F \neq 0, K \geqslant 0)-$Structure on Cotangent and Tangent Bundle

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Abstract

This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of $F(K,1)-$structure satisfying $F^K + F = 0$. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of $F(K,1)-$structure in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the horizontal lifts of $F(K,1)-$structure. In the second part, all results obtained in the first section were obtained according to the complete and horizontal lifts of $F(K,1)-$structure in tangent bundle $T(M^n)$.

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1. Introduction

The investigation for the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example [14]. There are a lot of structures on $n-$dim. differentiable manifold M^n. Firstly, Ishıhara and Yano [7] have obtained the integrability conditions of a structure F satisfying $F^3 + F = 0$. Gouli-Andreou [1] has studied the integrabilty conditions of a structure F satisfying $F^5 + F = 0$. Later, R. Nivas and C.S. Prasad [10] studied on the form $F_a(5,1)-$structure. In 1989, V. C. Gupta [6] studied on more generalized form $F(K,1)-$structure satisfying $F^K + F = 0$, where K is a positive integer ≥ 2. This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of $F(K,1)-$structure satisfying $F^K + F = 0$. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of $F(K,1)-$structure in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the horizontal lifts of $F(K,1)-$structure. In the second part, all results obtained in the first section were obtained according to the complete and horizontal lifts of $F(K,1)-$structure in tangent bundle $T(M^n)$.
Let us consider an $n-$dimensional differentiable manifold M^n of class C^∞ equipped with a non-null tensor field $F(\neq 0)$ of type $(1,1)$ and of class C^∞ satisfying

$$F^K + F = 0, \tag{1.1}$$

where K is a positive integer ≥ 2.
Let us put $(1,1)$ tensor s and t

$$s = -F^{K-1}, t = I + F^{K-1}, \tag{1.2}$$

where I being the identity operator. Then we have the properties

$$s^2 = s, t^2 = t, s.t = t.s = 0, s + t = I.$$

Consequently, if there is a tensor field $F \neq 0$ satisfying (1.1), then there exist on M^n two complementary distributions S and T. Corresponding to s and t respectively. Let the rank of F be constant and be equal to r ewerywhere, then the dimensions of S and T are r and $n-r$, respectively. We call such a structure a '$F(K,1)-$structure of rank r' and the manifold M^n with this structure a '$F(K,1)-$manifold.'
In the manifold M^n endowed with $F^K + F = 0$, $(F \neq 0, K \geq 2)$ structure, the $(1,1)$ tensor field ψ given by $\psi = s - t = -I - 2F^{K-1}$ gives an almost product structure.

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1.1. Horizontal Lift of the Structure Satisfying $F^K+F=0$, $(F\neq 0, K\geq 0)$ on Cotangent Bundle

Let F,G be two tensor field of type $(1,1)$ on the manifold M^n. If F^H denotes the horizontal lift of F, we have [9, 14]

$$F^HG^H+G^HF^H=(FG+GF)^H. \tag{1.3}$$

Taking F and G identical, we get

$$(F^H)^2=(F^2)^H, \tag{1.4}$$

Continuing the above process of replacing G in equation (1.3) by some higher powers of F, we obtain

$$(F^K)^C=(F^C)^K,$$

where K is a positive integer ≥ 2. Also if G and H are tensors of the same type then

$$(G+H)^H=G^H+H^H$$

Taking horizontal lift on both sides of equation $F^K+F=0$, we get

$$(F^H)^K=(F^K)^H. \tag{1.5}$$

Since F gives on M^n the $F(K,1)-$structure, we have

$$F^K+F=0. \tag{1.6}$$

Taking horizontal lift, we obtain

$$(F^K)^H+F^H=0. \tag{1.7}$$

In view of (1.5) and (1.7), we can write [9]

$$(F^H)^K+F^H=0. \tag{1.8}$$

Proposition 1.1. *Let M^n be a Riemannian manifold with metric g, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle $T^*(M^n)$ of M^n satisfies the following*

$$\begin{aligned} i)\ [\omega^V,\theta^V] &= 0,\\ ii)\ \left[X^H,\omega^V\right] &= (\nabla_X\omega)^V,\\ iii)\ \left[X^H,Y^H\right] &= [X,Y]^H+\gamma R(X,Y)=[X,Y]^H+(pR(X,Y))^V \end{aligned} \tag{1.9}$$

for all $X,Y\in\Im_0^1(M^n)$ and $\omega,\theta\in\Im_1^0(M^n)$. (See [14] p. 238, p. 277 for more details).

2. Main Results

Definition 2.1. *Let F be a tensor field of type $(1,1)$ admitting $F^K+F=0$ structure in M^n. The Nijenhuis tensor of a $(1,1)$ tensor field F of M^n is given by*

$$N_F=[FX,FY]-F[X,FY]-F[FX,Y]+F^2[X,Y] \tag{2.1}$$

for any $X,Y\in\Im_0^1(M^n)$ [2, 11, 12]. The condition of $N_F(X,Y)=N(X,Y)=0$ is essential to integrability condition in these structures. The Nijenhuis tensor N_F is defined local coordinates by

$$N_{ij}^k\partial_k=(F_i^s\partial_sF_j^k-F_j^l\partial_lF_i^k-\partial_iF_j^lF_l^k+\partial_jF_i^sF_s^k)\partial_k \tag{2.2}$$

where $X=\partial_i$, $Y=\partial_j$, $F\in\Im_1^1(M^n)$.

