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Bivariate Generalized Exponential Sampling Series and Applications to Seismic Waves

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ABSTRACT. In this paper, we introduce the generalized exponential sampling series of bivariate functions and establish some pointwise and uniform convergence results, also in a quantitative form. Moreover, we study the pointwise asymptotic behaviour of the series. One of the basic tools is the Mellin–Taylor formula for bivariate functions, here introduced. A practical application to seismic waves is also outlined.

Keywords: Bivariate generalized exponential sampling series, Mellin–Taylor formulae, moments, magnitude.

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Dedicated to the memory of our close unforgettable Friend and Colleague Professor Domenico Candeloro, who survives in our hearts.

1. INTRODUCTION

The classical exponential sampling series of a function $f : \mathbb{R}^+ \to \mathbb{C}$, in one dimensional case, represents a tool of relevant interest for optical phenomena, for example the light scattering and Fraunhofer diffraction, see e.g. [17, 11, 18] and [23].

It is defined by

$$(E_{c,T}f)(x) := \sum_{k=-\infty}^{\infty} f(e^{k/T})\text{lin}_{c/T}(e^{-k}x^T), \quad T > 0, \ x \in \mathbb{R}^+.$$

The lin_c-function for $c \in \mathbb{R}$, $\text{lin}_c : \mathbb{R}^+ \to \mathbb{R}$, is defined, for $x \in \mathbb{R}^+ \setminus \{1\}$, by

$$\text{lin}_c(x) = \frac{x^{-c}}{2\pi i}\frac{x^{\pi i} - x^{-\pi i}}{\log x} = x^{-c}\text{sinc}(\log x) = \frac{x^{-c}}{2\pi}\int_{-\pi}^{\pi} x^{-it}dt,$$

with the continuous extension $\text{lin}_c(1) := 1$.

Here, the "sinc" function, as usual, is defined by

$$\text{sinc}(u) := \frac{\sin(\pi u)}{\pi u}, \ \ u \neq 0, \ \text{sinc}(0) = 1.$$

The exponential sampling theorem for Mellin band-limited functions states that

$$(E_{c,T}f)(x) = f(x)$$

at every point x, and the series is absolutely and uniformly convergent on every compact interval of $\mathbb{R}^+$. A rigorous treatment of this theorem by a mathematical point of view was given in [14, 3, 10]. This theorem represents a Mellin version of the classical Shannon sampling theorem of Fourier analysis ([20, 27]).

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Note that, the exponential sampling theorem was framed in the field of the Mellin transform theory, which was first introduced in [22] and then developed in a systematic way in [13].

Now, as it happens for the Fourier band-limited functions, the assumption that f is Mellin band-limited is very restrictive, due to the Mellin–Paley–Wiener theorem (see [4], [5]). Therefore, in a recent paper [7] it was studied a generalization of the one dimensional exponential sampling theorem where the lin_c function is replaced by an arbitrary function φ satisfying suitable assumptions in an analogous way as for the generalized sampling series of Fourier analysis (see [15, 16, 6, 21, 2, 1]). In this way, we obtained an approximate reconstruction of a not necessarily Mellin band-limited function f.

The aim of this paper is to introduce a multivariate version of the generalized exponential sampling theorem in order to obtain new interesting applications to the study of the seismic waves. In this respect, for a sake of simplicity, we limit ourselves to consider the two dimensional case, being the general case carried on analogously.

Our main theoretical results concern the pointwise and uniform convergence and the study of the pointwise order of approximation through a bivariate asymptotic Voronovskaja formula. Basic tools are a two dimensional Mellin–Taylor formula, established in Section 3 both in the local and global version, and a notion of logarithmic modulus of continuity here introduced as a generalization of one dimensional case (see [8, 9]).

In Section 6, we give two important examples of bivariate kernel functions satisfying the required assumptions, namely the bivariate Mellin splines and the Mellin–Fejer kernels.

The last section is devoted to the study of the magnitude of an earthquake through the behaviour of the seismic waves.

We wish to dedicate this paper to the memory of our very close friend and colleague Professor Domenico Candeloro who passed away in May. He was a fine mathematician who combined his deep mathematical culture with a great modesty, a trait of his character that makes him an unforgettable person.

2. Preliminaries

Let us denote by $\mathbb{N}^2$, $\mathbb{N}_0^2$ and $\mathbb{Z}^2$ the sets of vectors $\mathtt{k} = (k_1, k_2)$ with k_1, k_2 positive integers, nonnegative integers and integers respectively and we set $\|\mathtt{k}\| := k_1 + k_2$. Moreover, by $\mathbb{R}^2$ we will denote the two dimensional Euclidean space comprising all vectors (x_1, x_2) with $x_1, x_2 \in \mathbb{R}$.

Given $\mathtt{x} = (x_1, x_2), \mathtt{y} = (y_1, y_2) \in \mathbb{R}^2$ we will say that $\mathtt{x} > \mathtt{y}$ if and only if $x_i > y_i$ for $i = 1, 2$ and we will denote by $\mathtt{1} := (1, 1)$, $\mathtt{0} := (0, 0)$ and by $\mathbb{R}_+^2$ the space of all vectors $\mathtt{x} > \mathtt{0}$.

Given $\mathtt{x}, \mathtt{y} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ we put as usual: $\mathtt{x} + \mathtt{y} := (x_1 + y_1, x_2 + y_2)$ and $\alpha\mathtt{x} := (\alpha x_1, \alpha x_2)$.

We will employ the following notations: $\mathtt{x}\mathtt{y} := (x_1 y_1, x_2 y_2)$, $\dfrac{\mathtt{x}}{\mathtt{y}} := (\dfrac{x_1}{y_1}, \dfrac{x_2}{y_2})$ (for $y_1, y_2 \neq 0$), $[\mathtt{x}] := (|x_1|, |x_2|)$, $\alpha^{\mathtt{x}} := (\alpha^{x_1}, \alpha^{x_2})$ with $\alpha > 0$, and $\mathtt{x}^{\mathtt{y}} := \prod_{i=1}^{2} x_i^{y_i}$, $\log(\mathtt{x}) := (\log x_1, \log x_2)$ with $\mathtt{x} > 0$.

We set $\|\mathtt{x}\| := \sqrt{x_1^2 + x_2^2}$, and the Euclidean distance $d(\mathtt{x}, \mathtt{y}) := \|\mathtt{x} - \mathtt{y}\|$.

For $\mathtt{w} = (w_1, w_2) \in \mathbb{R}_+^2$, by $\mathtt{w} \to \infty$ we mean $\underline{w} := \min\{w_1, w_2\} \to +\infty$.

