Dicle Universitesi Ziya Gokalp Egitim Fakultesi Dergisi, 11, 160-172

The Novikov-Veselov(NV) Equation as an example of Invariant Form Equations

Halis YILMAZ

Abstract

In this paper we will consider two scalar differential operators L and M. We call them scalar because their coefficients are functions rather than matrices. Our aim is to obtain some gauge invariant equations. In order to achieve this goal we firstly find the gauge invariants of L and M by using the gauge transformation on L and M. And then we apply 'L-M-f triad' representation $[L, M] + fL = 0$ *which gives us some linear and nonlinear equations such as the NV equation in invariant form.*

Özet

Bu makalede L ve M'yi iki skaler diferansiyel operatör olarak gözönüne alacağız. Burada L ve M skaler operatörlerdir, çünkü bu operatörlerin katsayıları matrisler değil fonksiyonlardır. Bizim amacımız 'gauge invariant' denklemlerini elde etmektir. Bu amaca ulaşmak için ilk önce L ve M üzerinde 'gauge transformasyon' kullanarak L ve M nin 'gauge invariant' larını bulacağız. Daha sonra 'L-M-f triad' denklemine, $[L, M] + fL = 0$ *, başvurarak bazı lineer ve lineer olmayan, örneğin Novikov-Veselov(NV), denklemleri invariant form'da elde edilir.*

1 Introduction

Let L and M be two operator functions such that

$$L = \partial_x \partial_y + a\partial_x + b\partial_y + c \tag{1a}$$

[1]Dr. Halis Yilmaz, Department of Mathematics, University of Glasgow, Glasgow/UK, e-mail: halis@halisyilmaz.com

$$M=\partial_t+\partial_x^3+u\partial_x^2+v\partial_x+w \tag{1b}$$

We will consider 'L-M-f triad' representation [1] :

$$[L,M]+fL=0$$

where f is an operator function to be found.

2 Invariants for L and M

We will use the gauge transformation [2] on L and M in order to get invariants for L and M:

$$g^{-1}Lg=\tilde{L}$$

$$g^{-1}Mg=\tilde{M}$$

2.1 Invariants for $L=\partial_x\partial_y+a\partial_x+b\partial_y+c$

$$g^{-1}Lg=\tilde{L}$$

Therefore,

$$g^{-1}\left(\partial_x\partial_y+a\partial_x+b\partial_y+c\right)g=\partial_x\partial_y+\tilde{a}\partial_x+\tilde{b}\partial_y+\tilde{c}\Rightarrow$$

$$g^{-1}\partial_x(\partial_y.g)+g^{-1}a(\partial_x.g)+g^{-1}b(\partial_y.g)+c=\partial_x\partial_y+\tilde{a}\partial_x+\tilde{b}\partial_y+\tilde{c}\Rightarrow$$

$$g^{-1}\partial_x(g_y+g\partial_y)+g^{-1}a(g_x+g\partial_x)+g^{-1}b(g_y+g\partial_y)+c=\partial_x\partial_y+\tilde{a}\partial_x+\tilde{b}\partial_y+\tilde{c}\Rightarrow$$

$$\partial_x\partial_y+(a+g^{-1}g_y)\partial_x+(b+g^{-1}g_x)\partial_y+c+g^{-1}g_{xy}+ag^{-1}g_x+bg^{-1}g_y$$

$$=\partial_x\partial_y+\tilde{a}\partial_x+\tilde{b}\partial_y+\tilde{c}$$

Therefore, we obtain

$$\tilde{a}=a+g^{-1}g_y \tag{2}$$

$$\tilde{b}=b+g^{-1}g_x \tag{3}$$

$$\tilde{c}=c+g^{-1}g_{xy}+ag^{-1}g_x+bg^{-1}g_y \tag{4}$$

So,

$$\tilde{a} = a + g^{-1}g_y \Rightarrow$$

$$g^{-1}g_y = \tilde{a} - a \tag{5}$$

Similarly from the equation (3), we have

$$g^{-1}g_x = \tilde{b} - b \tag{6}$$

Since $(g^{-1}g_y)_x = (g^{-1}g_x)_y$,

$$\tilde{a}_x - a_x = \tilde{b}_y - b_y \Rightarrow$$

$$\tilde{a}_x - \tilde{b}_y = a_x - b_y$$

Let

$$I = a_x - b_y \tag{7}$$

This is an invariant for the operator function L, where L is defined in (1a). Now, consider the equation (5)