2.1. The Nijenhuis Tensors of $(F^K)^H$ on Cotangent Bundle $T^*(M^n)$

Theorem 2.2. *The Nijenhuis tensors of* $(F^K)^H$ *and* F *denote by* $\tilde{N}$ *and* N*, respectively. Thus, taking account of the definition of the Nijenhuis tensor, the formulas (1.9) stated in Proposition 1.1 and the structure* $(F^K)^H+F^H=0$*, we find the following results of computation.*

$$\begin{aligned} i)\ \tilde{N}_{(F^K)^H(F^K)^H}\left(X^H,Y^H\right) &= \{[FX,FY]-F[FX,Y]-F[X,FY]\\ &\quad +F^2[X,Y]\}^H+\gamma\{R(FX,FY)-R(FX,Y)F\\ &\quad -R(X,FY)F+R(X,Y)(F)^2\}.\end{aligned}$$

$$ii)\ \tilde{N}_{(F^K)^H(F^K)^H}\left(X^H,\omega^V\right) = \{\omega\circ(\nabla_{FX}F)-(\omega\circ(\nabla_X F)F\}^V,$$

$$iii)\ \tilde{N}_{(F^K)^H(F^K)^H}\left(\omega^V,\theta^V\right) = 0.$$

Proof. *i)*The Nijenhuis tensor $\tilde{N}_{(F^K)^H(F^K)^H}(X^H,Y^H)$ of the horizontal lift $(F^K)^H$ vanishes if F is an almost complex structure i.e., $F^2=-I$ and $R(FX,FY)=R(X,Y)$.

$$\begin{aligned}\tilde{N}_{(F^K)^H(F^K)^H}(X^H,Y^H) &= [(F^K)^H X^H,(F^K)^H Y^H]-(F^K)^H[(F^K)^H X^H,Y^H]\\ &\quad -(F^K)^H[X^H,(F^K)^H Y^H]+(F^K)^H(F^K)^H[X^H,Y^H]\\ &= [F^H X^H,F^H Y^H]-F^H[F^H X^H,Y^H]\\ &\quad -F^H[X^H,F^H Y^H]+(F^H)^2[X^H,Y^H]\\ &= \{[FX,FY]-F[FX,Y]-F[X,FY]\\ &\quad +F^2[X,Y]\}^H+\gamma\{R(FX,FY)-R(FX,Y)F\\ &\quad -R(X,FY)F+R(X,Y)(F)^2\}.\end{aligned}$$

$(F^K)^H$ is integrable if the curvature tensor R of ∇ satisfies $R(FX,FY)=R(X,Y)$ and F is an almost complex structure, then we get $R(FX,Y)=-R(X,FY)$. Hence using $F^2=-I$, we find $R(FX,FY)-R(FX,Y)F-R(X,FY)F+R(X,Y)F^2=0$. Therefore, it follows $\tilde{N}_{(F^K)^H(F^K)^H}(X^H,Y^H)=0$.
*ii)*The Nijenhuis tensor $\tilde{N}_{(F^K)^H(F^K)^H}(X^H,\omega^V)$ of the horizontal lift $(F^K)^H$ vanishes if $\nabla F=0$.

$$\begin{aligned}\tilde{N}_{(F^K)^H(F^K)^H}(X^H,\omega^V) &= [(F^K)^H X^H,(F^K)^H\omega^V]-(F^K)^H[(F^K)^H X^H,\omega^V]\\ &\quad -(F^K)^H[X^H,(F^K)^H\omega^V]+(F^K)^H(F^K)^H[X^H,\omega^V]\\ &= [(FX)^H,(\omega\circ F)^V]-F^H[(FX)^H,\omega^V]\\ &\quad -F^H[X^H,(\omega\circ F)^V]+(F^H)^2(\nabla_X\omega)^V\\ &= \{\omega\circ(\nabla_{FX}F)-(\omega\circ(\nabla_X F)F\}^V,\end{aligned}$$

We now suppose $\nabla F=0$, then we see $\tilde{N}_{(F^K)^H(F^K)^H}\left(X^H,\omega^V\right)=0$, where $F\in\Im_1^1(M^n)$, $X\in\Im_0^1(M^n)$, $\omega\in\Im_1^0(M^n)$.
*iii)*The Nijenhuis tensor $\tilde{N}_{(F^K)^H(F^K)^H}(\omega^V,\theta^V)$ of the horizontal lift $(F^K)^H$ vanishes.
Because of $[\omega^V,\theta^V]=0$ for $\omega\circ F,\theta\circ F,\omega,\theta\in\Im_1^0(M^n)$ on $T^*(M^n)$, the Nijenhuis tensor $\tilde{N}_{(F^K)^H(F^K)^H}(\omega^V,\theta^V)$ of the horizontal lift $(F^K)^H$ vanishes. □

2.2. Tachibana Operators Applied to Vector and Covector Fields According to Lifts of $F^K+F=0$ Structure on $T^*(M^n)$

Definition 2.3. *Let* $\varphi\in\Im_1^1(M^n)$*, and* $\Im(M^n)=\sum_{r,s=0}^{\infty}\Im_s^r(M^n)$ *be a tensor alebra over* R*. A map* $\phi_\varphi\mid_{r+s>0}:\overset{*}{\Im}(M^n)\to\Im(M^n)$ *is called as Tachibana operatör or* ϕ_φ*operatör on* M^n *if*

a) ϕ_φ is linear with respect to constant coefficient,
b) $\phi_\varphi:\overset{*}{\Im}(M^n)\to\Im_{s+1}^r(M^n)$ for all r and s,
c) $\phi_\varphi(K\overset{C}{\otimes}L)=(\phi_\varphi K)\otimes L+K\otimes\phi_\varphi L$ for all $K,L\in\overset{*}{\Im}(M^n)$,
d) $\phi_{\varphi X}Y=-(L_Y\varphi)X$ for all $X,Y\in\Im_0^1(M^n)$, where L_Y is the Lie derivation with respect to Y (see [3, 5, 8]),
e)