Let J be an interval, bounded or not. We will denote by $C(J)$ the space of all continuous and bounded functions on J, by $C_c(J)$ the space of all continuous functions with compact support. Moreover, for $m \in \mathbb{N}$, by $C^{(m)}(J)$ we denote the subspace of $C(J)$ comprising all functions f with the derivatives up to the order m in $C(J)$.

Now, we introduce the following notion of continuity. We will say that a function $f : J \to \mathbb{C}$ is log-uniformly continuous on J if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|f(\mathtt{x}) - f(\mathtt{y})| <$

ε, whenever $\mathrm{x}, \mathrm{y} \in \mathbb{R}^2_+$ with $\|\log(\mathrm{x}) - \log(\mathrm{y})\| \leq \delta(\varepsilon)$. We will denote by $\mathcal{C}(J)$ the space of all log-uniformly continuous and bounded functions on J.

Note that for compact intervals $J \subset \mathbb{R}^2_+$ the notion of log-uniform continuity is equivalent to the classical uniform continuity.

Finally, we will say that a function $f : J \to \mathbb{C}$ belongs to $C^{(m)}(J)$ locally at a point $\mathrm{x} \in J$ if there is a neighbourhood I of x such that f is $(m-1)$-times differentiable on I and the derivative $f^{(m)}(\mathrm{x})$ exists.

3. Two-dimensional Mellin–Taylor formulae

We begin with the notion of partial derivatives of a function $f : \mathbb{R}^2_+ \to \mathbb{C}$ in the Mellin frame. The first partial Mellin derivative of f with respect to the variable x_i, $i = 1, 2$ at the point $\mathtt{x} = (x_1, x_2)$ is given by

$$\Theta_{x_i} f(\mathtt{x}) := x_i \frac{\partial f(\mathtt{x})}{\partial x_i}.$$

For a given $\mathtt{k} = (k_1, k_2) \in \mathbb{N}^2_0$ we define the partial Mellin derivatives of order $r = \|\mathtt{k}\| = k_1 + k_2$ at the point $\mathtt{x}$ as

$$\Theta^r_{x_1^{k_1} x_2^{k_2}} f(\mathtt{x}) := \Theta^{k_1}_{x_1}(\Theta^{k_2}_{x_2} f)(\mathtt{x}). \tag{3.1}$$

We will put $\Theta^1_{x_i} f(\mathtt{x}) := \Theta_{x_i} f(\mathtt{x})$ e $\Theta^0_{x_i} f(\mathtt{x}) := f(\mathtt{x})$.

Note that for example,

$$\Theta^2_{x_i} = \Theta_{x_i}(\Theta_{x_i} f)(\mathtt{x}) = x_i \frac{\partial f(\mathtt{x})}{\partial x_i} + x_i^2 \frac{\partial^2 f(\mathtt{x})}{\partial x_i^2}.$$

In order to extend the one-dimensional Mellin–Taylor formulae introduced in [8] to the bivariate case we will use the following notation. For a given $\mathtt{x} = (x_1, x_2)$ and $\mathtt{t} = (t_1, t_2)$ we set

$$\begin{aligned} &(\Theta_{x_1} \log t_1 + \Theta_{x_2} \log t_2)^m f(x_1, x_2) \\ := &\sum_{k=0}^{m} \binom{m}{k} \Theta^{m-k}_{x_1}(\Theta^k_{x_2} f)(x_1, x_2) \log^{m-k} t_1 \log^k t_2 \end{aligned} \tag{3.2}$$

with $m \in \mathbb{N}$ and $f \in C^{(m)}(\mathbb{R}^2_+)$ locally in (x_1, x_2).

For example for $m = 2$ we obtain

$$\begin{aligned} &(\Theta_{x_1} \log t_1 + \Theta_{x_2} \log t_2)^2 f(x_1, x_2) = \\ &\Theta^2_{x_1} f(x_1, x_2) \log^2 t_1 + 2\Theta_{x_1}(\Theta_{x_2} f)(x_1, x_2) \log t_1 \log t_2 + \Theta^2_{x_2} f(x_1, x_2) \log^2 t_2. \end{aligned}$$

We have the following proposition.

Proposition 3.1. *Let $f : \mathbb{R}^2_+ \to \mathbb{C}$ be a function in $C^{(m)}(\mathbb{R}^2_+)$ with $m \in \mathbb{N}$. Then for $\mathtt{x} = (x_1, x_2) \in \mathbb{R}^2_+$ and $\mathtt{t} = (t_1, t_2) \in \mathbb{R}^2_+$, we have*

$$\begin{aligned} f(t_1 x_1, t_2 x_2) = &f(x_1, x_2) + (\Theta_{x_1} \log t_1 + \Theta_{x_2} \log t_2) f(x_1, x_2) + \\ &\frac{1}{2!}(\Theta_{x_1} \log t_1 + \Theta_{x_2} \log t_2)^2 f(x_1, x_2) + \ldots + \\ &\frac{1}{(m-1)!}(\Theta_{x_1} \log t_1 + \Theta_{x_2} \log t_2)^{m-1} f(x_1, x_2) + R_m(t_1, t_2), \end{aligned}$$

with Lagrange remainder

$$R_m(t_1,t_2) = \frac{1}{m!}(\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)^m f(\theta,\eta),$$

where (θ,η) *is a suitable point in the segment* L_{t_1,t_2} *with end points* (x_1,x_2), (t_1x_1,t_2x_2).

Proof. We prove the case $m=2$. For the general case one can apply (3.2).

Let us take the function $F(t) = f({t_1}^{\log t}x_1, {t_2}^{\log t}x_2)$ with $t\in[1,e]$. Applying the one dimensional Mellin–Taylor formula with Lagrange remainder, we obtain

$$F(t) = F(1) + \Theta F(1)\log t + \frac{\Theta^2 F(\tilde t)}{2}\log^2 t$$

with $\tilde t \in]1,e[$. We have

$$\Theta F(t) = \frac{\partial f}{\partial x_1}({t_1}^{\log t}x_1, {t_2}^{\log t}x_2)x_1{t_1}^{\log t}\log t_1 + \frac{\partial f}{\partial x_2}({t_1}^{\log t}x_1, {t_2}^{\log t}x_2)x_2{t_2}^{\log t}\log t_2$$

and for $t=1$ we obtain

$$\Theta F(1) = \Theta_{x_1}f(x_1,x_2)\log t_1 + \Theta_{x_2}f(x_1,x_2)\log t_2 = (\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)f(x_1,x_2).$$