$$g^{-1}g_y = \tilde{a} - a$$

If we take derivative both sides w.r.t. x, then we obtain

$$g^{-1}g_{xy} = (g^{-1}g_y)_x + (g^{-1}g_x)(g^{-1}g_y)$$

By substituting the above expression in the equation (4), we obtain

$$\tilde{c} = c + (g^{-1}g_y)_x + (g^{-1}g_x)(g^{-1}g_y) + ag^{-1}g_x + bg^{-1}g_y$$

Since $g^{-1}g_x = \tilde{b} - b$ and $g^{-1}g_y = \tilde{a} - a$,

$$\tilde{c} = c + (\tilde{a} - a)_x + (\tilde{a} - a)(\tilde{b} - b) + a(\tilde{b} - b) + b(\tilde{a} - a) \Rightarrow$$

$$\tilde{a} + \tilde{a}\tilde{b} - \tilde{c} = a_x + ab - c$$

Let

$$J = a_x + ab - c \tag{8}$$

where J is an invariant for the operator function L.

2.2 Invariants for $M = \partial_t + \partial_x^3 + u\partial_x^2 + v\partial_x + w$

In KP equation, we have already obtained the invariants for M:

$$g^{-1}Mg = \tilde{M} \Rightarrow$$

$$g^{-1}(\partial_t + \partial_x^3 + u\partial_x^2 + v\partial_x + w)g = \partial_t + \partial_x^3 + \tilde{u}\partial_x^2 + \tilde{v}\partial_x + \tilde{w}$$

Therefore, we obtain

$$\tilde{u} = u + 3g^{-1}g_x \tag{9}$$

$$\tilde{v} = v + 2ug^{-1}g_x + 3g^{-1}g_{xx} \tag{10}$$

$$\tilde{w} = w + g^{-1}g_t + vg^{-1}g_x + ug^{-1}g_{xx} + g^{-1}g_{xxx} \tag{11}$$

From the equation (9), we have

$$g^{-1}g_x = \frac{1}{3}(\tilde{u} - u) \tag{12}$$

By doing the exactly same calculation like we did in the KP equation, we obtain

$$P = u_x + \frac{1}{3}u^2 - v \tag{13}$$

$$R = u_t + (u_{xx} - \frac{2}{9}u^3 + uv)_x - 3w_x \tag{14}$$

where P and R are invariants for the operator function M.

Summarise all invariants for L and M:

$$\begin{aligned} I &= a_x - b_y \\ J &= a_x + ab - c \\ P &= u_x + \frac{1}{3}u^2 - v \\ R &= u_t + (u_{xx} - \frac{2}{9}u^3 + uv)_x - 3w_x \end{aligned}$$

3 L-M-f triad representation

$$[L, M] + fL = 0 \tag{15}$$

where $L = \partial_x\partial_y + a\partial_x + b\partial_y + c$, $M = \partial_t + \partial_x^3 + u\partial_x^2 + v\partial_x + w$ and f is an operator function which must be found [3].

First of all we will find $[L, M]$ which will help us to suggest the operator function $f(\partial_x, \partial_y)$.

$$[L, M] = \left[\partial_x\partial_y + a\partial_x + b\partial_y + c, \partial_t + \partial_x^3 + u\partial_x^2 + v\partial_x + w\right]$$

$$= (\partial_x\partial_y + a\partial_x + b\partial_y + c).(\partial_t + \partial_x^3 + u\partial_x^2 + v\partial_x + w)$$
$-(\partial_t + \partial_x^3 + u\partial_x^2 + v\partial_x + w).(\partial_x\partial_y + a\partial_x + b\partial_y + c)$

