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Absolute Summability Factors Related to the Summability Method $|\bar{N}, p_n, \theta|(\mu)$

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ABSTRACT: By (A, B), we denote the set of all sequences ϵ such that $\sum a_n \epsilon_n$ is summable B whenever $\sum a_n$ is summable A where A and B are two summability methods. In this study, applying the main theorems in (Gökçe and Sarıgöl, 2018) to summability factors, we characterize the sets $(|\bar{N}, p_n, \theta|(\mu), |\bar{N}, q_n|)$ and $(|\bar{N}, p_n, \theta|(\mu), |\bar{N}, q_n, \psi|(\lambda))$. Also, in the special case, we get some well-known results.

Keywords: Absolute weighted summability, summability factors, matrix transformations, sequence spaces.

$|\bar{N}, p_n, \theta|(\mu)$ Toplanabilme Metodu ile İlgili Mutlak Toplanabilme Çarpanları

ÖZET: A ve B iki toplanabilme metodu olmak üzere $\sum a_n$, A toplanabilir iken $\sum a_n \epsilon_n$, B toplanabilir olacak şekildeki bütün ϵ dizilerinin kümesi (A, B) ile gösterilir ve ϵ dizisine toplanabilme çarpanı adı verilir. Bu çalışmada, (Gökçe ve Sarıgöl, 2018) tarafından verilen teoremler yardımıyla $(|\bar{N}, p_n, \theta|(\mu), |\bar{N}, q_n|)$ ve $(|\bar{N}, p_n, \theta|(\mu), |\bar{N}, q_n, \varphi|(\lambda))$ toplanabilme çarpanları kümeleri karakterize edilmiştir. Ayrıca özel durumlarda, bilinen bazı sonuçlar elde edilmiştir.

Anahtar Kelimeler: Mutlak ağırlıklı ortalama toplanabilme, toplanabilme çarpanı, matris dönüşümleri, dizi uzayları.

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INTRODUCTION

Let $\sum a_v$ be a given infinite series with partial sum s_n, $\theta = (\theta_n)$ be any sequence of positive real numbers and $\mu = (\mu_n)$ be any bounded sequence of positive real numbers. If

$$\sum_{n=1}^{\infty} \theta_n^{\mu_{n-1}} |A_n(s) - A_{n-1}(s)|^{\mu_n} < \infty \tag{1}$$

where

$$A_n(s) = \sum_{v=0}^{\infty} a_{nv} s_v,$$

then the series $\sum a_v$ is said to be summable $|A, \theta|(\mu)$ (Gökçe and Sarıgöl, 2018).

Let (p_n) be a sequence of nonnegative numbers with $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$ $(P_{-1} = p_{-1} = 0)$. If we take the weighted mean matrix instead of A, the summability $|A, \theta|(\mu)$ is reduced to the summability $|\bar{N}, p_n, \theta|(\mu)$, and also the space of all series summable by $|\bar{N}, p_n, \theta|(\mu)$ is defined as follows (Gökçe and Sarıgöl, 2018)

$$|\bar{N}_p^{\theta}|(\mu) = \left\{ a = (a_v) : \sum_{n=1}^{\infty} \theta_n^{\mu_{n-1}} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \right|^{\mu_n} < \infty \right\}.$$

One gives the weighted mean matrix by

$$a_{nv} = \begin{cases} p_v / P_n, & 0 \le v \le n \\ 0, & v > n. \end{cases}$$

The series-to-sequence transformations corresponding to the summability $|\bar{N}, p_n, \theta|(\mu)$

$$T_0 = a_0 \theta_0^{1/\mu_0^*}, T_n = \theta_n^{1/\mu_n^*} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, n \ge 1 \tag{2}$$

define the sequence (T_n). Also, a few calculations show that its inverse transformation is as follows:

$$a_n = \theta_n^{-1/\mu_n^*} \frac{P_n}{p_n} T_n - \theta_{n-1}^{-1/\mu_{n-1}^*} \frac{P_{n-2}}{p_{n-1}} T_{n-1}, n \ge 0. \tag{3}$$

Now, we assume that $0 < \inf \mu_n < \infty$ and μ_n^* is conjugate of μ_n, i.e., $1/\mu_n^* + 1/\mu_n = 1$ for $\mu_n > 0$, $1/\mu_n^* = 0$ for $\mu_n = 1$ in the whole paper.