Analogously for $\Theta^2F(t) = tF'(t) + t^2F''(t)$, we have

$$\begin{aligned}\Theta^2 F(t) =& \frac{\partial^2 f}{\partial x_1^2}({t_1}^{\log t}x_1, {t_2}^{\log t}x_2)x_1^2{t_1}^{2\log t}\log^2 t_1 + \frac{\partial^2 f}{\partial x_2^2}({t_1}^{\log t}x_1, {t_2}^{\log t}x_2)x_2^2{t_2}^{2\log t}\log^2 t_2 + \\ & 2\frac{\partial^2 f}{\partial y\partial x_1}({t_1}^{\log t}x_1, {t_2}^{\log t}x_2)x_1x_2(t_1t_2)^{\log t}\log t_1\log t_2 + \\ & \frac{\partial f}{\partial x_1}({t_1}^{\log t}x_1, {t_2}^{\log t}x_2)x_1{t_1}^{\log t}\log^2 t_1 + \frac{\partial f}{\partial x_2}({t_1}^{\log t}x_1, {t_2}^{\log t}x_2)x_2{t_2}^{\log t}\log^2 t_2\end{aligned}$$

and for $t=\tilde t$

$$\begin{aligned}\frac{\Theta^2F(\tilde t)}{2} =& \frac{1}{2}\Big\{\Big(\frac{\partial^2 f}{\partial x_1^2}(\theta,\eta)\theta^2 + \frac{\partial f}{\partial x_1}(\theta,\eta)\theta\Big)\log^2 t_1 + \Big(\frac{\partial^2 f}{\partial x_2^2}(\theta,\eta)\eta^2 + \frac{\partial f}{\partial x_2}(\theta,\eta)\eta\Big)\log^2 t_2 \\ & + 2\frac{\partial^2 f}{\partial x_1\partial x_2}(\theta,\eta)\theta\eta\log t_1\log t_2\Big\},\end{aligned}$$

with $(\theta,\eta) = (t_1^{\log\tilde t}x_1, t_2^{\log\tilde t}x_2)\in L_{t_1,t_2}$.

Now, using the definition of the partial Mellin derivative, we have the formulae

$$\frac{\partial^2 f}{\partial x_1^2}(\theta,\eta)\theta^2 = [\Theta^2_{x_1}f(\theta,\eta) - \Theta_{x_1}f(\theta,\eta)],$$

$$\frac{\partial^2 f}{\partial x_2^2}(\theta,\eta)\eta^2 = [\Theta^2_{x_2}f(\theta,\eta) - \Theta_{x_2}f(\theta,\eta)],$$

$$\frac{\partial^2 f}{\partial x_1\partial x_2}(\theta,\eta)\theta\eta = \Theta_{x_1}(\Theta_{x_2}f)(\theta,\eta),$$

then

$$\frac{\Theta^2F(\tilde t)}{2} = \frac{1}{2}(\Theta_{x_1}\log t_1 + \Theta_{x_2}\log t_2)^2 f(\theta,\eta).$$

So the assertion follows. □

By Proposition 3.1, we can deduce a local version of the Mellin–Taylor formula, namely a formula with the Peano remainder. It is based on the following proposition

Proposition 3.2. *Under the same assumptions and notations of Proposition 3.1 there holds*

$$\lim_{(t_1,t_2)\to(1,1)} \frac{(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2)^m f(\theta,\eta)-(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2)^m f(x_1,x_2)}{(\log^2 t_1+\log^2 t_2)^{m/2}}=0.$$

Proof. We consider the case $m=2$, the general case is carried on in a similar way. Setting

$$I := \left|(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2)^2 f(\theta,\eta)-(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2)^2 f(x_1,x_2)\right|,$$

we have

$$\begin{aligned} I &\le |\Theta^2_{x_1} f(\theta,\eta)-\Theta^2_{x_1} f(x_1,x_2)|\log^2 t_1\\ &+\ 2|\Theta_{x_1}(\Theta_{x_2} f)(\theta,\eta)-\Theta_{x_1}(\Theta_{x_2} f)(x_1,x_2)||\log t_1\log t_2|\\ &+\ |\Theta^2_{x_2} f(\theta,\eta)-\Theta^2_{x_2} f(x_1,x_2)|\log^2 t_2, \end{aligned}$$

and hence

$$\begin{aligned}\frac{I}{\log^2 t_1+\log^2 t_2} \le & |\Theta^2_{x_1} f(\theta,\eta)-\Theta^2_{x_1} f(x_1,x_2)|+|\Theta_{x_1}(\Theta_{x_2} f)(\theta,\eta)-\Theta_{x_1}(\Theta_{x_2} f)(x_1,x_2)|+\\ & |\Theta^2_{x_2} f(\theta,\eta)-\Theta^2_{x_2} f(x_1,x_2)|.\end{aligned}$$

Taking into account that $(\theta,\eta)\in L_{t_1,t_2}$ the assertion follows from the assumption $f\in C^{(2)}(\mathbb{R}^2_+)$. □

By Proposition 3.2, we can write the local form of the Mellin–Taylor formula as

$$\begin{aligned} f(t_1x,t_2y) =& f(x_1,x_2)+(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2)f(x_1,x_2)+\\ & \frac{1}{2!}(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2)^2 f(x_1,x_2)+\ldots+\\ & \frac{1}{m!}(\Theta_{x_1}\log t_1+\Theta_{x_2}\log t_2)^m f(x_1,x_2)+R_m(t_1,t_2),\end{aligned}$$

with the Peano remainder

$$R_m(t_1,t_2)=H(t_1,t_2)(\log^2 t_1+\log^2 t_2)^{m/2},$$

where $H(t_1,t_2)$ is a bounded function such that $\lim_{(t_1,t_2)\to(1,1)} H(t_1,t_2)=0$.

Remark 3.1. Note that setting $\mathtt{t}=(t_1,t_2)$, $\mathtt{x}=(x_1,x_2)$ we can write

$$R_m(\mathtt{t})=H(\mathtt{t})\|\log(\mathtt{tx})-\log\mathtt{x}\|^m.$$

Moreover, it is not difficult to see that the local version of the Mellin–Taylor formula can be proved under the more general assumptions $f\in C^{(m)}(\mathbb{R}^2_+)$ locally at the point $\mathtt{x}$.