After doing some operator calculations, we obtain

$[L, M] = -c_t - c_{xxx} + w_{xy} + aw_x + bw_y - uc_{xx} - vc_x$
$+ (w_y + v_{xy} + av_x + bv_y - a_t - a_{xxx} - 3c_{xx} - ua_{xx} - 2uc_x - va_x)\,\partial_x$
$+(u_{xy} + v_y + au_x + bu_y - 3a_{xx} - 3c_x - 2ua_x)\partial_x^2 + (v_x - 2ub_x - 3b_{xx})\partial_x\partial_y$
$+(w_x - b_t - b_{xxx} - ub_{xx} - vb_x)\partial_y + (u_y - 3a_x)\partial_x^3 + (u_x - 3b_x)\partial_x^2\partial_y$

Now,

$$fL = f.(\partial_x\partial_y + a\partial_x + b\partial_y + c)$$

Let us choose $f = m\partial_x + n\partial_y + k$. Then

$$fL = (m\partial_x + n\partial_y + k).(\partial_x\partial_y + a\partial_x + b\partial_y + c)$$

By doing some simple operator calculations on the above equation, we obtain

$fL = mc_x + nc_y + kc + (ma_x + mc + na_y + ka)\partial_x + (mb_x + nb_y + nc + kb)\partial_y$
$+(mb + na + k)\partial_x\partial_y + ma\partial_x^2 + nb\partial_y^2 + m\partial_x^2\partial_y + n\partial_x\partial_y^2$
So if we substitute $[L, M]$ and fL in the equation (15), then we obtain the following expression
$-c_t - c_{xxx} + w_{xy} + aw_x + bw_y - uc_{xx} - vc_x + mc_x + nc_y + kc + (w_y + v_{xy} + av_x + bv_y - a_t - a_{xxx} - 3c_{xx} - ua_{xx} - 2uc_x - va_x + ma_x + mc + na_y + ka)\partial_x + (w_x - b_t - b_{xxx} - ub_{xx} - vb_x + mb_x + nb_y + nc + kb)\partial_y + (u_{xy} + v_y + au_x + bu_y - 3a_{xx} - 3c_x - 2ua_x + ma)\partial_x^2 + (v_x - 2ub_x - 3b_{xx} + mb + na + k)\partial_x\partial_y + (u_y - 3a_x)\partial_x^3 + (u_x - 3b_x + m)\partial_x^2\partial_y + nb\partial_y^2 + n\partial_x\partial_y^2 = 0$

In the above equation, we can easily see that $n = 0$.
Therefore, we obtain the following equations:

$$u_x - 3b_x + m = 0 \quad (16)$$
$$v_x - 2ub_x - 3b_{xx} + mb + k = 0 \quad (17)$$
$$u_y - 3a_x = 0 \quad (18)$$
$$u_{xy} + v_y + au_x + bu_y - 3a_{xx} - 3c_x - 2ua_x + ma = 0 \quad (19)$$
$$w_x - b_t - b_{xxx} - ub_{xx} - vb_x + mb_x + kb = 0 \quad (20)$$
$$w_y + v_{xy} + av_x + bv_y - a_t - a_{xxx} - 3c_{xx} - ua_{xx} - 2uc_x - va_x + h = 0 \quad (21)$$
$$-c_t - c_{xxx} + w_{xy} + aw_x + bw_y - uc_{xx} - vc_x + mc_x + kc = 0 \quad (22)$$

where $h = ma_x + mc + ka$ for simplicity. So, from the equation (16) we have

$$m = 3b_x - u_x$$

and the equation (17) gives us

$$k = -v_x + 2ub_x + 3b_{xx} - mb$$

Since $m = 3b_x - u_x$,

$$k = -v_x + 2ub_x + 3b_{xx} - 3bb_x - bu_x$$

Therefore,

$$f = m\partial_x + n\partial_y + k \Rightarrow$$

$$f = (3b_x - u_x)\partial_x - v_x + 2ub_x + 3b_{xx} - 3bb_x + bu_x \qquad (23)$$

From the equation (18) we have

$$u_y = 3a_x$$

So,
if we substitute $u_{xy} = 3a_{xx}$ and $m = 3b_x - u_x$ in the equation (19), we obtain

$$3ab_x - 2ua_x + bu_y - 3c_x + v_y = 0$$

Similarly by substituting $m = 3b_x - u_x$ and $k = -v_x + 2ub_x + 3b_{xx} - 3bb_x - bu_x$ in the equation (20), we obtain

$$-b_t - b_{xxx} - ub_{xx} - u_x b_x - b_x v - bv_x + 3b_x^2 + 3bb_{xx} - 3b^2 b_x + b^2 u_x + 2ubb_x + w_x = 0$$