$$\begin{aligned}(\phi_{\varphi X}\eta)Y &= (d(\iota_Y\eta))(\varphi X)-(d(\iota_Y(\eta o\varphi)))X+\eta((L_Y\varphi)X)\\ &= \phi X(\iota_Y\eta)-X(\iota_{\varphi Y}\eta)+\eta((L_Y\varphi)X)\end{aligned}\tag{2.3}$$

for all $\eta\in\Im_1^0(M^n)$ and $X,Y\in\Im_0^1(M^n)$, where $\iota_Y\eta=\eta(Y)=\eta\overset{C}{\otimes}Y$, $\overset{*}{\Im}{}_s^r(M^n)$ the module of all pure tensor fields of type (r,s) on M^n with respect to the affinor field, $\overset{C}{\otimes}$ is a tensor product with a contraction C [2, 4, 11] (see [12] for applied to pure tensor field).

Remark 2.4. *If* $r=s=0$*, then from c), d) and e) of Definition2.3 we have* $\phi_{\varphi X}(\iota_Y\eta)=\phi X(\iota_Y\eta)-X(\iota_{\varphi Y}\eta)$ *for* $\iota_Y\eta\in\Im_0^0(M^n)$*, which is not well-defined* $\phi_\varphi-$*operator. Different choices of* Y *and* η *leading to same function* $f=\iota_Y\eta$ *do get the same values. Consider* $M^n=R^2$

with standard coordinates x,y. Let $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the function $f = 1$. This may be written in many different ways as $\iota_Y \eta$. Indeed taking $\eta = dx$, we may choose $Y = \frac{\partial}{\partial_x}$ or $Y = \frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y}$. Nov the right-hand side of $\phi_{\varphi X}(\iota_Y \eta) = \phi X(\iota_Y \eta) - X(\iota_{\varphi Y}\eta)$ is $(\phi X)1 - 0 = 0$ in the first case, and $(\phi X)1 - Xx = -Xx$ in the second case. For $X = \frac{\partial}{\partial_x}$, the latter expression is $-1 \neq 0$. Therefore, we put $r+s > 0$ [11].

Remark 2.5. *From d) of Definition2.3 we have*

$$\phi_{\varphi X} Y = [\varphi X, Y] - \varphi[X,Y]. \tag{2.4}$$

By virtue of

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X \tag{2.5}$$

for any $f,g \in \Im_0^0(M^n)$, we see that $\phi_{\varphi X} Y$ is linear in X, but not Y [11].

Theorem 2.6. *Let $(F^K)^H$ be a tensor field of type $(1,1)$ on $T^*(M^n)$. If the Tachibana operator ϕ_φ applied to vector fields according to horizontal lifts of $F^K + F = 0$ structure defined by (1.7) on $T^*(M^n)$, then we get the following results.*

$$\begin{aligned} i)\ \phi_{(F^K)^H X^H} Y^H &= ((L_Y F)X)^H + (PR(Y,FX))^V \\ &\quad -((PR(Y,X))\circ F)^V, \end{aligned}$$

$$ii)\ \phi_{(F^K)^H X^H} \omega^V = ((\nabla_X \omega)\circ F)^V - (\nabla_{(FX)}\omega)^V,$$

$$iii)\ \phi_{(F^K)^H \omega^V} X^H = (\omega \circ (\nabla_X F))^V,$$

$$iv)\ \phi_{(F^K)^H \omega^V} \theta^V = 0,$$

where horizontal lifts $X^H, Y^H \in \Im_0^1(T^*(M^n))$ of $X,Y \in \Im_0^1(M^n)$ and the vertical lift $\omega^V, \theta^V \in \Im_0^1(T^*(M^n))$ of $\omega, \theta \in \Im_1^0(M^n)$ are given, respectively.

Proof. *i)*

$$\begin{aligned} \phi_{(F^K)^H X^H} Y^H &= -(L_{Y^H}(F^K)^H)X^H \\ &= -L_{Y^H}(F^K)^H X^H + (F^K)^H L_{Y^H} X^H \\ &= L_{Y^H} F^H X^H - F^H([Y,X]^H + (PR(Y,X))^V) \\ &= ((L_Y F)X)^H + (PR(Y,FX))^V - ((PR(Y,X))\circ F)^V \end{aligned}$$

ii)

$$\begin{aligned} \phi_{(F^K)^H X^H} \omega^V &= -(L_{\omega^V}(F^K)^H)X^H \\ &= -L_{\omega^V}(F^K)^H X^H + (F^K)^H L_{\omega^V} X^H \\ &= L_{\omega^V}(FX)^H + F^H(\nabla_X \omega)^V \\ &= -(\nabla_{(FX)}\omega)^V + ((\nabla_X \omega)\circ F)^V \\ &= ((\nabla_X \omega)\circ F)^V - (\nabla_{(FX)}\omega)^V \end{aligned}$$

iii)

$$\begin{aligned} \phi_{(F^K)^H \omega^V} X^H &= -(L_{X^H}(F^K)^H)\omega^V \\ &= -L_{X^H}(F^K)^H \omega^V + (F^K)^H L_{X^H}\omega^V \\ &= L_{X^H}(\omega\circ F)^V - F^H(\nabla_X \omega)^V \\ &= (\nabla_X(\omega\circ F))^V - ((\nabla_X \omega)\circ F)^V \\ &= (\omega\circ(\nabla_X F))^V \end{aligned}$$

vi)

$$\begin{aligned} \phi_{(F^K)^H \omega^V} \theta^V &= -(L_{\theta^V}(F^K)^H)\omega^V \\ &= -L_{\theta^V}(F^K)^H \omega^V + (F^K)^H(L_{\theta^V}\omega^V) \\ &= L_{\theta^V}(\omega\circ F)^V \\ &= 0 \end{aligned}$$