4. Bivariate generalized sampling operator

Let $\varphi\colon \mathbb{R}^2_+\to\mathbb{R}$ be a continuous function such that

(φ.1) $\sum\limits_{\mathtt{k}\in\mathbb{Z}^2}\varphi(e^{-\mathtt{k}}\mathtt{x})=\sum\limits_{(k_1,k_2)\in\mathbb{Z}^2}\varphi(e^{-k_1}x_1,e^{-k_2}x_2)=1$ for every $\mathtt{x}=(x_1,x_2)\in\mathbb{R}^2_+$;

(φ.2) there holds

$$M_0(\varphi):=\sup_{\mathtt{x}\in\mathbb{R}^2_+}\sum_{\mathtt{k}\in\mathbb{Z}^2}|\varphi(e^{-\mathtt{k}}\mathtt{x})|<+\infty;$$

(φ.3) $\lim\limits_{r\to+\infty}\sum\limits_{\|\mathtt{k}-\log(\mathtt{x})\|\ge r}|\varphi(e^{-\mathtt{k}}\mathtt{x})|=0$, uniformly with respect to $\mathtt{x}$.

Let Φ be the class of all functions φ satisfying the above assumptions.

Let $\mathtt{j} = (j_1, j_2) \in \mathbb{N}_0^2$ and let $\nu = [\![\mathtt{j}]\!]$. For $\mathtt{x} \in \mathbb{R}_+^2$, we define the moments of order $\mathtt{j}$ of $\varphi \in \Phi$ as

$$m_{\mathtt{j}}^{\nu}(\varphi, \mathtt{x}) := \sum_{\mathtt{k}\in\mathbb{Z}^2} \varphi(e^{-\mathtt{k}}\mathtt{x}) \log^{\mathtt{j}}(e^{\mathtt{k}}\mathtt{x}^{-1}) = \sum_{\mathtt{k}\in\mathbb{Z}^2} \varphi(e^{-\mathtt{k}}\mathtt{x})(\mathtt{k} - \log(\mathtt{x}))^{\mathtt{j}}$$
$$= \sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi(e^{-k_1}x_1, e^{-k_2}x_2)(k_1 - \log x_1)^{j_1}(k_2 - \log x_2)^{j_2}.$$

The absolute moments of order $\mathtt{j}$ of $\varphi \in \Phi$ are defined as

$$M_{\mathtt{j}}^{\nu}(\varphi, \mathtt{x}) := \sum_{\mathtt{k}\in\mathbb{Z}^2} |\varphi(e^{-\mathtt{k}}\mathtt{x})|[\log(e^{\mathtt{k}}\mathtt{x}^{-1})]^{\mathtt{j}} = \sum_{\mathtt{k}\in\mathbb{Z}^2} |\varphi(e^{-\mathtt{k}}\mathtt{x})|[\mathtt{k} - \log(\mathtt{x})]^{\mathtt{j}}$$
$$= \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\varphi(e^{-k_1}x_1, e^{-k_2}x_2)||k_1 - \log x_1|^{j_1}\, |k_2 - \log x_2|^{j_2}.$$

Finally, we set $M_{\mathtt{j}}^{\nu}(\varphi) := \sup_{\mathtt{x}\in\mathbb{R}_+^2} M_{\mathtt{j}}^{\nu}(\varphi, \mathtt{x})$.

Let $\varphi \in \Phi$. For any $\mathtt{w} > \mathtt{0}$, $\mathtt{w} = (w_1, w_2)$ and $f\colon \mathbb{R}_+^2 \to \mathbb{C}$, we define the generalized exponential series as

$$(E_{\mathtt{w}}^{\varphi} f)(\mathtt{x}) := \sum_{\mathtt{k}\in\mathbb{Z}^2} f(e^{\frac{\mathtt{k}}{\mathtt{w}}})\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}}) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} f(e^{\frac{k_1}{w_1}}, e^{\frac{k_2}{w_2}})\varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2}) \tag{4.3}$$

for $\mathtt{x} = (x_1, x_2) \in \mathbb{R}_+^2$ and for any function $f \in \operatorname{dom} E_{\mathtt{w}}^{\varphi}$, being $\operatorname{dom} E_{\mathtt{w}}^{\varphi}$ the set of all functions f for which the series is absolutely convergent on every $\mathtt{x}$. Using the conditions of the class Φ, it is easy to see that the above operator is well defined as an absolutely convergent series, for any bounded function f. In particular $C(\mathbb{R}_+^2) \subset \operatorname{dom} E_{\mathtt{w}}^{\varphi}$, for any $\mathtt{w} > \mathtt{0}$.

We begin with the following pointwise convergence theorem.

Theorem 4.1. *Let $f \in C(\mathbb{R}_+^2)$ and $\varphi \in \Phi$. Then*

$$\lim_{\mathtt{w}\to\infty} \sum_{\mathtt{k}\in\mathbb{Z}^2} f(e^{\frac{\mathtt{k}}{\mathtt{w}}})\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}}) = f(\mathtt{x}), \quad \mathit{for} \quad \mathtt{x} \in \mathbb{R}_+^2. \tag{4.4}$$

Proof. Since $\varphi \in \Phi$, we have

$$\Big| \sum_{\mathtt{k}\in\mathbb{Z}^2} f(e^{\frac{\mathtt{k}}{\mathtt{w}}})\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}}) - f(\mathtt{x})\Big| \le \sum_{\mathtt{k}\in\mathbb{Z}^2} |f(e^{\frac{\mathtt{k}}{\mathtt{w}}}) - f(\mathtt{x})||\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}})|.$$

For a fixed $\varepsilon > 0$, by the continuity of f at $\mathtt{x}$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|\log(\mathtt{x}) - \log(e^{\frac{\mathtt{k}}{\mathtt{w}}})\| = \|\log(\mathtt{x}) - \frac{\mathtt{k}}{\mathtt{w}}\| < \delta$, then $|f(\mathtt{x}) - f(e^{\frac{\mathtt{k}}{\mathtt{w}}})| < \varepsilon$. We write

$$\sum_{\mathtt{k}\in\mathbb{Z}^2} |f(e^{\frac{\mathtt{k}}{\mathtt{w}}}) - f(\mathtt{x})||\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}})|$$
$$= \Big\{ \sum_{\|\frac{\mathtt{k}}{\mathtt{w}} - \log(\mathtt{x})\| < \delta} + \sum_{\|\frac{\mathtt{k}}{\mathtt{w}} - \log(\mathtt{x})\| \ge \delta} \Big\} |f(e^{\frac{\mathtt{k}}{\mathtt{w}}}) - f(\mathtt{x})||\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}})| =: I_1 + I_2.$$

Now by assumption $(\varphi.2)$, we have immediately $I_1 \leq M_0(\varphi)\,\varepsilon$. As to I_2 by the boundedness of f and $(\varphi.3)$, taking into account that

$$\|\frac{\mathtt{k}}{\mathtt{w}} - \log(\mathtt{x})\| \leq \frac{\|\mathtt{k} - \log(\mathtt{x}^{\mathtt{w}})\|}{\underline{w}},$$

we have, for sufficiently large $\underline{w}$,

$$I_2 = \sum_{\|\frac{\mathtt{k}}{\mathtt{w}} - \log(\mathtt{x})\| \geq \delta} |f(e^{\frac{\mathtt{k}}{\mathtt{w}}}) - f(\mathtt{x})||\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}})| \leq 2\|f\|_\infty \sum_{\|\mathtt{k} - \log(\mathtt{x}^{\mathtt{w}})\| \geq \delta \underline{w}} |\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}})| < 2\|f\|_\infty \varepsilon,$$

and so the assertion follows. □

Using essentially the same reasoning employed in the previous theorem, we can prove the following uniform convergence result.

