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# Improvement in Exponential Estimators of Population Mean using Information on Auxiliary Attribute

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**Abstract:** This paper proposes new exponential estimators combining ratio estimators for estimate population mean of study variable using information about population proportion possessing certain attributes. It is obtained mean square error (MSE) equations for all proposed ratio exponential estimators and is shown that all proposed exponential estimators are always more efficient than the ratio estimators. In addition, these results are supported by an application with original data sets.

# Yardımcı Niteliğe İlişkin Bilgi Kullanılarak Kitle Ortalamasının Üstel Tahmin Edicilerindeki Gelişme

**Anahtar Kelimeler**
Üstel Tamin Ediciler,
Basit Ragele Örnekleme,
Hata Kareler Ortalama
Yardımcı Özellik
Etkinlik

**Öz:** Bu makale, belli özelliğe sahip olan kitle oranı hakkındaki bilgiyi kullanarak çalışma değişkeninin ortalama tahmini için önerilen tahmin edicileri birleştiren yeni üstel tahminciler önermektedir. Önerilen tüm üstel tahmin ediciler için hata kareler ortalaması (HKO) denklemleri elde edilmiştir ve önerilen bütün üstel tahmin edicilerin, oran tahmincilerinden her zaman daha verimli olduğu gösterilmiştir. Ek olarak, bu sonuçlar orijinal veri setleri içeren bir uygulama tarafından desteklenmektedir.

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## 1. Introduction

There are many situation when auxiliary information is available in the form of attributes. For example sex is a good auxiliary attribute while dealing with height, and the breed of a cow is a good auxiliary attribute while estimating milk production [1], crop variety is used as an auxiliary attribute in estimating the yield of wheat [2], etc. There are some recent studies on the estimators using the information of the auxiliary attribute in Literature, such as, Shabbir and Gupta [3], Koyuncu [4], Malik and Singh [5], Zaman [6, 7] and Zaman and Kadilar [8]

The Naik and Gupta [1] estimator for the population mean $\bar{Y}$ of the variate of study, which make use of information regarding the population proporiton posseessing certain attribute, is defined by

$$\bar{y}_{NG} = \frac{\bar{y}}{p} P \tag{1.1}$$

Let $y_i$ be $i$th characteristic of the population and $\phi_i$ is the case of possessing certain attributes. If $i$th unit has the desired characteristic, it takes the value 1, if not then the value 0. That is;

$$\phi_i = \begin{cases} 1 \quad , & if\ ith\ unit\ of\ the\ population\ possesses\ attribute \\ 0 \quad , & otherwise \end{cases}$$

where $\bar{y}$ the sample mean of the study variable and $a = \sum_{i=1}^{n} \phi_i$ be the the total count of the units that possess certain attribute sample. $p = \frac{a}{n}$ shows the ratio of these units and it is assumed that the population proportion $P$ of the form of attribute $\phi$ is known.

The MSE of the Naik and Gupta [1] estimator is

$$MSE(\bar{y}_{NG}) \cong \frac{1-f}{n}\bar{Y}^2\left(C_y^2 - 2\rho_{pb}C_yC_p + C_p^2\right) \tag{1.2}$$

where, $f = \frac{n}{N}$; N is the number of units in the population; $C_p$ is the population coefficient of variation the form of attribute and $C_y$ is the population coefficient of variation of the study variable.

Following Bahl and Tuteja [9] , Zaman and Kadilar [10] proposed ratio exponential estimators in order to estimate population mean of study variable $y$, using information about population proportion possessing certain attributes in simple random sampling;

$$t_{ZK} = \bar{y}exp\left[\frac{(kP+l)-(kp+l)}{(kP+l)+(kp+l)}\right] \tag{1.3}$$

where $k(\neq 0)$, $l$ are either real number or the functions of the known parameters of the attribute such as $C_p$, $\beta_2(\phi)$ ve $\rho_{pb}$. The following table presents estimators of the population mean which can be obtained by suitable choice of constants $k$ and $l$