□

2.3. The Purity Conditions of Sasakian Metric with Respect to $(F^K)^H$

Definition 2.7. *A Sasakian metric* ^Sg *is defined on* $T^*(M^n)$ *by the three equations*

$$^Sg(\omega^V,\theta^V)=(g^{-1}(\omega,\theta))^V=g^{-1}(\omega,\theta)o\pi, \tag{2.6}$$

$$^Sg(\omega^V,Y^H)=0, \tag{2.7}$$

$$^Sg(X^H,Y^H)=(g(X,Y))^V=g(X,Y)\circ\pi. \tag{2.8}$$

For each $x\in M^n$ the scalar product $g^{-1}=(g^{ij})$ is defined on the cotangent space $\pi^{-1}(x)=T_x^*(M^n)$ by

$$g^{-1}(\omega,\theta)=g^{ij}\omega_i\theta_j,$$

where $X,Y\in\Im_0^1(M^n)$ and $\omega,\theta\in\Im_1^0(M^n)$. Since any tensor field of type $(0,2)$ on $T^*(M^n)$ is completely determined by its action on vector fields of type X^H and ω^V (see [14], p.280), it follows that ^Sg is completely determined by equations (2.6), (2.7) and (2.8).

Theorem 2.8. *Let* $(T^*(M^n),^Sg)$ *be the cotangent bundle equipped with Sasakian metric* ^Sg *and a tensor field* $(F^K)^H$ *of type* $(1,1)$ *defined by (1.7). Sasakian metric* ^Sg *is pure with respect to* $(F^K)^H$ *if* $F=I$ *(*I *= identity tensor field of type* $(1,1)$*).*

Proof. We put

$$S(\tilde{X},\tilde{Y})=^Sg((F^K)^H\tilde{X},\tilde{Y})-^Sg(\tilde{X},(F^K)^H\tilde{Y}).$$

If $S(\tilde{X},\tilde{Y})=0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ω^V,θ^V or X^H,Y^H, then $S=0$. By virtue of $F^K+F=0$ and (2.6), (2.7), (2.8), we get

i)

$$\begin{aligned}S(\omega^V,\theta^V)&={}^Sg((F^K)^H\omega^V,\theta^V)-^Sg(\omega^V,(F^K)^H\theta^V)\\&={}^Sg(-F^H\omega^V,\theta^V)-^Sg(\omega^V,-F^H\theta^V)\\&=-(^Sg((\omega\circ F)^V,\theta^V)-^Sg(\omega^V,(\theta\circ F)^V)).\end{aligned}$$

ii)

$$\begin{aligned}S(X^H,\theta^V)&={}^Sg((F^K)^HX^H,\theta^V)-^Sg(X^H,(F^K)^H\theta^V)\\&={}^Sg(-F^HX^H,\theta^V)-^Sg(X^H,-F^H\theta^V)\\&=-(^Sg((FX)^H,\theta^V)-^Sg(X^H,(\omega\circ F)^V))\\&=0.\end{aligned}$$

iii)

$$\begin{aligned}S(X^H,Y^H)&={}^Sg((F^K)^HX^H,Y^H)-^Sg(X^H,(F^K)^HY^H)\\&={}^Sg(-F^HX^H,Y^H)-^Sg(X^H,-F^HY^H)\\&=-(^Sg((FX)^H,Y^H)-^Sg(X^H,(FY)^H)).\end{aligned}$$

Thus, $F=I$, then ^Sg is pure with respect to $(F^K)^H$. □

2.4. Complete Lift of $F(K,1)-$Structure on Tangent Bundle $T(M^n)$

Let M^n be an $n-$dimensional differentiable manifold of class C^∞ and $T_P(M^n)$ the tangent space at a point p of M^n and

$$T(M^n)=\underset{p\in M^n}{U}T_P(M^n) \tag{2.9}$$

is the tangent bundle over the manifold M^n.
Let us denote by $T_s^r(M^n)$, the set of all tensor fields of class C^∞ and of type (r,s) in M^n and $T(M^n)$ be the tangent bundle over M^n. The complete lift of F^C of an element of $T_1^1(M^n)$ with local components F_i^h has components of the form [13]

$$F^C=\begin{bmatrix}F_i^h & 0\\ \delta_i^h & F_i^h\end{bmatrix}. \tag{2.10}$$

Now we obtain the following results on the complete lift of F satisfying $F^K+F=0$, $(F\neq0,K\geq0)$.
Let $F,G\in T_1^1(M^n)$. Then we have [13]

$$(FG)^C=F^CG^C. \tag{2.11}$$

Replacing G by F in (2.11) we obtain

$$(FF)^C=F^CF^C \text{ or } (F^2)^C=(F^C)^2. \tag{2.12}$$

Now putting $G=F^4$ in (2.11) since G is $(1,1)$ tensor field therefore F^4 is also $(1,1)$ so we obtain $(FF^4)^C=F^C(F^4)^C$ which in view of (2.12) becomes

$$(F^5)^C=(F^C)^5.$$

Continuing the above process of replacing G in equation (2.11) by some higher powers of F, we obtain

$$(F^K)^C=(F^C)^K,$$

where K is a positive integer ≥ 2. Also if G and H are tensors of the same type then

$$(G+H)^C=G^C+H^C \tag{2.13}$$

Taking complete lift on both sides of equation $F^K+F=0$, we get

$$(F^K+F)^C=0$$

Using (2.13) and $I^C=I$, we get

$$(F^K)^C+F^C=0 \tag{2.14}$$

$$(F^C)^K+F^C=0.$$

Let F satisfying $(1,1)$ be an $F-$structure of rank r in M^n. Then the complete lifts $s^C=-(F^{K-1})^C$ of s and $t^C=I+(F^{K-1})^C$ of t are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions S^C and T^C determined by s^C and t^C, respectively.