Theorem 4.2. *Let $f \in \mathcal{C}(\mathbb{R}^2_+)$ and $\varphi \in \Phi$,then*

$$\lim_{\mathtt{w} \to \infty} \Big\| \sum_{\mathtt{k} \in \mathbb{Z}^2} f(e^{\frac{\mathtt{k}}{\mathtt{w}}})\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}}) - f(\mathtt{x})\Big\|_\infty = 0. \tag{4.5}$$

5. Estimation of the error of approximation

We premise the following notion. The logarithmic modulus of continuity of $f \in \mathcal{C}(\mathbb{R}^2_+)$ is defined, for $\delta > 0$, by

$$\omega(f, \delta) := \sup\{|f(\mathtt{x}) - f(\mathtt{y})| : \mathtt{x}, \mathtt{y} \in \mathbb{R}^2_+, \quad \|\log(\mathtt{x}) - \log(\mathtt{y})\| \leq \delta\}.$$

This modulus satisfies all the properties of the one dimensional logarithmic modulus of continuity (see [9]). In particular, it is a monotone increasing function of $\delta > 0$ and the following inequality holds, for $\lambda > 0$

$$\omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta). \tag{5.6}$$

We have the following theorem.

Theorem 5.3. *If $f \in \mathcal{C}(\mathbb{R}^2_+)$, $\varphi \in \Phi$, and*

$$D := \sup_{\mathtt{x} \in \mathbb{R}^2_+} \sum_{\mathtt{k} \in \mathbb{Z}^2} |\varphi(e^{-\mathtt{k}}\mathtt{x})|\|\mathtt{k} - \log \mathtt{x}\| < +\infty,$$

then for $\mathtt{w} > 0$ and $\delta > 0$, we have

$$\Big|\sum_{\mathtt{k} \in \mathbb{Z}^2} f(e^{\frac{\mathtt{k}}{\mathtt{w}}})\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}}) - f(\mathtt{x})\Big| \leq M_0(\varphi)\omega(f, \delta) + D\frac{\omega(f, \delta)}{\delta \underline{w}}.$$

Proof. Using that $\varphi \in \Phi$, (5.6) and the inequality

$$\|\frac{\mathtt{k}}{\mathtt{w}} - \log(\mathtt{x})\| \leq \frac{\|\mathtt{k} - \log(\mathtt{x}^{\mathtt{w}})\|}{\underline{w}},$$

we have

$$|E^{\varphi}_{\mathtt{w}} f(\mathtt{x}) - f(\mathtt{x})| \leq \omega(f, \delta)\Big(\sum_{\mathtt{k} \in \mathbb{Z}^2} |\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}})| + \sum_{\mathtt{k} \in \mathbb{Z}^2} |\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}})| \frac{\|\mathtt{k} - \log(\mathtt{x}^{\mathtt{w}})\|}{\delta \underline{w}}\Big)$$

$$\leq \omega(f, \delta)M_0(\varphi) + \frac{\omega(f, \delta)}{\delta \underline{w}} D$$

and so the assertion follows. □

As a corollary we can prove

Corollary 5.1. *Under the assumptions of Teorem 5.3 there holds*

$$\Big| \sum_{\mathtt{k}\in\mathbb{Z}^2} f(e^{\frac{\mathtt{k}}{\mathtt{w}}})\varphi(e^{-\mathtt{k}}\mathtt{x}^{\mathtt{w}}) - f(\mathtt{x})\Big| \leq C(\varphi)\omega\Big(f, \frac{1}{\underline{w}}\Big). \tag{5.7}$$

Proof. Applying Theorem 5.3 with $\delta = \dfrac{1}{\underline{w}}$, we obtain

$$|E^{\varphi}_{\mathtt{w}} f(\mathtt{x}) - f(\mathtt{x})| \leq \omega\Big(f, \frac{1}{\underline{w}}\Big)M_0(\varphi) + \omega\Big(f, \frac{1}{\underline{w}}\Big)D.$$

Setting $C(\varphi) = M_0(\varphi) + D$ we have the assertion. □

Now, we obtain estimations of the order of approximation under some local regularity assumptions on the function f. In order to do that we will need further assumptions on the kernel function φ, i.e., there exists $\ell \in \mathbb{N}$ such that for every $\mathtt{j} \in \mathbb{N}^2_0$, $[\![\mathtt{j}]\!] \leq \ell$

(φ.4) $m^{[\![\mathtt{j}]\!]}_{\mathtt{j}}(\varphi, \mathtt{x}) =: m^{[\![\mathtt{j}]\!]}_{\mathtt{j}}(\varphi)$ is independent of $\mathtt{x}$;

(φ.5) $M^{[\![\mathtt{j}]\!]}_{\mathtt{j}}(\varphi) < +\infty$ and

$$\lim_{r\to+\infty} \sum_{\|\mathtt{k}-\log(\mathtt{x})\|>r} |\varphi(e^{-\mathtt{k}}\mathtt{x})|\, \|(\mathtt{k} - \log(\mathtt{x}))\|^{\ell} = 0,$$

uniformly with respect to $\mathtt{x}$.

Remark 5.2. Since for any $\mathtt{j} = (j_1, j_2) \in \mathbb{N}^2_0$ and any vector $\mathtt{v} = (v_1, v_2) \in \mathbb{R}^2_+$ we have $[\mathtt{v}]^{\mathtt{j}} \leq \|\mathtt{v}\|^{\mathtt{j}}$, we deduce immediately that assumption (φ.5) implies that

$$\lim_{r\to+\infty} \sum_{\|\mathtt{k}-\log(\mathtt{x})\|>r} |\varphi(e^{-\mathtt{k}}\mathtt{x})|\, [\mathtt{k} - \log(\mathtt{x})]^{[\![\mathtt{j}]\!]} = 0,$$

uniformly with respect to $\mathtt{x}$, for every $\mathtt{j}$ with $[\![\mathtt{j}]\!] \leq \ell$.

We denote by Φ_ℓ the set of functions φ satisfying conditions (φ.1), (φ.4), (φ.5).

We have the following result, in which we assume $\ell = 2$.