**Table 1.** The Proposed Estimators by Zaman and Kadilar [10]

| Estimators | Values of $k$ | $l$ |
|---|---|---|
| $\boldsymbol{t_{ZK1} = \bar{y}exp\left(\frac{P-p}{P+p}\right)}$ Singh et al. (2007) estimator | 1 | 0 |
| $\boldsymbol{t_{ZK2} = \bar{y}exp\left(\frac{P-p}{P+p+2\beta_2(\phi)}\right)}$ | 1 | $\beta_2(\phi)$ |
| $\boldsymbol{t_{ZK3} = \bar{y}exp\left(\frac{P-p}{P+p+2C_p}\right)}$ | 1 | $C_p$ |
| $\boldsymbol{t_{ZK4} = \bar{y}exp\left(\frac{P-p}{P+p+2\rho_{pb}}\right)}$ | 1 | $\rho_{pb}$ |
| $\boldsymbol{t_{ZK5} = \bar{y}exp\left[\frac{\beta_2(\phi)(P-p)}{\beta_2(\phi)(P+p)+2C_p}\right]}$ | $\beta_2(\phi)$ | $C_p$ |
| $\boldsymbol{t_{ZK6} = \bar{y}exp\left[\frac{C_p(P-p)}{C_p(P+p)+2\beta_2(\phi)}\right]}$ | $C_p$ | $\beta_2(\phi)$ |
| $\boldsymbol{t_{ZK7} = \bar{y}exp\left[\frac{C_p(P-p)}{C_p(P+p)+2\rho_{pb}}\right]}$ | $C_p$ | $\rho_{pb}$ |
| $\boldsymbol{t_{ZK8} = \bar{y}exp\left[\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2C_p}\right]}$ | $\rho_{pb}$ | $C_p$ |
| $\boldsymbol{t_{ZK9} = \bar{y}exp\left[\frac{\beta_2(\phi)(P-p)}{\beta_2(\phi)(P+p)+2\rho_{pb}}\right]}$ | $\beta_2(\phi)$ | $\rho_{pb}$ |
| $\boldsymbol{t_{ZK10} = \bar{y}exp\left[\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2\beta_2(\phi)}\right]}$ | $\rho_{pb}$ | $\beta_2(\phi)$ |

In Table 1, $C_p$, $\beta_2(\phi)$ and $\rho_{pb}$ are, respectively, coefficient of variation belonging to ratio of units possessing certain attributes, coefficient of population kurtosis and population correlation coefficient between ratio of units possessing certain attributes and study variable. $\bar{y}$ and $p$ are sample mean belonging to study variable and sample proportion possessing certain attributes, respectively.

The MSE and bias of this ratio estimator is as follows;

$$B(t_{ZKi}) \cong \frac{1-f}{n}\bar{Y}\left(\lambda_i^2C_p^2 - \lambda_i\rho_{pb}C_yC_p\right) \tag{1.4}$$

$$MSE(t_{ZKi}) \cong \frac{1-f}{n}\bar{Y}^2\left[\lambda_i^2C_p^2 - 2\lambda_i\rho_{pb}C_yC_p + C_y^2\right] \quad, i = 2, \dots, 10 \tag{1.5}$$

where, $\lambda_1 = \frac{1}{2}$; $\lambda_2 = \frac{P}{2(P+\beta_2(\phi))}$; $\lambda_3 = \frac{P}{2(P+C_p)}$; $\lambda_4 = \frac{P}{2(P+\rho_{pb})}$; $\lambda_5 = \frac{\beta_2(\phi)P}{2(\beta_2(\phi)P+C_p)}$;

$\lambda_6 = \frac{C_p P}{2(C_p P + \beta_2(\phi))}$; $\lambda_7 = \frac{C_p P}{2(C_p P + \rho_{pb})}$; $\lambda_8 = \frac{\rho_{pb} P}{2(\rho_{pb} P + C_p)}$; $\lambda_9 = \frac{\beta_2(\phi) P}{2(\beta_2(\phi) P + \rho_{pb})}$; $\lambda_{10} = \frac{\rho_{pb} P}{2(\rho_{pb} P + \beta_2(\phi))}$

## 2. Suggested Estimators

Following Kadilar and Cingi [11], it is proposed the exponential estimators combining ratio exponential estimators $tZK_1$ and $t_{ZKi}$ $(i = 2,3,\dots,10)$ as follows;

$$t_{ZKi}^* = \omega t_1 + (1-\omega) z t_i \;;\; (i = 2,3,\dots,10) \tag{2.1}$$

where $\omega$ is a real constant to be determined such that the MSE of $t_{ZKi}^*$ is minimum.