Note that the summability $|\bar{N}, p_n, \theta|(\mu)$ reduces to some well-known methods in special case of μ and θ. For example, if we take $\mu_n = k$ for all $n \ge 0$, then we have the summability $|\bar{N}, p_n, \theta|_k$ (Sarıgöl, 2011) and the summability $|\bar{N}, p_n|_k$ with $\theta_n = P_n/p_n$ (Orhan and Sarıgöl, 1993).

MATERIALS AND METHODS

Let A and B be two summability methods. If $\sum a_n \epsilon_n$ is summable B whenever $\sum a_n$ is summable A, then it is said that ϵ is summability factor of type (A, B), denoted by $\epsilon \in (A, B)$. In this paper, applying the main theorems in (Gökçe and Sarıgöl, 2018) to summability factors, we characterize the sets $(|\bar{N}, p_n, \theta|(\mu), |\bar{N}, q_n|)$ and $(|\bar{N}, p_n, \theta|(\mu), |\bar{N}, q_n, \psi|(\lambda))$ where (θ_n) and (ψ_n) are sequences of positive numbers and (μ_n) and (λ_n) are arbitrary bounded sequences of positive numbers. Also, in the special case, we get some well-known results.

Definition 2.1 Let f and g be any real valued functions defined on some unbounded subset of the positive real numbers. Then, $f(x) = O(g(x))$ if and only if there exists a positive real number M and a real number x_0 such that $|f(x)| \le M g(x)$ for all $x \ge x_0$.

Lemma 2.2 Let $k \geq 1$ and (p_n) be a sequence of positive numbers. If $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$ $(P_{-1} = p_{-1} = 0)$, then

$$\frac{1}{kP_{v-1}^k} \leq \sum_{n=v}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \leq \frac{1}{P_{v-1}^k},$$

(Sarıgöl, 2016).

Theorem 2.3 Let $A = (a_v)$ be an infinite matrix of complex numbers and (θ_n) be a sequence of positive numbers. If $\mu = (\mu_n)$ is an arbitrary bounded sequence of positive numbers such that $\mu_n > 1$ for all n, then $A \in \left(\left|\bar{N}_p^\theta\right|(\mu), \left|\bar{N}_q\right|\right)$ if and only if there exists an integer $M > 1$ such that, for $n = 0,1, \ldots,$

$$\sup_m \left|\frac{M^{-1} P_m a_{nm}}{\theta_m^{1/\mu_m^*} p_m}\right|^{\mu_m^*} < \infty, \tag{4}$$

$$\sum_{v=0}^{\infty} |M^{-1} \hat{a}_{nv}|^{\mu_v^*} < \infty, \tag{5}$$

$$\sum_{v=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{M^{-1} q_n}{Q_n Q_{n-1}} \left|\sum_{j=1}^{n} Q_{j-1} \hat{a}_{jv}\right|\right)^{\mu_v^*} < \infty \tag{6}$$

where

$$\hat{a}_{nv} = \frac{P_v}{\theta_v^{1/\mu_v^*} p_v} \left(a_{nv} - \frac{P_{v-1}}{P_v} a_{n,v+1}\right),$$

(Gökçe and Sarıgöl, 2018).

Theorem 2.4 Let $A = (a_{nv})$ be an infinite matrix of complex numbers, (θ_n) and (ψ_n) be sequences of positive numbers. If $\mu = (\mu_n)$ and $\lambda = (\lambda_n)$ are any bounded sequences of positive numbers such that $\mu_n \leq 1$ and $\lambda_n \geq 1$ for all n, then, $A \in \left(\left|\bar{N}_p^\theta\right|(\mu), \left|\bar{N}_q^\psi\right|(\lambda)\right)$ if and only if there exists an integer $M > 1$ such that, for $n = 0, 1, \ldots,$

$$\sup_v |\hat{a}_{nv}|^{\mu_v} < \infty, \tag{7}$$

$$\sup_m \left|\frac{P_m a_{nm}}{\theta_m^{1/\mu_m^*} p_m}\right| < \infty, \tag{8}$$

and

$$\sup_v \sum_{n=1}^{\infty} \left|\frac{\psi_n^{1/\lambda_n^*} q_n M^{-1/\mu_v}}{Q_n Q_{n-1}} \sum_{j=1}^{n} Q_{j-1} \hat{a}_{jv}\right|^{\lambda_n} < \infty, \tag{9}$$

(Gökçe and Sarıgöl, 2018).