Theorem 5.4. *Let $f\colon \mathbb{R}^2_+ \to \mathbb{C}$ be a function such that $f \in C^{(2)}(\mathbb{R}^2_+)$ locally at the point $\mathtt{x} = (x_1, x_2) \in \mathbb{R}^2_+$. If $\varphi \in \Phi_2$, then for $\mathtt{w} = (w_1, w_2) > 0$,*

$$(E^{\varphi}_{(w_1,w_2)} f)(x_1, x_2) - f(x_1, x_2) = \sum_{\nu=1}^{2} \sum_{[\![\mathtt{h}]\!]=\nu} \Big(\frac{\Theta^{\nu} f(x_1, x_2}{\nu!} \frac{m^{\nu}_{h}(\varphi)}{\mathtt{w}^{\mathtt{h}}}\Big) + o(\underline{w}^{-2}).$$

Proof. Since $\varphi \in \Phi_2$, applying the Mellin–Taylor formula of the second order with local remainder, we can write

$$(E^{\varphi}_{(w_1,w_2)}f)(x_1,x_2) - f(x_1,x_2) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})\Bigg((\Theta_{x_1}\log(\frac{e^{\frac{k_1}{w_1}}}{x_1})+$$
$$\Theta_{x_2}\log(\frac{e^{\frac{k_2}{w_2}}}{x_2}))f(x_1,x_2) + \frac{1}{2}(\Theta_{x_1}\log(\frac{e^{\frac{k_1}{w_1}}}{x_1}) + \Theta_{x_2}\log(\frac{e^{\frac{k_2}{w_2}}}{x_2}))^2 f(x_1,x_2)+$$
$$H(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2})(\log^2(\frac{e^{\frac{k_1}{w_1}}}{x_1}) + \log^2(\frac{e^{\frac{k_2}{w_2}}}{x_2}))\Bigg)$$
$$= \frac{1}{w_1}\Theta_{x_1}f(x_1,x_2)m^1_{(1,0)}(\varphi) + \frac{1}{w_2}\Theta_{x_2}f(x_1,x_2)m^1_{(0,1)}(\varphi) + \frac{1}{2}\frac{1}{w_1^2}\Theta^2_{x_1}f(x,y)m^2_{(2,0)}(\varphi)+$$
$$\frac{1}{2}\frac{1}{w_2^2}\Theta^2_{x_2}f(x_1,x_2)m^2_{(0,2)}(\varphi) + \frac{1}{w_1}\frac{1}{w_2}\Theta_{x_1}(\Theta_{x_2}f)(x_1,x_2)m^2_{(1,1)}(\varphi)+$$
$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})H\Big(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2}\Big)\Big(\log^2\Big(\frac{e^{\frac{k_1}{w_1}}}{x_1}\Big) + \log^2\Big(\frac{e^{\frac{k_2}{w_2}}}{x_2}\Big)\Big)$$
$$= \sum_{\nu=1}^{2}\sum_{|h|=\nu}\Big(\frac{\Theta^\nu f(x_1,x_2)}{\nu!}\frac{m^\nu_h(\varphi)}{\mathtt{w}^{\mathtt{h}}}\Big)+$$
$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})H\Big(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2}\Big)\Big(\log^2\Big(\frac{e^{\frac{k_1}{w_1}}}{x_1}\Big) + \log^2\Big(\frac{e^{\frac{k_2}{w_2}}}{x_2}\Big)\Big).$$

Here, $H(t_1,t_2)$ tends to zero, for $(t_1,t_2) \to (1,1)$. Thus, for a given, $\varepsilon > 0$ there is $\delta > 0$ such that

$$H\Big(\frac{e^{k_1/w_1}}{x_1}, \frac{e^{k_2/w_2}}{x_2}\Big) < \varepsilon,$$

whenever $\|\log(e^{\mathtt{k}/\mathtt{w}}) - \log(\mathtt{x})\| < \delta$.

Setting

$$R := \sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})H\Big(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2}\Big)\Big(\log^2\Big(\frac{e^{\frac{k_1}{w_1}}}{x_1}\Big) + \log^2\Big(\frac{e^{\frac{k_2}{w_2}}}{x_2}\Big)\Big),$$

taking into account that

$$\|\frac{\mathtt{k}}{\mathtt{w}} - \log(\mathtt{x})\| \le \frac{\|\mathtt{k} - \log(\mathtt{x}^{\mathtt{w}})\|}{\underline{w}}$$

as in Theorem 4.1, we obtain

$$\underline{w}^2|R| \le \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})|\,\Big|H\Big(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2}\Big)\Big|\|\mathtt{k} - \log(\mathtt{x}^{\mathtt{w}})\|^2$$
$$\le \Bigg\{\sum_{\|\frac{\mathtt{k}}{\mathtt{w}}-\log(\mathtt{x})\|<\delta} + \sum_{\|\mathtt{k}-\log(\mathtt{x}^{\mathtt{w}})\|\ge\delta\underline{w}}\Bigg\}|\varphi(e^{-k_1}x_1^{w_1}, e^{-k_2}x_2^{w_2})|\Big|H\Big(\frac{e^{\frac{k_1}{w_1}}}{x_1}, \frac{e^{\frac{k_2}{w_2}}}{x_2}\Big)\Big|\|\mathtt{k} - \log(\mathtt{x}^{\mathtt{w}})\|^2$$
$$=: \quad S_1 + S_2.$$

As to S_1, denoting $M_2(\varphi) := \max_{\|\mathtt{j}\|=2} M^2_{\mathtt{j}}(\varphi)$, we have easily $S_1 \leq 3M_2(\varphi)\varepsilon$, while for S_2, by $(\varphi.5)$ and the boundedness of H, we obtain $S_2 \leq \|H\|_\infty\, \varepsilon$. Thus the proof is complete. □

6. Some examples

In [7] the one-dimensional generalized exponential sampling series was introduced, in which the generating (one-dimensional) kernel $\phi : \mathbb{R}^+ \to \mathbb{R}$ satisfies the assumptions:

$(\phi.1)$ $\sum_{k\in\mathbb{Z}} \phi(e^{-k}x) = 1$ for every $x \in \mathbb{R}^+$,

$(\phi.2)$ $\sup_{x\in\mathbb{R}^+} \sum_{k\in\mathbb{Z}} |\phi(e^{-k}x)| < +\infty$;

$(\phi.3)$ $\lim_{r\to+\infty} \sum_{|k-\log(x)|>r} |\phi(e^{-k}x)| = 0$, uniformly with respect to x.