Proposition 2.9. *The* $(1,1)$ *tensor field* $\tilde{\psi}$ *given by* $\tilde{\psi}=s^C-t^C=-2(F^{K-1})^C-I$ *gives an almost product structure on* $T(M^n)$.

Proof. For $s^C=-(F^{K-1})^C$, $t^C=I+(F^{K-1})^C$ and $\tilde{\psi}=s^C-t^C=-2(F^{K-1})^C-I$, we have

$$\begin{aligned}
\tilde{\psi}^2 &= 4(F^{2K-2})^C+4(F^{K-1})^C+I\\
&= 4(F^K)^C(F^{K-2})^C+4(F^{K-1})^C+I\\
&= -(4F^{K-1})^C+4(F^{K-1})^C+I\\
&= I,
\end{aligned}$$

where $\tilde{\psi}\in\Im_1^1(T(M^n)), I$ = identity tensor field of type $(1,1)$. □

2.5. Horizontal Lift of $F(K,1)-$Structure on Tangent Bundle $T(M^n)$

Let F_i^h be the component of F at A in the coordinate neighbourhood U of M^n. Then the horizontal lift F^H of F is also a tensor field of type $(1,1)$ in $T(M^n)$ whose components $\tilde{F}_B^A$ in $\pi^{-1}(U)$ are given by

$$F^H=F^C-\gamma(\nabla F)=\begin{pmatrix} F_i^h & 0\\ -\Gamma_t^h F_i^t+\Gamma_i^t F_t^h & F_i^h\end{pmatrix}. \tag{2.15}$$

Let F, G be two tensor fields of type $(1,1)$ on the manifold M. If F^H denotes the horizontal lift of F, we have

$$(FG)^H=F^HG^H. \tag{2.16}$$

Taking F and G identical, we get

$$(F^H)^2=(F^2)^H. \tag{2.17}$$

Multiplying both sides by F^H and making use of the same (2.17) , we get

$$(F^H)^3=(F^3)^H$$

Thus it follows that

$$(F^H)^4=(F^4)^H,\ (F^H)^5=(F^5)^H \tag{2.18}$$

and so on. Taking horizontal lift on both sides of equation $F^K+F=0$ we get

$$(F^K)^H+F^H=0 \tag{2.19}$$

view of (2.18), we can write

$$(F^H)^K+F^H=0.$$

2.6. The Structure $(F^K)^C + F^C = 0$ on Tangent Bundle $T(M^n)$

Definition 2.10. *Let X and Y be any vector fields on a Riemannian manifold (M^n, g), we have [14]*

$$\begin{aligned} \left[X^H, Y^H\right] &= [X,Y]^H - (R(X,Y)u)^V, \\ \left[X^H, Y^V\right] &= (\nabla_X Y)^V, \\ \left[X^V, Y^V\right] &= 0, \end{aligned}$$

where R is the Riemannian curvature tensor of g defined by

$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

In particular, we have the vertical spray u^V and the horizontal spray u^H on $T(M^n)$ defined by

$u^V = u^i(\partial_i)^V = u^i\partial_{\bar{i}}$, $u^H = u^i(\partial_i)^H = u^i\delta_i$,

where $\delta_i = \partial_i - u^j\Gamma^s_{ji}\partial_{\bar{s}}$. u^V is also called the canonical or Liouville vector field on $T(M^n)$.

Theorem 2.11. *The Nijenhuis tensors $\tilde{N}_{(F^K)^C(F^K)^C}(X^C, Y^C)$, $\tilde{N}_{(F^K)^C(F^K)^C}(X^C, Y^V)$, $\tilde{N}_{(F^K)^C(F^K)^C}(X^V, Y^V)$ of the complete lift $(F^K)^C$ vanishes if the Nijenhuis tensor of the F is zero.*

Proof. In consequence of Definition 2.1 and the formulations in Definition 2.10, the Nijenhuis tensors of $\left(F^K\right)^C$ are given by
i)

$$\begin{aligned} \tilde{N}_{(F^K)^C(F^K)^C}\left(X^C, Y^C\right) &= [\left(F^K\right)^C X^C, \left(F^K\right)^C Y^C] - \left(F^K\right)^C [\left(F^K\right)^C X^C, Y^C] \\ &\quad - \left(F^K\right)^C [X^C, \left(F^K\right)^C Y^C] + \left(F^K\right)^C \left(F^K\right)^C \left[X^C, Y^C\right] \\ &= [(FX)^C, (FY)^C] + F^C[(FX)^C, Y^C] \\ &\quad - F^C[X^C, (FY)^C] + F^C F^C \left[X^C, Y^C\right] \\ &= N_F(X,Y)^C \end{aligned}$$

ii)