Lemma 2.5 Let $a, b \in \mathbb{C}$, $k \geq 0$ and $c_k = 1$ for $k \leq 1$, $c_k = 2^{k-1}$ for $k > 1$. Then,

$$|a+b|^k \leq c_k(|a|^k + |b|^k),$$

(Mitrinovic, 1970).

RESULTS AND DISCUSSION

In this section, firstly we give main theorems and then, by making special chooses for ψ, θ, μ and λ, we obtain certain well-known corollaries.

Theorem 3.1 Let (θ_n) be a sequence of positive numbers and (μ_n) be an arbitrary bounded sequence of positive numbers with $\mu_n > 1$ for all n. Then, $\epsilon \in (|\bar{N}, p_n, \theta|(\mu), |\bar{N}, q_n|)$ if and only if

$$\sum_{v}^{\infty}\left(M^{-1}\frac{P_v q_v}{Q_v p_v}\theta_v^{-1/\mu_v^*}|\epsilon_v|\right)^{\mu_v^*} < \infty \tag{10}$$

$$\sum_{v}^{\infty}\left(M^{-1}\theta_v^{-1/\mu_v^*}\left|\frac{P_v}{p_v}\Delta\epsilon_v + \epsilon_{v+1}\right|\right)^{\mu_v^*} < \infty \tag{11}$$

where $\Delta\epsilon_v = \epsilon_v - \epsilon_{v+1}$ for all $v \geq 0$.

Proof. Take the diagonal matrix W instead of A in Theorem 2.3. Then, (4) and (5) are directly satisfied. Also, using Lemma 2.2, we get

$$\sum_{v=0}^{\infty}\frac{M^{-1/\mu_v^*}}{\theta_v}\left(\frac{q_v P_v}{Q_v p_v}|\epsilon_v| + \sum_{n=v+1}^{\infty}\frac{q_n}{Q_n Q_{n-1}}\left|\frac{P_v}{p_v}\left(Q_{v-1}\epsilon_v - \frac{P_{v-1}}{P_v}Q_v\epsilon_{v+1}\right)\right|\right)^{\mu_v^*}$$

$$= \sum_{v=0}^{\infty}\frac{M^{-1/\mu_v^*}}{\theta_v}\left(\frac{q_v P_v}{Q_v p_v}|\epsilon_v| + \left|\frac{P_v}{Q_v p_v}Q_{v-1}\epsilon_v - \frac{P_{v-1}}{p_v}\epsilon_{v+1}\right|\right)^{\mu_v^*}$$

$$= \sum_{v=0}^{\infty}\frac{M^{-1/\mu_v^*}}{\theta_v}\left(\frac{q_v P_v}{Q_v p_v}|\epsilon_v| + \left|\frac{P_v}{p_v}\Delta\epsilon_v - \frac{P_v q_v}{Q_v p_v}\epsilon_v + \epsilon_{v+1}\right|\right)^{\mu_v^*} < \infty.$$

So, it can be seen immediately that the condition (6) is reduced to the condition (10) and the following condition:

$$\sum_{v=0}^{\infty}\left(\frac{M^{-1}}{\theta_v^{1/\mu_v^*}}\left|\frac{P_v}{p_v}\Delta\epsilon_v - \frac{P_v q_v}{Q_v p_v}\epsilon_v + \epsilon_{v+1}\right|\right)^{\mu_v^*} < \infty.$$

Since $\mu_v^* > 1$ for all v, it can be written that

$$\left(M^{-1}\theta_v^{-1/\mu_v^*}\left|\frac{P_v}{p_v}\Delta\epsilon_v + \epsilon_{v+1}\right|\right)^{\mu_v^*} = \left(\frac{M^{-1}}{\theta_v^{1/\mu_v^*}}\left|\frac{P_v}{p_v}\Delta\epsilon_v - \frac{P_v q_v}{Q_v p_v}\epsilon_v + \frac{P_v q_v}{Q_v p_v}\epsilon_v + \epsilon_{v+1}\right|\right)^{\mu_v^*}$$

$$\leq 2^{H-1}\left\{\left(\frac{M^{-1}}{\theta_v^{1/\mu_v^*}}\left|\frac{P_v}{p_v}\Delta\epsilon_v - \frac{P_v q_v}{Q_v p_v}\epsilon_v + \epsilon_{v+1}\right|\right)^{\mu_v^*} + \left(\frac{M^{-1}}{\theta_v^{1/\mu_v^*}}\left|\frac{P_v q_v}{Q_v p_v}\epsilon_v\right|\right)^{\mu_v^*}\right\}$$

where $H = \sup_v\{\mu_v^*\}$. So, it can be obtained that

$$\sum_{v}^{\infty}\left(\frac{M^{-1}}{\theta_v^{1/\mu_v^*}}\left|\frac{P_v}{p_v}\Delta\epsilon_v + \epsilon_{v+1}\right|\right)^{\mu_v^*} < \infty$$

which completes proof.