We will denote by Ψ the set comprising all functions $\phi : \mathbb{R}^+ \to \mathbb{R}$ satisfying $(\phi.1), (\phi.2)$ and $(\phi.3)$. Using a product of two such functions, we can construct examples of two-dimensional kernel $\varphi \in \Phi$. In this respect we have the following proposition.

Proposition 6.3. *If $\phi_1, \phi_2 \in \Psi$ are bounded, then $\Gamma(x_1, x_2) := \phi_1(x_1)\phi_2(x_2) \in \Phi$.*

Proof. Assumptions $(\varphi.1)$ and $(\varphi.2)$ are immediate, applying $(\phi.1)$ and $(\phi.2)$. As to $(\varphi.3)$, note that if $\|(k_1 - \log(x_1), k_2 - \log(x_2))\| > r$, then $r < \|(k_1 - \log(x_1), k_2 - \log(x_2))\| \leq |k_1 - \log(x_1)| + |k_2 - \log(x_2)|$. Therefore,

$$\lim_{r\to+\infty} \sum_{||(k_1-\log(x_1),k_2-\log(y))||>r} |\varphi_1(e^{-k_1}x)\varphi_2(e^{-k_2}x_2)|$$
$$\leq \lim_{r\to+\infty} \sum_{|k_1-\log(x_1)|+|k_2-\log(x_2)|>r} |\varphi_1(e^{-k_1}x)\varphi_2(e^{-k_2}x_2)|.$$

For a fixed $\mathtt{x} = (x_1, x_2) \in \mathbb{R}^2_+$ and $r > 0$, we set

$$A := \{(k_1, k_2) : |k_1 - \log(x_1)| + |k_2 - \log(x_2)| > r\};$$

$$B_1 := \{(k_1, k_2) : |k_1 - \log(x_1)| > \frac{r}{2}\}, \quad B_2 := \{(k_1, k_2) : |k_2 - \log(x_2)| > \frac{r}{2}\}$$

and $B := B_1 \cup B_2$. Since $A \subset B$, we have

$$\lim_{r\to+\infty} \sum_{(k_1,k_2)\in A} |\varphi_1(e^{-k_1}x_1)\varphi_2(e^{-k_2}x_2)| \leq \lim_{r\to+\infty} \sum_{(k_1,k_2)\in B} |\varphi_1(e^{-k_1}x_1)\varphi_2(e^{-k_2}x_2)|$$
$$\leq \lim_{r\to+\infty} \sum_{(k_1,k_2)\in B_1} |\varphi_1(e^{-k_1}x_1)\varphi_2(e^{-k_2}x_2)| + \lim_{r\to+\infty} \sum_{(k_1,k_2)\in B_2} |\varphi_1(e^{-k_1}x_1)\varphi_2(e^{-k_2}x_2)|.$$

By the boundedness of the functions ϕ_1 and ϕ_2 and by $(\phi.3)$, we obtain easily $(\varphi.3)$. □

Making use of Proposition 6.3, we costruct some box-type kernel, using the product of two classical one-dimensional kernels.

Example 6.1. Denoting by r_+ the positive part of a number $r \in \mathbb{R}$, for $n \in \mathbb{N}$, we define the (one-dimensional) Mellin spline of order n, as (see [7, 10]

$$B_n(x) := \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (\frac{n}{2} + \log x - j)^{n-1}_+, \quad (x \in \mathbb{R}^+). \tag{6.8}$$

These kernel functions are the Mellin version of the classical central B-splines see [25]. The functions B_n are compactly supported, and satisfy all the assumptions of the class Ψ (see [7]). Using these functions we can define, for $n, m \in \mathbb{N}$,

$$B_{n,m}(x_1, x_2) := B_n(x_1)\, B_m(x_2). \tag{6.9}$$

By Proposition 6.3, the kernel $B_{n,m} \in \Phi$. In particular, for $n = m = 2$, we obtain

$$B_{2,2}(x_1, x_2) = \begin{cases} (1 + \log x_1)(1 + \log x_2), & e^{-1} < x_1, x_2 < 1 \\ (1 - \log x_1)(1 + \log x_2), & 1 < x_1 < e, e^{-1} < x_2 < 1 \\ (1 + \log x_1)(1 - \log x_2), & e^{-1} < x_1 < 1, 1 < x_2 < e \\ (1 - \log x_1)(1 - \log x_2), & 1 < x_1, x_2 < e \\ 0, & \text{otherwise.} \end{cases}$$

Example 6.2. Another interesting example, is given by the product of two one-dimensional Mellin–Fejer kernels, which are defined for any $\rho > 0$, $c \in \mathbb{R}$ and $x \in \mathbb{R}^+$ by (see [7, 10])

$$F^c_\rho(x) := \begin{cases} \dfrac{x^{-c}}{2\pi}\, \rho\, \mathrm{sinc}^2\!\left(\dfrac{\rho}{\pi} \log \sqrt{x}\right), & x \neq 1 \\ \dfrac{\rho}{2\pi}, & x = 1 \end{cases}. \tag{6.10}$$

We have $F^c_\rho \in \Psi$ and using again Proposition 6.3, we see that the kernel

$$F^{c_1,c_2}_{\rho_1,\rho_2}(x_1, x_2) = F^{c_1}_{\rho_1}(x_1)\, F^{c_2}_{\rho_2}(x_2) \qquad (x_1, x_2) \in \mathbb{R}^2_+, \tag{6.11}$$

with $\rho_1, \rho_2 > 0$, $c_1, c_2 \in \mathbb{R}$, belongs to the class Φ.

Remark 6.3. As remarked in [7], the one-dimensional Mellin–Fejer kernel does not satisfy the (one-dimensional) moment condition

$$\widetilde{M}_1(\phi) := \sup_{x \in \mathbb{R}^+} \sum_{k=-\infty}^{\infty} |\varphi(e^{-k}x)|\, |k - \log x| < +\infty.$$

Therefore, in [7, 10] a suitable modification of the kernel was studied, introducing the so-called Mellin–Jackson kernels, which satisfy the above condition. Thus, one can obtain other interesting examples, by considering the product of two Mellin–Jackson kernels.