In the equation (21) we know that $h = ma_x + mc + ka$.
Since $m = 3b_x - u_x$ and $k = -v_x + 2ub_x + 3b_{xx} - 3bb_x - bu_x$,
the equation (21) becomes
$-a_t - a_{xxx} - ua_{xx} - u_x a_x + 3ab_{xx} + 3a_x b_x - 3abb_x + abu_x$
$+2uab_x - 3c_{xx} - 2uc_x - u_x c + 3b_x c + v_{xy} - va_x + bv_y + w_y = 0$

Finally, if we substitute the same m and k in the equation (22), we have
$-c_t - c_{xxx} - uc_{xx} - u_x c_x + 3b_{xx}c + 3b_x c_x - v_x c - vc_x + 2ucb_x$
$+bcu_x - 3bcb_x + w_{xy} + aw_x + bw_y = 0$
Hence,

$$[L, M] + fL = 0, \ (15)$$

where $L = \partial_x\partial_y + a\partial_x + b\partial_y + c$, $M = \partial_t + \partial_x^3 + u\partial_x^2 + v\partial_x + w$,
and $f = (3b_x - u_x)\partial_x - v_x + 2ub_x + 3b_{xx} - 3bb_x + bu_x$ (23), gives the following equations

$$u_y - 3a_x = 0 \,(24)$$

$$3ab_x - 2ua_x + bu_y - 3c_x + v_y = 0 \,(25)$$

$$-b_t - b_{xxx} - ub_{xx} - u_x b_x - b_x v - bv_x + 3b_x^2 + 3bb_{xx} - 3b^2 b_x + b^2 u_x$$
$$+2ubb_x + w_x = 0 \,(26)$$

$$-a_t - a_{xxx} - ua_{xx} - u_x a_x + 3ab_{xx} + 3a_x b_x - 3abb_x + abu_x + 2uab_x$$
$$-3c_{xx} - 2uc_x - u_x c + 3b_x c + v_{xy} - va_x + bv_y + w_y = 0 \,(27)$$

$$-c_t - c_{xxx} - uc_{xx} - u_x c_x + 3b_{xx} c + 3b_x c_x - v_x c - vc_x + 2ucb_x$$
$$+bcu_x - 3bcb_x + w_{xy} + aw_x + bw_y = 0 \,(28)$$

So, the equation (24) gives us

$$u_y = 3a_x \Rightarrow$$

$$u = 3\int a_x dy + \mu(x,t)$$

where μ is an arbitrary function of x and t .
Or

$$u = 3b + 3\int I dy + \mu(x,t) \qquad (29)$$

since $a_x = I + b_y$.

$$3ab_x - 2ua_x + bu_y - 3c_x + v_y = 0 \quad (25)$$

From the equation (8) and the equation (13), we have

$$c = a_x + ab - J \qquad (30)$$
$$v = u_x + \frac{1}{3}u^2 - P \qquad (31)$$

respectively.

By substituting c and v in the equation (25), we obtain

$$-2ua_x + bu_y - 3a_{xx} - 3a_x b + 3J_x + u_{xy} + \frac{2}{3}uu_y - P_y = 0$$

Since $u_y = 3a_x$, the above equation becomes

$$P_y = 3J_x \tag{32}$$

this is one of the invariant equations obtained from the equation (15).
Now, let us consider the equation (26) which can be written as the following form

$$-b_t + (-b_{xx} - ub_x - bv + 3bb_x - b^3 + b^2u)_x + w_x = 0$$

From the equation (14), we have

$$w_x = \frac{1}{3}u_t + \frac{1}{3}(u_{xx} + uu_x + \frac{1}{9}u^3 - uP)_x - \frac{1}{3}R \tag{33}$$