$$\begin{aligned} \tilde{N}_{(F^K)^C(F^K)^C}\left(X^C, Y^V\right) &= [\left(F^K\right)^C X^C, \left(F^K\right)^C Y^V] - \left(F^K\right)^C [\left(F^K\right)^C X^C, Y^V] \\ &\quad - \left(F^K\right)^C [X^C, \left(F^K\right)^C Y^V] + \left(F^K\right)^C \left(F^K\right)^C \left[X^C, Y^V\right] \\ &= [(FX)^C, (FY)^V] - F^C[(FX)^C, Y^V] \\ &\quad - F^C[X^C, (FY)^V] + \left(F^2\right)^C [X,Y]^V \\ &= N_F(X,Y)^V \end{aligned}$$

iii) Because of $\left[X^V, Y^V\right] = 0$ and $X, Y \in M$, easily we get

$\tilde{N}_{(F^K)^C(F^K)^C}\left(X^V, Y^V\right) = 0.$

□

2.7. The Purity Conditions of Sasakian Metric with Respect to $(F^K)^C$ on $T(M^n)$

Definition 2.12. *The Sasaki metric ${}^S g$ is a (positive definite) Riemannian metric on the tangent bundle $T(M^n)$ which is derived from the given Riemannian metric on M^n as follows [11]:*

$$\begin{aligned} {}^S g\left(X^H, Y^H\right) &= g(X,Y), \\ {}^S g\left(X^H, Y^V\right) &= {}^S g\left(X^V, Y^H\right) = 0, \\ {}^S g\left(X^V, Y^V\right) &= g(X,Y) \end{aligned} \tag{2.20}$$

for all $X, Y \in \Im^1_0(M^n)$.

Theorem 2.13. *The Sasaki metric ${}^S g$ is pure with respect to $\left(F^K\right)^C$ if $\nabla F = 0$ and $F = I$, where I=ıdentity tensor field of type $(1,1)$.*

Proof. $S(\widetilde{X},\widetilde{Y}) = {}^{S}g((F^{K})^{C}\widetilde{X},\widetilde{Y}) - {}^{S}g(\widetilde{X},(F^{K})^{C}\widetilde{Y})$ if $S(\widetilde{X},\widetilde{Y}) = 0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form X^{V},Y^{V} or X^{H},Y^{H} then $S = 0$.

i)

$$\begin{aligned} S\left(X^{V},Y^{V}\right) &= {}^{S}g(\left(F^{K}\right)^{C}X^{V},Y^{V}) - {}^{S}g(X^{V},\left(F^{K}\right)^{C}Y^{V}) \\ &= -{}^{S}g((FX)^{V},Y^{V}) + {}^{S}g(X^{V},(FY)^{V})\} \\ &= -(g(FX,Y))^{V} + (g(X,FY))^{V}\} \end{aligned}$$

ii)

$$\begin{aligned} S\left(X^{V},Y^{H}\right) &= {}^{S}g(\left(F^{K}\right)^{C}X^{V},Y^{H}) - {}^{S}g(X^{V},\left(F^{K}\right)^{C}Y^{H}) \\ &= {}^{S}g(X^{V},(FY)^{H} + \left(\nabla_{\gamma}F\right)Y^{H}) \\ &= {}^{S}g\left(X^{V},\left(\nabla_{\gamma}F\right)Y^{H}\right) \\ &= {}^{S}g(X^{V},(((\nabla F)u)Y)^{V}) \\ &= (g(X,((\nabla F)u)Y))^{V} \end{aligned}$$

iii)

$$\begin{aligned} S\left(X^{H},Y^{H}\right) &= {}^{S}g(\left(F^{K}\right)^{C}X^{H},Y^{H}) - {}^{S}g(X^{H},\left(F^{K}\right)^{C}Y^{H}) \\ &= -{}^{S}g(F^{C}X^{H},Y^{H}) + {}^{S}g(X^{H},F^{C}Y^{H}) \\ &= -{}^{S}g((FX)^{H} + \left(\nabla_{\gamma}F\right)X^{H},Y^{H}) \\ &\quad + {}^{S}g(X^{H},(FY)^{H} + \left(\nabla_{\gamma}F\right)Y^{H}) \\ &= -g((FX),Y)^{V} + g(X,(FY))^{V}\} \end{aligned}$$

□

Theorem 2.14. *Let ϕ_{φ} be the Tachibana operator and the structure $\left(F^{K}\right)^{C} + F^{C} = 0$ defined by Definition 2.3 and (2.14), respectively. If $L_{Y}F = 0$, then all results with respect to $\left(F^{K}\right)^{C}$ is zero, where $X,Y \in \Im_{0}^{1}(M^{n})$, the complete lifts $X^{C},Y^{C} \in \Im_{0}^{1}(T(M^{n}))$ and the vertical lift $X^{V},Y^{V} \in \Im_{0}^{1}(T(M^{n}))$.*

$$\begin{aligned} i)\ \phi_{(F^{K})^{C}X^{C}}Y^{C} &= ((L_{Y}F)X)^{C} \\ ii)\ \phi_{(F^{K})^{C}X^{C}}Y^{V} &= ((L_{Y}F)X)^{V} \\ iii)\ \phi_{(F^{K})^{C}X^{V}}Y^{C} &= ((L_{Y}F)X)^{V} \\ iv)\ \phi_{(F^{K})^{C}X^{V}}Y^{V} &= 0 \end{aligned}$$