7. Seismic waves and exponential sampling

Due to their nature and their way of propagation, seismic waves can be modeled using exponential functions. Seismic waves, due to the continuous and natural movements of the terrestrial plates, originate from a point at a certain depth into the ground, this point called hypocenter, and develop for several kilometers, being attenuated thanks to the elastic properties of the medium they cross. The projection of the hypocenter on the heart surface is called epicenter and it is the point of maximum amplitude of the seismic wave. For a given direction $\theta \in [-\pi, \pi]$ in the horizontal plane, the definition of amplitude $A(R, \theta)$ of a seismic wave, used in this work, is the variation, measured in mm, registered by a standard Wood-Anderson seismograph at a certain distance R from the epicenter. Connected with the definition of $A(R, \theta)$ is the formalization of the Local Magnitude $LM(R, \theta)$ [12], needed to measure the intensity of an earthquake according to the Richter scale [19, 24]:

$$LM(R, \theta) := \log_{10} A(R, \theta) - \log_{10} A_0(R, \theta), \tag{7.12}$$

where $\log_{10} A_0(R, \theta)$ is a calibration function such that, for $R = 100\,Km$ and in whatever direction θ, $\log_{10} A_0(100, \theta) = 3$. We assume the model to be symmetric with respect to the epicenter that is located in the center of the axis ((0,0) coordinates).

The model derived in [12], based on the dataset provided by ISNet (Irpinia Seismic Network), has been used to test how the mathematical theory approximates real data. ISNet is a network of 27 stations located in the South of Italy, along the Apennines chain [26]. In the ISNet dataset the epicenter distance R has been substituted with the hypocenter distance, committing an error of less than 1%, approximation possible thanks to the reduced depth (<20 Km) of the hypocenter in the ISNet data. To approximate the real data the following model has been used:

$$LM(R, \theta) = \log_{10} A(R, \theta) - \alpha \log_{10} R - kR - \beta \tag{7.13}$$

where $\alpha = -1.79$, $\beta = 0.58$, $k = 0$. The values of the parameters have been achieved considering a minimization criteria, according to the elastic structural parameters characterizing the area monitored by the ISNet network. In the light of the previous considerations, the model assumes the form (see [12] for all the details):

$$LM(R, \theta) = \log_{10} A(R, \theta) + 1.79 \log_{10} R - 0.58 \tag{7.14}$$

from which, in case of invertibility, we can write the inverse formulation:

$$A(R, \theta, LM) = 10^{(LM(R,\theta) - 1.79 \log_{10} R + 0.58)},$$

or equivalently in cartesian coordinates:

$$A(R_1, R_2, LM) = 10^{(LM(R_1,R_2) - 1.79 \log_{10} \arctan(R_1/R_2) + 0.58)},$$

where R_1 and R_2 are, respectively, the horizontal and vertical cartesian axis such that $R = \arctan(R_1/R_2)$, $R_1 = R\cos(\theta)$, $R_2 = R\sin(\theta)$. Fixed a value for $LM(R_1, R_2)$, it is possible to calculate $A(R_1, R_2)$ and to approximate it with an exponential sampling operator, defined as:

$$(E^{\varphi}_{(w_1,w_2)} A)(R_1, R_2) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} A(e^{\frac{k_1}{w_1}}, e^{\frac{k_2}{w_2}})\varphi(e^{-k_1} {R_1}^{w_1}, e^{-k_2} {R_2}^{w_2}), \tag{7.15}$$

with R_1, R_2 integers.

Finally, a quantification of the reconstruction absolute mean error AME has been provided introducing the following error estimator

$$AME := \frac{1}{N_1 N_2} \sum_{R_1=0}^{N_1} \sum_{R_2=0}^{N_2} |(E^{\varphi}_{(w_1,w_2)} A)(R_1, R_2) - A(R_1, R_2)|.$$

In the previous expression N_1 and N_2 are the number of points in the two main cartesian axis directions, $N_1 \times N_2$ being the total number of the samples in the grid and $R_1 \in [0, N_1]$, $R_2 \in [0, N_2]$.

Chosen $LM = 2.7$ in the ISNet dataset, we achieve, for bivariate Mellin–Fejer kernels, the numerical values shown in table 1, where only a single row of the approximating matrix is reported for practical reasons. Other rows of the same matrix exhibit the same trend. Using Mellin Splines kernels, we achieve better approximation results (see table 2). In any case, increasing w_1, w_2, N_1, N_2 we observe as AME decreases, whatever kernel being used in the approximation formula.

A	N=15, w=5	N=30, w=10	N=60, w=20
918.4869	2016.2750	1420.2633	1182.1876
907.7103	1545.3763	1213.2350	1064.3751
890.3316	1319.0114	1090.6358	990.5768
867.1510	1169.1624	998.5332	932.5291
839.1579	1046.1715	926.9082	880.6653
807.4335	945.1902	861.7557	832.4209
773.0601	862.8483	803.7728	786.1427
737.0491	794.8578	752.2213	741.7739
700.2923	737.3727	704.8035	698.9664
663.5358	687.5171	660.3791	658.5373
627.3745	643.3295	618.9350	619.9301
592.2586	603.5319	580.7086	583.3547
558.5099	567.3041	545.7103	549.1905
526.3414	534.1130	513.6886	517.2242
495.8780	503.5975	484.2704	487.1994
467.1756	475.4960	457.0936	459.1393
440.2385	449.6028	431.8738	433.1069
415.0331	425.7436	408.4140	408.9969
391.4999	403.7617	386.5841	386.5923
369.5621	383.5117	366.2922	365.7069

TABLE 1. In the first column of the table the values of the 15^{th} row of the real data for $LM = 2.7$. In the following columns, from left to right, the reconstructed data with bivariate Mellin–Fejer kernels with $N = N_1 = N_2, w = w_1 = w_2$ respectively equal to $15, 30, 60$ and $5, 10, 20$.

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A	N=15, w=5	N=30, w=10	N=60, w=20
918.4869	919.7995	919.1974	918.8313
907.7103	908.9871	908.3652	908.0278
890.3316	891.5131	890.9341	890.6223
867.1510	867.3503	867.5869	867.3689
839.1579	838.9015	839.5563	839.3507
807.4335	806.7726	807.6950	807.5622
773.0601	772.5943	772.9821	773.1665
737.0491	736.8876	737.0027	737.0632
700.2923	699.0600	700.4001	700.3419
663.5358	663.8368	663.6755	663.5998
627.3745	626.2857	627.4080	627.3887
592.2586	592.0042	592.1529	592.2225
558.5099	557.9946	558.3477	558.4872
526.3414	525.8575	526.2417	526.3176
495.8780	495.8853	495.8670	495.8693
467.1756	467.2693	467.1553	467.1692
440.2385	440.3887	440.2862	440.2415
415.0331	415.0450	415.0472	415.0398
391.4999	392.0046	391.6462	391.5060
369.5621	370.4046	369.5769	369.5696

TABLE 2. In the first column of the table the values of the 15^{th} row of the real data for $LM = 2.7$. In the following columns, from left to right, the reconstructed data with the bivariate Mellin-Splines $B_{2,2}$ with $N = N_1 = N_2, w = w_1 = w_2$ respectively equal to $15, 30, 60$ and $5, 10, 20$.

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