It is obtained the MSE and bias equations for these proposed estimators using Taylor series as; (for details, please see the Appendix A)

$$MSE(t_{ZKi}^*) \cong \frac{1-f}{n}\bar{Y}^2\left[C_p^2\left(\frac{\omega}{2} + \lambda_i - \omega\lambda_i\right)^2 - 2\rho_{pb}C_yC_p\left(\frac{\omega}{2} + \lambda_i - \omega\lambda_i\right) + C_y^2\right] \tag{2.2}$$

where, $\lambda_2 = \frac{P}{2(P+\beta_2(\phi))}$; $\lambda_3 = \frac{P}{2(P+C_p)}$; $\lambda_4 = \frac{P}{2(P+\rho_{pb})}$; $\lambda_5 = \frac{\beta_2(\phi)P}{2(\beta_2(\phi)P + C_p)}$; $\lambda_6 = \frac{C_p P}{2(C_p P + \beta_2(\phi))}$

$\lambda_7 = \frac{C_p P}{2(C_p P + \rho_{pb})}$; $\lambda_8 = \frac{\rho_{pb} P}{2(\rho_{pb} P + C_p)}$; $\lambda_9 = \frac{\beta_2(\phi) P}{2(\beta_2(\phi) P + \rho_{pb})}$; $\lambda_{10} = \frac{\rho_{pb} P}{2(\rho_{pb} P + \beta_2(\phi))}$

We can have the optimal values of $\omega$ (2.2) by following equations: (for details, please see the Appendix B).

$$\omega_{opt} = \frac{2\left(\rho_{pb}\frac{C_y}{C_p} - \lambda_i\right)}{(1-2\lambda_i)} \tag{2.3}$$

It is obtained minimum MSE of the proposed estimators using the optimal equations of $\omega$ in (2.3). All proposed estimators have the same minimum MSE as follows:

$$MSE_{min}(t_{ZKi}^*) \cong \frac{1-f}{n}\bar{Y}^2\left[C_y^2\left(1-\rho_{pb}^2\right)\right] \;;\; i = 2,3,\dots,10 \tag{2.4}$$

## 3. Efficiency Comparisons

In this section, it is compared the MSE of the proposed exponential estimators in (2.4) with the MSE of the Naik-Gupta [1] estimator, the ratio exponential estimator suggested by Singh et al. [12] and ratio estimators listed in Table1.

Comparing the MSE of the proposed estimators, given in (2.1), with the ratio estimator suggested by Naik-Gupta [1], given in (1.1), we have the following conditions;

$$MSE(t_{ZKi}^*) < MSE(\bar{y}_{NG}) \quad i = 2,3,\dots,10$$

$$\frac{1-f}{n}\bar{Y}^2\left[C_y^2\left(1-\rho_{pb}^2\right)\right] < \frac{1-f}{n}\bar{Y}^2\left(C_y^2 - 2\rho_{pb}C_yC_p + C_p^2\right)$$

$$\left(\rho_{pb}C_y - C_p\right)^2 > 0 \tag{3.1}$$

When the conditions (3.1) is satisfied, the proposed exponential estimators are more efficient than the ratio estimator suggested by Naik-Gupta [1].

Comparing the MSE of the proposed exponential estimators, given in (2.1), with the MSE of the ratio exponential estimator suggested by Singh et al. [12], given in (1.3), we have the following conditions;

$$MSE_{min}(t_{ZKi}^*) < MSE(t_{ZK1})$$

$$\frac{1-f}{n}\bar{Y}^2\left[C_y^2\left(1-\rho_{pb}{}^2\right)\right] < \frac{1-f}{n}\bar{Y}^2\left[\frac{C_p^2}{4}-\rho_{pb}C_yC_p+C_y^2\right]$$

$$\left(\rho_{pb}C_y-\frac{C_p}{2}\right)^2 > 0 \qquad (3.2)$$

When the conditions (3.2) is satisfied, the proposed exponential estimators are more efficient than the ratio estimator suggested by Singh et al. [12].