Proof. *i*)

$$\begin{aligned} \phi_{(F^{K})^{C}X^{C}}Y^{C} &= -(L_{Y^{C}}\left(F^{K}\right)^{C})X^{C} \\ &= L_{Y^{C}}(FX)^{C} - F^{C}L_{Y^{C}}X^{C} \\ &= ((L_{Y}F)X)^{C} \end{aligned}$$

ii)

$$\begin{aligned} \phi_{(F^{K})^{C}X^{C}}Y^{V} &= -(L_{Y^{V}}\left(F^{K}\right)^{C})X^{C} \\ &= -L_{Y^{V}}\left(F^{K}\right)^{C}X^{C} + \left(F^{K}\right)^{C}L_{Y^{V}}X^{C} \\ &= L_{Y^{V}}(FX)^{C} - F^{C}L_{Y^{V}}X^{C} \\ &= ((L_{Y}F)X)^{V} \end{aligned}$$

iii)

$$\begin{aligned} \phi_{(F^{K})^{C}X^{V}}Y^{C} &= -(L_{Y^{C}}\left(F^{K}\right)^{C})X^{V} \\ &= -L_{Y^{C}}\left(F^{K}\right)^{C}X^{V} + \left(F^{K}\right)^{C}L_{Y^{C}}X^{V} \\ &= L_{Y^{C}}(FX)^{V} - F^{C}L_{Y^{C}}X^{V} \\ &= ((L_{Y}F)X)^{V} \end{aligned}$$

iv)

$$\begin{aligned}\phi_{(F^K)^C X^V} Y^V &= -(L_{Y^V}\left(F^K\right)^C)X^V \\ &= -L_{Y^V}\left(F^K\right)^C X^V + \left(F^K\right)^C L_{Y^V} X^V \\ &= 0\end{aligned}$$

□

Theorem 2.15. *If $L_Y F = 0$ for $Y \in M^n$, then its complete lift Y^C to the tangent bundle is an almost holomorfic vector field with respect to the structure $\left(F^K\right)^C + F^C = 0$.*

Proof. i)

$$\begin{aligned}(L_{Y^C}\left(F^K\right)^C)X^C &= L_{Y^C}\left(F^K\right)^C X^C - \left(F^K\right)^C L_{Y^C} X^C \\ &= -L_{Y^C}(FX)^C + F^C L_{Y^C} X^C \\ &= -((L_Y F)X)^C\end{aligned}$$

ii)

$$\begin{aligned}(L_{Y^C}\left(F^K\right)^C)X^V &= L_{Y^C}\left(F^K\right)^C X^V - \left(F^K\right)^C L_{Y^C} X^V \\ &= -L_{Y^C}(FX)^V + F^C L_{Y^C} X^V \\ &= -((L_Y F)X)^V\end{aligned}$$

□

2.8. The Structure $\left(F^K\right)^H + F^H = 0$ on Tangent Bundle $T(M^n)$

Theorem 2.16. *The Nijenhuis tensor $\tilde{N}_{(F^K)^H (F^K)^H}\left(X^H, Y^H\right)$ of the horizontal lift $\left(F^K\right)^H$ vanishes if the Nijenhuis tensor of the F is zero and $\{-(\hat{R}(FX,FY)u) + (F(\hat{R}(FX,Y)u)) + (F(R(X,FY)u)) - (F^2(\hat{R}(X,Y)u))\}^V = 0$.*

Proof.

$$\begin{aligned}\tilde{N}_{(F^K)^H (F^K)^H}\left(X^H, Y^H\right) &= [\left(F^K\right)^H X^H, \left(F^K\right)^H Y^H] - \left(F^K\right)^H [\left(F^K\right)^H X^H, Y^H] \\ &\quad - \left(F^K\right)^H [X^H, \left(F^K\right)^H Y^H] + \left(F^K\right)^H \left(F^K\right)^H \left[X^H, Y^H\right] \\ &= [(FX)^H, (FY)^H] - (F)^H [(FX)^H, Y^H] \\ &\quad - (F)^H [X^H, (FY)^H] + (F)^H (F)^H \left[X^H, Y^H\right] \\ &= (N_F(X,Y))^H - (\hat{R}(FX,FY)u)^V \\ &\quad + (F(\hat{R}(FX,Y)u))^V + (F(\hat{R}(X,FY)u))^V \\ &\quad - (F^2(\hat{R}(X,Y)u))^V.\end{aligned}$$

□

If $N_F(X,Y) = 0$ and $\{-\hat{R}(FX,FY)u + (F(\hat{R}(FX,Y)u)) + (F(\hat{R}(X,FY)u)) - (F^2(\hat{R}(X,Y)u))\}^V = 0$, then we get $N_{(F^K)^H (F^K)^H}\left(X^H, Y^H\right) = 0$, where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$ defined by $\hat{\nabla}_X Y = \nabla_Y X + [X,Y]$ (see [14] p.88-89).

Theorem 2.17. *The Nijenhuis tensor $\tilde{N}_{(F^K)^H (F^K)^H}\left(X^H, Y^V\right)$ of the horizontal lift $(F^K)^H$ vanishes if the Nijenhuis tensor of the F is zero and $\nabla F = 0$.*

Proof.

$$\begin{aligned}\tilde{N}_{(F^K)^H (F^K)^H}\left(X^H, Y^V\right) &= [\left(F^K\right)^H X^H, \left(F^K\right)^H Y^V] - \left(F^K\right)^H [\left(F^K\right)^H X^H, Y^V] \\ &\quad - \left(F^K\right)^H [X^H, \left(F^K\right)^H Y^V] + \left(F^K\right)^H \left(F^K\right)^H \left[X^H, Y^V\right] \\ &= [FX + FY]^V - (F[FX,Y])^V - (F[X,FY])^V \\ &\quad + ((F)^2 [X,Y])^V + (\nabla_{FY} FX)^V - (F(\nabla_Y FX))^V \\ &\quad - (F(\nabla_{FY} X))^V + ((F)^2 \nabla_Y X)^V \\ &= (N_F(X,Y))^V + ((\nabla_{FY} F)X)^V - (F((\nabla_Y F)X))^V.\end{aligned}$$

□

Theorem 2.18. *The Nijenhuis tensor* $\tilde{N}_{(F^K)^H(F^K)^H}\left(X^V,Y^V\right)$ *of the horizontal lift* $(F^K)^H$ *vanishes.*

Proof. Because of $\left[X^V,Y^V\right]=0$ for $X,Y\in M^n$, easily we get

$\tilde{N}_{(F^K)^H(F^K)^H}\left(X^V,Y^V\right)=0.$

□

Theorem 2.19. *The Sasakian metric* ${}^Sg$ *is pure with respect to* $\left(F^K\right)^H$ *if* $F=I$*, where* $I=$*ıdentity tensor field of type* $(1,1)$.