Theorem 3.2 Let (θ_n) and (ψ_n) be any sequences of positive numbers. If (μ_n) and (λ_n) are any bounded sequences of positive numbers such that $\mu_n \leq 1$ and $\lambda_n \geq 1$ for all n, then $\epsilon \in (|\bar{N}, p_n, \theta|(\mu), |\bar{N}, q_n, \varphi|(\lambda))$ if and only if

$$\sup_v \left|\psi_n^{1/\lambda_n^*} M^{-1/\mu_v} \theta_v^{-1/\mu_v^*} \frac{P_v q_v}{Q_v p_v} \epsilon_v\right|^{\lambda_n} < \infty \quad (12)$$

$$\sup_v \sum_{n=v+1}^{\infty} \left|\psi_n^{1/\lambda_n^*} M^{-1/\mu_v} \theta_v^{-1/\mu_v^*} \frac{q_n}{Q_n Q_{n-1}} \left(\frac{Q_{v-1} P_v}{p_v} \epsilon_v - \frac{Q_v P_{v-1}}{p_v} \epsilon_{v+1}\right)\right|^{\lambda_n} < \infty \quad (13).$$

Proof. If we take the diagonal matrix W instead of A in Theorem 2.4, then (7) and (8) are directly satisfied. Moreover, the condition (9) can be written as

$$\sup_v \left\{\left|\frac{\psi_v^{1/\lambda_v^*}}{M^{1/\mu_v} \theta_v^{1/\mu_v^*}} \frac{q_v P_v}{Q_v p_v} \epsilon_v\right|^{\lambda_v} + \sum_{n=v+1}^{\infty} \left|\frac{\psi_n^{1/\lambda_n^*}}{M^{1/\mu_v} \theta_v^{1/\mu_v^*}} \frac{q_n}{Q_n Q_{n-1}} \left(\frac{Q_{v-1} P_v}{p_v} \epsilon_v - \frac{Q_v P_{v-1}}{p_v} \epsilon_{v+1}\right)\right|^{\lambda_n}\right\} < \infty.$$

So, this completes the proof.

Corollary 3.3 Assume that (θ_n) and (ψ_n) are any sequences of positive numbers and $k \geq 1$. Then, necessary and sufficient conditions for $\epsilon \in (|\bar{N}, p_n, \theta|, |\bar{N}, q_n, \psi|_k)$ are

$$\sup_v \left|\psi_v^{1/k^*} \frac{P_v q_v}{Q_v p_v} \epsilon_v\right|^k < \infty,$$

$$\sup_v \sum_{n=v+1}^{\infty} \left|\psi_n^{1/k^*} \frac{q_n}{Q_n Q_{n-1}} \left(\frac{Q_{v-1} P_v}{p_v} \epsilon_v - \frac{Q_v P_{v-1}}{p_v} \epsilon_{v+1}\right)\right|^k < \infty.$$

Corollary 3.4 Let (θ_n) be any sequences of positive numbers and $k > 1$. Then, $\epsilon \in (|\bar{N}, p_n, \theta|_k, |\bar{N}, q_n|)$ if and only if

$$\sum_v^{\infty} \left(\frac{P_v q_v}{Q_v p_v} \theta_v^{-1/k^*} |\epsilon_v|\right)^{k^*} < \infty$$

$$\sum_v^{\infty} \left(\theta_v^{-1/k^*} \left|\frac{P_v}{p_v} \Delta\epsilon_v + \epsilon_{v+1}\right|\right)^{k^*} < \infty.$$

Following two theorems have been given by (Sarıgöl and Orhan, 1995).