Comparing the MSE of the proposed exponential estimators, given in (2.1), with the MSE of the ratio exponential estimator suggested by Zaman and Kadilar [10], given in Table 1, we have the following conditions;

$$MSE_{min}(t_{ZKi}^*) < MSE(t_{ZKi})$$

$$\frac{1-f}{n}\bar{Y}^2\left[C_y^2\left(1-\rho_{pb}{}^2\right)\right] < \frac{1-f}{n}\bar{Y}^2\left[\lambda_i^2C_p^2-2\lambda_i\rho_{pb}C_yC_p+C_y^2\right]$$

$$\left(\lambda_iC_p-\rho_{pb}C_y\right)^2 > 0 \qquad (3.3)$$

It is inferred that all proposed exponential estimators are more efficient that all ratio estimators in given in Table 1 in all conditions, because the condition given in (3.3) is always satisfied.

## 4. Numerical Illustration

It is used the teacher and wdbc data sets to calculate efficiency of estimators which are given in Table 2 and Table 3. In this section, we use the data set in Zaman et al. [13] in order to compare the efficiencies between the proposed estimators, given in (2.1), with the ratio estimators, given in Section 1, based on MSE equations. The MSE of these estimators are computed as given in (1.2), (1.5) and (2.3) and these estimators are compared to each other with respect to their MSE values.
The data is defined as following;

$$y = the\ number\ of\ teachers$$

$$\phi_i = \begin{cases} 1 & , \quad if\ the\ number\ of\ teachers\ is\ more\ than\ 60 \\ 0 & , \quad otherwise \end{cases}$$

**Table 2.** Population 1 Data Statistics

| | | | | |
|---|---|---|---|---|
| N:111 | $\bar{Y}$: 29.279 | $\lambda_2$:0.0146 | $\lambda_6$:0.0382 | $\lambda_{10}$:0.0117 |
| n: 30 | $P$: 0.117 | $\lambda_3$:0.0203 | $\lambda_7$:0.1441 | |
| $\beta_2(\phi)$: 3.898 | $C_y$: 0.872 | $\lambda_4$:0.0640 | $\lambda_8$:0.0164 | |
| $\rho_{pb}$: 0.797 | $C_p$: 2.758 | $\lambda_5$:0.0709 | $\lambda_9$:0.1819 | |

As second example, the data for the empirical study is taken from population data set considered by Sukhatme [14]

The data is defined as following;

$$y = Number\ of\ villages\ in\ the\ circles$$

$$\phi_i = \begin{cases} 1 & , \quad if\ A\ circle\ consisting\ more\ than\ five\ vilages \\ 0 & , \quad otherwise \end{cases}$$

**Table 3.** Population 2 Data Statistics

| | | | | |
|---|---|---|---|---|
| N:89 | $\bar{Y}$: 3.3596 | $\lambda_2$:0.0171 | $\lambda_6$:0.0433 | $\lambda_{10}$:0.0132 |

| | | | |
|---|---|---|---|
| n: 20 | $P$: 0.1236 | $\lambda_3$:0.0221 | $\lambda_7$:0.1508 |
| $\beta_2(\phi)$: 3.4917 | $C_y$: 0.6008 | $\lambda_4$:0.0695 | $\lambda_8$:0.0171 |
| $\rho_{pb}$: 0.766 | $C_p$: 2.6779 | $\lambda_5$:0.0694 | $\lambda_9$:0.1802 |

In Tables 2 and 3, it is observed the statistics about the populations. Note that the sample sizes as $n = 30, n = 20$ and use simple random sampling [15]. We would like to recall that sample size has no effect on efficiency comparisons of estimators, as shown in Section 3.

**Tablo4.** MSE values of the Ratio Estimators

| | MSE | |
|---|---|---|
| **Estimator** | **Population 1** | **Population 2** |
| $t_{NG}$ | 94.532 | 2.2168 |
| $t_{ZK1}$ | 15.5403 | 0.4030 |
| $t_{ZK2}$ | 14.7247 | 0.1404 |
| $t_{ZK3}$ | 14.2948 | 0.1356 |
| $t_{ZK4}$ | 11.3891 | 0.0981 |
| $t_{ZK5}$ | 10.9827 | 0.0981 |
| $t_{ZK6}$ | 13.0318 | 0.1171 |
| $t_{ZK7}$ | 7.6304 | 0.0666 |
| $t_{ZK8}$ | 14.5909 | 0.1404 |
| $t_{ZK9}$ | 6.5614 | 0.0654 |
| $t_{ZK10}$ | 14.9436 | 0.1442 |
| **Proposed** | **5.7840** | **0.0652** |

In Table 4, values of MSE, which are computed using equations presented in Sections 1 and 2, are given. When we examine Table 4, it is observed that the proposed exponential estimators have the smallest MSE value among all ratio estimators given Section 1. This is an expected results, as mentioned in Section 3.