If we substitute w_x and $v = u_x + \frac{1}{3}u^2 - P$ in the equation (26), then we obtain

$$\frac{1}{3}(u-3b)_t - \frac{1}{3}R + \left[\frac{1}{3}(u-3b)_{xx} + \frac{1}{3}(u-3b)(u-3b)_x - \frac{1}{3}(u-3b)P + \frac{1}{27}(u-3b)^3\right]_x = 0$$

since $u^3 - 27b^3 = (u-3b)^3 + 9ub(u-3b)$.
From the equation (29), we have

$$u - 3b = 3\int I dy + \mu(x,t)$$

By choosing $\mu = 0$ and then substituting

$$u - 3b = 3\int I dy \tag{34}$$

in the above equation, we obtain

$$\int I_t dy - \frac{1}{3}R + \left[\int I_{xx}dy + 3\int I dy \int I_x dy - P\int I dy + (\int I dy)^3\right]_x = 0 \tag{35}$$

which is an invariant equation.
Similarly, consider the rewritten form of the equation (27) which is
$-a_t + (-a_{xx} - ua_x + 3ab_x - 3c_x + v_y)_x + 3(c - ab)b_x + (ab - c)u_x$
$+2u(ab_x - c_x) - a_x v + bv_y + w_y = 0$
From the equations (30) and (31), we have

$$\begin{aligned} c &= a_x + ab - J \\ v &= u_x + \frac{1}{3}u^2 - P \end{aligned}$$

Therefore,

$$\begin{aligned} c_x &= a_{xx} + a_x b + ab_x - J_x \\ v_y &= 3a_{xx} + 2ua_x - P_y \end{aligned}$$

since $u_y = 3a_x$.
In the equation (33), if we integrate both sides w.r.t. x, then we have

$$w = \frac{1}{3}\int u_t dx + \frac{1}{3}(u_{xx} + uu_x + \frac{1}{9}u^3 - uP) - \frac{1}{3}\int R dx + \beta(y,t)$$

where β is an arbitrary function of y and t.
By taking derivative both sides w.r.t. y, we obtain

$$w_y = a_t + a_{xxx} + a_x u_x + ua_{xx} + \frac{1}{3}u^2 a_x - a_x P - \frac{1}{3}uP_y - \frac{1}{3}\int R_y dx + \beta_y \quad (36)$$

Hence, if we substitute c, c_x, v_y and w_y in the equation (27), we obtain

$$3J_{xx} - P_{xy} - 3b_x J + u_x J + 2uJ_x - bP_y - \frac{1}{3}uP_y - \frac{1}{3}\int R_y dx + \beta_y = 0$$

The above equation implies

$$3J_{xx} - P_{xy} - \frac{1}{3}\int R_y dx + (u - 3b)_x J + 2uJ_x - (b + \frac{1}{3}u)P_y + \beta_y = 0$$

By substituting the equation (32), $P_y = 3J_x$, in the last equation above, gives

$$\frac{1}{3}\int R_y dx = [(u - 3b)J]_x + \beta_y = 0$$

Since $u - 3b = 3\int I dy$ (34),

$$\int R_y dx = 9[J\int I dy]_x + 3\beta_y$$

By taking derivative w.r.t x, we have

$$R_y = 9\left[J\int I dy\right]_{xx} \tag{37}$$

which is the third invariant equation.
Finally, we consider the equation (28) which can be rewritten as the following form

$$-c_t+(-c_{xx}-uc_x+3b_xc-vc+w_y)_x+2ub_xc+bu_xc-3bb_xc+aw_x+bw_y = 0$$

From the equations (30), (31), (33), and (36), we have

$$\begin{aligned} c &= a_x + ab - J \\ v &= u_x + \frac{1}{3}u^2 - P \\ w_x &= \frac{1}{3}u_t + \frac{1}{3}(u_{xx} + uu_x + \frac{1}{9}u^3 - uP)_x - \frac{1}{3}R \\ w_y &= a_t + a_{xxx} + a_xu_x + ua_{xx} + \frac{1}{3}u^2a_x - a_xP - \frac{1}{3}uP_y - \frac{1}{3}\int R_y dx + \beta_y \end{aligned}$$

In the equation (30), if we take derivative w.r.t. t, x, xx, then we obtain

$$\begin{aligned} c_t &= a_{xt} + a_tb + ab_t - J_t \\ c_x &= a_{xx} + a_xb + ab_x - J_x \\ c_{xx} &= a_{xxx} + a_{xx}b + 2a_xb_x + ab_{xx} - J_{xx} \end{aligned}$$

respectively.