Proof. $S(\widetilde{X},\widetilde{Y})={}^Sg((F^K)^H\widetilde{X},\widetilde{Y})-{}^Sg(\widetilde{X},(F^K)^H\widetilde{Y})$ if $S(\widetilde{X},\widetilde{Y})=0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form X^V,Y^V or X^H,Y^H then $S=0$.
i)

$$\begin{aligned}S\left(X^V,Y^V\right) &= {}^Sg(\left(F^K\right)^H X^V,Y^V)-{}^Sg(X^V,\left(F^K\right)^H Y^V)\\ &= -{}^Sg((FX)^V,Y^V)+{}^Sg(X^V,(FY)^V)\\ &= -(g(FX,Y))^V+(g(X,FY))^V\}\end{aligned}$$

ii)

$$\begin{aligned}S\left(X^V,Y^H\right) &= {}^Sg(\left(F^K\right)^H X^V,Y^H)-{}^Sg(X^V,\left(F^K\right)^H Y^H)\\ &= {}^Sg(X^V,(FY)^H)\\ &= 0\end{aligned}$$

iii)

$$\begin{aligned}S\left(X^H,Y^H\right) &= {}^Sg(\left(F^K\right)^H X^H,Y^H)-{}^Sg(X^H,\left(F^K\right)^H Y^H)\\ &= -({}^Sg(FX)^H,Y^H)+{}^Sg(X^H,(FY)^H)\\ &= -(g(FX),Y)^V+(g(X,(FY)^H))^V\end{aligned}$$

□

Theorem 2.20. *Let* ϕ_φ *be the Tachibana operator and the structure* $\left(F^K\right)^H+F^H=0$ *defined by Definition 2.3 and (2.19), respectively. if* $L_YF=0$ *and* $F=I$*, then all results with respect to* $\left(F^K\right)^H$ *is zero, where* $X,Y\in\Im_0^1(M^n)$*, the horizontal lifts* $X^H,Y^H\in\Im_0^1(T(M^n))$ *and the vertical lift* $X^V,Y^V\in\Im_0^1(T(M^n))$.

$$\begin{aligned}i)\ \phi_{(F^K)^HX^H}Y^H &= -((L_YF)X)^H+(\hat{R}(Y,FX)u)^V\\ &\quad -(F(\hat{R}(Y,X)u))^V,\\ ii)\ \phi_{(F^K)^HX^H}Y^V &= ((L_YF)X)^V-((\nabla_YF)X)^V,\\ iii)\ \phi_{(F^K)^HX^V}Y^H &= ((L_YF)X)^V+(\nabla_{FX}Y)^V-(F(\nabla_XY))^V,\\ iv)\ \phi_{(F^K)^HX^V}Y^V &= 0.\end{aligned}$$

Proof. *i*)

$$\begin{aligned}\phi_{(F^K)^HX^H}Y^H &= -(L_{Y^H}\left(F^K\right)^H)X^H\\ &= -L_{Y^C}\left(F^K\right)^H X^H+\left(F^K\right)^H L_{Y^H}X^H\\ &= [Y,FX]^H-\gamma\hat{R}[Y,FX]\\ &\quad -(F[Y,X])^H+F^H(\hat{R}(Y,X)u)^V\\ &= -((L_YF)X)^H+(\hat{R}(Y,FX)u)^V\\ &\quad -(F(\hat{R}(Y,X)u))^V\end{aligned}$$

ii)

$$\begin{aligned}\phi_{(F^K)^HX^H}Y^V &= -(L_{Y^V}\left(F^K\right)^H)X^H\\ &= -L_{Y^V}\left(F^KX\right)^H+\left(F^K\right)^H L_{Y^V}X^H\\ &= [Y,FX]^V-(\nabla_YFX)^V\\ &\quad -(F[Y,X])^V+(F(\nabla_YX))^V\\ &= ((L_YF)X)^V-((\nabla_YF)X)^V\end{aligned}$$

iii)

$$\begin{aligned}
\phi_{(F^K)^H X^V} Y^H &= -(L_{Y^H}\left(F^K\right)^H)X^V \\
&= -L_{Y^H}\left(F^K X\right)^V + \left(F^K\right)^H L_{Y^H} X^V \\
&= -[Y,FX]^V + (\nabla_{FX}Y)^V \\
&\quad -(F[Y,X])^H - (F(\nabla_X Y))^V \\
&= ((L_Y F)X)^V + (\nabla_{FX}Y)^V - (F(\nabla_X Y))^V
\end{aligned}$$

iv)

$$\begin{aligned}
\phi_{(F^K)^H X^V} Y^V &= -(L_{Y^V}\left(F^K\right)^H)X^V \\
&= L_{Y^V}(FX)^V - F^H L_{Y^V} X^V \\
&= 0
\end{aligned}$$

□

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