From the result of this numerical illustration, it is deduced that all proposed exponential estimators are more efficient than all ratio estimators for this data set.

## 5. Conclusions

It is developed new exponential estimators combining ratio estimators considered is Section 1 using information about population proportion possessing certain attributes in simple random sampling and obtained minimum MSE equation for proposed estimators. Theoretically, It is demonstrated that all proposed exponential estimators are always more efficient than all ratio estimators given Section 1. These theoretical results are supported by an application with original data sets.

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## Appendices

### Appendix A.

In general, Taylor series method for k variables can be given as;

$$h(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k) = h(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_k) + \sum_{j=1}^{k} d_j(\bar{x}_j - \bar{X}_j) + R_k(\bar{X}_k, \alpha) + O_k$$

where

$$d_j = \frac{\partial h(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k)}{\partial \alpha_j}$$

and

$$R_k(\bar{X}_k, \alpha) = \sum_{j=1}^{k}\sum_{j=1}^{k} \frac{1}{2!} \frac{\partial^2 h(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_k)}{\partial \bar{X}_i \bar{X}_j} (\bar{x}_j - \bar{X}_j)(\bar{x}_i - \bar{X}_i) + O_k$$

where $O_k$ represents the terms in the expansion of the Taylor series of more than the second degree [9]. When it is omitted the term $R_k(\bar{X}_k, \alpha)$, we obtain Taylor series method for two variables as follows;

$$h(p, \bar{y}) - h(P, \bar{Y}) \cong \frac{\partial h(c,d)}{\partial c}\bigg|_{P,\bar{Y}} (p - P) + \frac{\partial h(c,d)}{\partial d}\bigg|_{\bar{Y},P} (\bar{y} - \bar{Y})$$

where, $h(p,\bar{y}) = t_{ZKi}^{*}$ and $h(P,\bar{Y}) = \bar{Y}$
MSE equations of the proposed estimators given in (2.1) compute as follows:

$$t_{ZKi}^{*} - \bar{Y} \cong \left.\frac{\partial\left(\omega\bar{y}exp\left[\frac{P-p}{P+p}\right] + (1-\omega)\bar{y}exp\left[\frac{(kP+l)-(kp+l)}{(kP+l)+(kp+l)}\right]\right)}{\partial p}\right|_{P,\bar{Y}} (p-P)$$

$$+ \left.\frac{\partial\left(\omega\bar{y}exp\left[\frac{P-p}{P+p}\right] + (1-\omega)\bar{y}exp\left[\frac{(kP+l)-(kp+l)}{(kP+l)+(kp+l)}\right]\right)}{\partial y}\right|_{P,\bar{Y}} (\bar{y}-\bar{Y})$$

$$\cong \left(\frac{-\omega\bar{Y}}{2P} + (1-\omega)\left(\frac{-\bar{Y}k}{2kP+2l}\right)\right)(p-P) + (\bar{y}-\bar{Y})$$

$$E(t_{ZKi}^{*} - \bar{Y})^2 \cong \left[\left(\frac{\omega^2\bar{Y}^2}{4P^2} + (1-\omega)^2\left(\frac{\bar{Y}^2k^2}{(2kP+2l)^2}\right) + 2\left(\frac{-\omega\bar{Y}}{2P}\right)(1-\omega)\left(\frac{-\bar{Y}k}{2kP+2l}\right)\right)V(p) - 2\left(\frac{\omega\bar{Y}}{2P} + \frac{\bar{Y}k}{2kP+2l} - \frac{\omega\bar{Y}k}{2kP+2l}\right)Cov(p,\bar{y}) + V(\bar{y})\right]$$

$$\cong \bar{Y}^2\left[\left(\frac{\omega^2}{4P^2} + (1-\omega)^2\left(\frac{k^2}{(2kP+2l)^2}\right) + \frac{2\omega(1-\omega)k}{2P(2kP+2l)}\right)V(p) - \frac{2}{\bar{Y}}\left(\frac{\omega}{2P} + \frac{k}{2kP+2l} - \frac{\omega k}{2kP+2l}\right)Cov(p,\bar{y}) + \frac{V(\bar{y})}{\bar{Y}^2}\right]$$