Corollary 3.5 Let $1 \leq k < \infty$. Then, necessary and sufficient conditions for $\epsilon \in (|\bar{N}, p_n|, |\bar{N}, q_n|_k)$ are

a. $\epsilon_n = O(1)$
b. $\Delta\epsilon_n = O(p_n/P_n)$
c. $\epsilon_n = O((p_n/P_n)(Q_n/q_n)^{1/k})$

as $n \to \infty$, where $\Delta\epsilon_n = \epsilon_n - \epsilon_{n+1}$.

Proof. In Theorem 3.2, we take $\mu_v = 1, \theta_v = \frac{P_v}{p_v}, \lambda_v = k \geq 1$ and $\psi_n = \frac{Q_v}{q_v}$ for all v. Then, the condition (12) is reduced to (c). By Lemma 2.2, (13) can be arranged as

$$\frac{P_v}{p_v} \Delta\epsilon_v - \frac{P_v q_v}{Q_v p_v} \epsilon_v + \epsilon_{v+1} = O(1).$$

Moreover, since $\frac{P_\nu q_\nu}{Q_\nu p_\nu}\epsilon_\nu = O\left(\left(\frac{q_\nu}{Q_\nu}\right)^{1/k^*}\right) = O(1)$, the last condition is equivalent to

$$\frac{P_\nu}{p_\nu}\Delta\epsilon_\nu + \epsilon_{\nu+1} = O(1)$$

which completes the proof.

Corollary 3.6 Let $1 < k < \infty$. Then, $\epsilon \in (|\bar{N}, p_n|_k, |\bar{N}, q_n|)$ if and only if

a. $\sum_{\nu=1}^{\infty}(p_\nu/P_\nu)\left|\frac{P_\nu}{p_\nu}\Delta\epsilon_\nu + \epsilon_{\nu+1}\right|^{k^*} < \infty,$

b. $\sum_{\nu=1}^{\infty}(p_\nu/P_\nu)\left(\frac{P_\nu q_\nu}{Q_\nu p_\nu}|\epsilon_\nu|\right)^{k^*} < \infty$

where $1/k + 1/k^* = 1$ for $k > 1$.

Proof. If we take $\mu_\nu = k, \theta_\nu = \frac{P_\nu}{p_\nu}$ for all ν in Theorem 3.1, the conditions (10) and (11) are reduced to (a) and (b).

Corollary 3.7 $\epsilon \in (|\bar{N}, p_n|, |\bar{N}, q_n|)$ if and only if

a. $\epsilon_n = O(1)$
b. $\Delta\epsilon_\nu = O(p_n/P_n)$
c. $\epsilon_n = O(p_n Q_n / P_n q_n)$ as $n \to \infty$.

CONCLUSION

Let $(\theta_n), (\psi_n)$ be sequences of positive numbers and (μ_n), (λ_n) be any bounded sequences of positive numbers. In this study, applying the main theorems in (Gökçe and Sarıgöl, 2018) to summability factors, we obtain the characterizations of the sets $(|\bar{N}, p_n, \theta|(\mu), |\bar{N}, q_n|)$ and $(|\bar{N}, p_n, \theta|(\mu), |\bar{N}, q_n, \psi|(\lambda))$. Also, in the special case, we get some well-known results.

REFERENCES

Gökçe F, Sarıgöl M A, 2018. A new series space $|\bar{N}_P^\theta|(\mu)$ and matrix transformations with applications. Kuwait Journal of Science, 45(4): 1-8.

Grosse-Erdmann KG, 1993. Matrix transformations between the sequence spaces of Maddox. Journal of Mathematical Analysis and Applications, 180(1): 223-238.

Mitrinovic DS, 1970. Analytic Inequalties. Springer-Verlag, Berlin.

Orhan C, Sarıgöl MA, 1993. On absolute weighted mean summability. The Rocky Mountain Journal of Mathematics, 23(3): 1091-1097.

Sarıgöl MA, 2016. Norms and compactness of operators on absolute weighted mean summable series. Kuwait Journal of Science, 43(4): 68-74.

Sarıgöl MA, 2013. An inequality for matrix operators and its applications. Journal of Classical Analysis, 2(2): 145-150.

Sarıgöl MA, 2011. Matrix transformatins on fields of absolute weighted mean summability. Studia Scientiarum Mathematicarum Hungarica, 48(3): 331-341.

Sarıgöl MA, Bor H, 1995. Characterization of absolute summability factors. Journal of Mathematical Analysis and Applications, 195 (2): 537-545.