By substituting c, c_t, c_x, c_{xx}, v, w_x, w_y in the equation (28), we obtain
$J_t+J_{xxx}-(JP)_x-\frac{1}{3}R_y-\frac{1}{3}aR-\frac{1}{3}b\int R_ydx+[(u-3b)_xJ]_x+\frac{2}{3}u(u-3b)_xJ$
$-b(u-3b)_xJ+\frac{1}{3}u(u-3b)J_x-\frac{1}{3}a(u-3b)_xP-\frac{1}{3}a(u-3b)P_x-\frac{1}{3}a(u-3b)_t$
$+\frac{1}{3}a(u-3b)_{xxx}+\frac{1}{3}au(u-3b)_{xx}-ab(u-3b)_{xx}-ab_x(u-3b)_x+\frac{1}{3}au_x(u-3b)_x$
$-\frac{2}{3}abu(u-3b)_x+ab^2(u-3b)_x+\frac{1}{9}au^2(u-3b)_x+\beta_yb = 0$

since $P_y = 3J_x$ (32).
If we substitute the equation (34)

$$u - 3b = 3\int I dy$$

in the above equation, then we obtain
$J_t + J_{xxx} - (JP)_x - \frac{1}{3}R_y + 3(J\int I_x dy)_x - \frac{1}{3}b\int R_y dx + u(J\int I dy)_x +$
$3J\int I dy\int I_x dy \qquad a[-\frac{1}{3}R - (P\int I dy)_x + \int I_t dy + \int I_{xxx} dy + 3\int I dy\int I_{xx} dy$
$+3(\int I_x dy)^2 + 3(\int I dy)^2\int I_x dy] + \beta_y b = 0$
By substituting the equation (37)

$$R_y = 9\left[J\int I dy\right]_{xx}$$

in the above equation, we have
$J_t + J_{xxx} - (JP)_x - 3\left[J\int I dy\right]_{xx} + 3(J\int I_x dy)_x + 3\left[J(\int I dy)^2\right]_x$
$a\left\{\int I_t dy - \frac{1}{3}R + \left[\int I_{xx} dy + 3\int I dy\int I_x dy + (\int I dy)^3\right]\right\} = 0$
since $u - 3b = 3\int I dy$.
We recall the equation (35) which is

$$\int I_t dy - \frac{1}{3}R + \left[\int I_{xx} dy + 3\int I dy\int I_x dy - P\int I dy + (\int I dy)^3\right]_x = 0$$

Therefore, the above equation becomes

$$J_t + J_{xxx} - (JP)_x - 3\left[J\int I dy\right]_{xx} + 3(J\int I_x dy)_x + 3\left[J(\int I dy)^2\right]_x = 0$$

Hence, we obtain the following invariant equation

$$J_t + J_{xxx} - (JP)_x - 3\left[J_x\int I dy\right]_x + 3\left[J(\int I dy)^2\right]_x = 0 \qquad (38)$$

4 Conclusion

Summarise all invariant equations:
$P_y = 3J_x$
$\int I_t dy - \frac{1}{3}R + \left[\int I_{xx} dy + 3\int I dy\int I_x dy - P\int I dy + (\int I dy)^3\right]_x = 0$
$R_y = 9\left[J\int I dy\right]_{xx}$

$J_t + J_{xxx} - (JP)_x - 3\left[J_x \int I dy\right]_x + 3\left[J(\int I dy)^2\right]_x = 0$
If we put $I = 0$ in the equation (38), then we obtain

$$J_t + J_{xxx} - (JP)_x = 0$$

which is the special case of the Novikov-Veselov(NV) equation, where $P_y = 3J_x$.

References

[1] B G Konopelchenko, The two dimensioanal second-order differential spectral problem: compatibility conditions, general Bts and integrable equations, Invers problems 4 (1988) 151-163

[2] C.Athorne, A $\mathbb{Z}^2 \times \mathbb{R}^3$ Toda system, Phys. Lett.A 206 (1995) 162-166

[3] S.P.Novikov and A.P.Veselov, Two-Dimensional Schrodinger operator: Inverse Scattering Transform and Evolutional Equations, Physica 18D (1986) 267-273