$$\cong \frac{1-f}{n}\bar{Y}^2\left[\left(\frac{\omega^2}{4} + (1-\omega)^2\lambda_i^2 + \omega(1-\omega)\lambda_i\right)C_p^2 - 2\rho_{pb}C_yC_p\left(\frac{\omega}{2} + \lambda_i - \omega\lambda_i\right) + C_y^2\right]$$

$$MSE(t_{ZKi}^{*}) \cong \frac{1-f}{n}\bar{Y}^2\left[\left(\frac{\omega}{2} + \lambda_i - \omega\lambda_i\right)^2 C_p^2 - 2\rho_{pb}C_yC_p\left(\frac{\omega}{2} + \lambda_i - \omega\lambda_i\right) + C_y^2\right]; i = 2,3,\ldots,10 \quad (A.1)$$

**Appendix B**

It has the optimal values of $\alpha$ by following equations:

$$\frac{\partial MSE(t_{ZKi}^{*})}{\partial\omega} = \frac{1-f}{n}\bar{Y}^2\left[2\left(\frac{\omega}{2} + \lambda_i - \omega\lambda_i\right)\left(\frac{1}{2} - \lambda_i\right)C_p^2 - 2\rho_{pb}C_yC_p\left(\frac{1}{2} - \lambda_i\right)\right] = 0$$

$$\left(\frac{\omega}{2} + \lambda_i - \omega\lambda_i\right)\left(\frac{1}{2} - \lambda_i\right)C_p^2 = \rho_{pb}C_yC_p\left(\frac{1}{2} - \lambda_i\right)$$

$$\frac{\omega}{2} + \lambda_i - \omega\lambda_i = \rho_{pb}\frac{C_y}{C_p}$$

$$\omega_{opt} = \frac{2\left(\rho_{pb}\frac{C_y}{C_p} - \lambda_i\right)}{(1-2\lambda_i)} \quad (B.1)$$

It is obtained minimum MSE of the proposed estimators using the optimal equations of $\omega_{opt}$ in $(B.1)$.

$$MSE_{min}(t_{ZKi}^{*}) \cong \frac{1-f}{n}\bar{Y}^2\left[\left(\frac{\omega_{opt}}{2} + \lambda_i - \omega_{opt}\lambda_i\right)^2 C_p^2 - 2\rho_{pb}C_yC_p\left(\frac{\omega_{opt}}{2} + \lambda_i - \omega_{opt}\lambda_i\right) + C_y^2\right] \quad (B.2)$$

$$\frac{\omega_{opt}}{2} + \lambda_i - \omega_{opt}\lambda_i = \omega_{opt}\left(\frac{1}{2} - \lambda_i\right) + \lambda_i = \frac{\rho_{pb}\frac{C_y}{C_p} - \lambda_i - 2\lambda_i\rho_{pb}\frac{C_y}{C_p} + 2\lambda_i^2}{1-2\lambda_i} + \lambda_i$$

$$= \frac{\rho_{pb}\frac{C_y}{C_p} - \lambda_i - 2\lambda_i\rho_{pb}\frac{C_y}{C_p} + 2\lambda_i^2 + \lambda_i - 2\lambda_i^2}{1-2\lambda_i}$$

$$= \frac{\rho_{pb}\frac{C_y}{C_p} - 2\lambda_i \rho_{pb}\frac{C_y}{C_p}}{1-2\lambda_i} = \frac{\rho_{pb}\frac{C_y}{C_p}(1-2\lambda_i)}{1-2\lambda_i} = \rho_{pb}\frac{C_y}{C_p} \qquad (B.3)$$

Using $(B.3)$ in $(B.1)$, we have

$$MSE_{min}(t^*_{ZKi}) \cong \frac{1-f}{n}\bar{Y}^2\left[\left(\rho_{pb}\frac{C_y}{C_p}\right)^2 C_p^2 - 2\rho_{pb}C_yC_p\left(\rho_{pb}\frac{C_y}{C_p}\right) + C_y^2\right]$$

$$MSE_{min}(t^*_{ZKi}) \cong \frac{1-f}{n}\bar{Y}^2\left[C_y^2\left(1-{\rho_{pb}}^2\right)\right] \;\; ; \; i = 2,3,\dots,10 \qquad (B.3)$$

In this paper, It is used the function given in (B.3) Equation to calculate MSE values of the proposed exponential estimators.