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HARDY-SOBOLEV-MAZYA INEQUALITY FOR NABLA TIME SCALE CALCULUS

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ABSTRACT

First part of the study is mainly about some missing concepts of the measure theory of nabla time scale calculus. In the second part, as an application of the nabla measure theory, especially the nabla Sobolev spaces, Hardy- Sobolev Mazya Inequality on nabla time scale is obtained and given.

Keywords: Time scale calculus, Measure theory, Dynamic inequalities

1. INTRODUCTION

Henri Lebesgue, Emile Borel, Maurice Fréchet and Johann Radon are very important scientists that contribute to the development of the measure theory. Probability theory and ergodic theory can be seen as the principal applications of measure theory and Lebesgue integration. By using Lebesgue integration, one can define integrals on more general subsets of R or R^n. Similarly to that, in a different space, when one can specify the measure, then the integral on more general subsets of this space can also be defined. Moreover, if Riemann and Lebesgue integrations are considered, the difference can be seen more easily. As the example of the measure theory for the other spaces the probability theory and ergodic theory can be taken into account. The measure of the whole set in measure theory is taken as 1 and the events are considered as the measurable sets. Measures that are invariant under a dynamical system is the subject of ergodic theory.

Besides, as time passes and according to the developments in other areas of engineering and basic sciences, some other spaces like that are both continuous and discrete become significant. Stephan Hilger is the first who dealt with these subject in his doctoral thesis [4] and the name of the spaces are known as time scale spaces. From 1990, many studies have been done on time scale calculus. Delta, nabla and diamond-alpha time scale calculi are defined. According to the similar needs in Eucledean spaces, Guseinov [3] Lebesgue $\Delta-$ measure on time scale spaces are defined. As its consequence, in the master thesis [1] , the delta measure theory notions and Lebesgue $\Delta-$ integral is defined.

In that sense, the main aim of this study is to be able give some more precise definitions and consequences for the nabla measure theory notions and Lebesgue $\nabla-$integral. In the following section, some basic notions of measure theory and time scale calculus is given.

2. PRELIMINARIES

In that section, the main concepts about the measure theory and the $\nabla-$time scales calculus will be given. The main resources for these informations are [5, 6, 7].

Definition 1. [5] Let X be a non-empty set and Ω is the non-empty collections of X. One can say that Ω is the σ $-$algebra if the followings are satisfied:

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- $\emptyset, X \in \Omega.$
- $If\ A \in \Omega\, then\, A^c \in \Omega.$
- $If\ \{A_n\}_{n\in N} \in \Omega, then\ \cup_{n=1}^{\infty} A_n \in \Omega.$

If Ω is only an algebra then first two conditions should be satisfied. When Ω is a σ −algebra, then (X, Ω) is measurable space.

Definition 2. [5] Assume that (X, Ω) is a measurable space. If a function $\mu: \Omega \to [0, \infty]$ is defined such that the followings are satisfied:

- μ is countably additive. In other words, $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$, where A_n's are disjoint.
- $\mu(\emptyset) = 0.$

Then (X, Ω, μ) is called a measure space.

Proposition 1. [7] Assume that (X, Ω) is measurable space, let A be a subset of X such that $A \in \Omega$. Also, suppose that f be a $[0, +\infty]$ −valued measurable function on X. Then, the sequence S_n exists that are simple functions on X and satisfies $S_1 \leq S_2 \leq \ldots$ and $f = lim_{n\to\infty} S_n$.

We can extend the set h that is a $\nabla -$ measurable function on the interval $(a, b] \subset T$ to the interval $(a, b]$ in R as follows:

$$\tilde{h}(t) = \begin{cases} h(t), if & x \in T \\ h(\rho(t_i), if & x \in (\rho(t_i), t_i) \end{cases} \tag{1}$$

The following equation is significant for the following main results:

$$\tilde{A} = A \cup \cup_{i\in I_A} (\rho(t_i), t_i)\,, \tag{2}$$

where t_i denotes the all left scattered points in $(a, b] \subset T$.

Theorem 1. Assume that $A \subset T/min\{T\}$ and $h(t)$ is a ∇ −measurable function, then

$$\int_A h(s)\nabla\, s = \int_{\tilde{A}} \tilde{h}(s)\nabla\, s,$$

the extension of $\tilde{h}(t)$ and A are done according to equations (1) and (2).

Proof. Proof is similar to the proof in [2].

Theorem 2. Assume that h is a ld-continuous function then h is ∇ −measurable.

Lemma 1. In a time scale T, the number of left scattered points can be at most countably many. If D_L is denoted as the left dense points in T and S_L is denoted as the left scattered points in T, then

$$T/D_L = S_L = \{t_i\}_{\{i\in N\}}.$$

is satisfied.

In addition to the above measure theoretical results some time scale calculus results are also important for this study. These are given here very shortly as it is seen in the following, but the references that include these are [6, 7, 9]. The very basic definition about the Δ and ∇-calculus can be found in the book [6]. For dynamic systems, inequalities are very important to get some desired results like existence, uniqueness...ect. Hardy-Sobolev-Mazya inequality is one of them. In that sense, when one deals with ∇ −calculus and dynamic systems, it is important to know the nabla version of the Hardy-Sobolev-Mazya inequality. To obtain this inequality some other inequalities like $\diamond_\alpha -$ Hölder, ∇ −Minkowski and ∇-Hölder inequalities can be found in the studies of [7] and [9], respectively.

The set T^{κ} is defined by $T \setminus (\rho(supT), supT]$ and the set T_{κ} is defined by $T \setminus (\sigma(infT), infT]$. The forward jump operator $\sigma: T \to T$ is defined by $\sigma(t) \coloneqq inf\{s \in T: s > t\}$, for $t \in T$. Similarly, the backward jump operator $\rho: T \to T$ is defined by $\rho(t) \coloneqq sup\{s \in T: s < t\}$, for $t \in T$. The forward graininess function $\mu : T \to R_0^+$ is defined by $\mu(t) \coloneqq \sigma(t) - t$, for $t \in T$. The backward graininess function $\nu : T \to R_0^+$ is defined by $\nu(t) \coloneqq t - \rho(t)$, for $t \in T$. Here it is assumed that $inf\emptyset = supT$ and $sup\emptyset = infT$.

For a function $f : T \to T$, we define the Δ −derivative of f at t $\in T^{\kappa}$, denoted by $f^{\Delta}(t)$ for all $\epsilon > 0$. There exists a neighborhood $U \subset T$ of $t \in T^{\kappa}$ such that

$$\left|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)(\sigma(t) - s)\right| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

For the same function define the ∇ −derivative of f at $t \in T_{\kappa}$, denoted by $f^{\nabla}(t)$, for all $\epsilon > 0$. There exists a neighborhood $V \subset T$ of $t \in T_{\kappa}$, such that

$$\left|(f(s) - f(\rho(t))) - f^{\Delta}(t)(s - \rho(t))\right| \leq \varepsilon|s - \rho(t)|,$$

for all $s \in V$.

A function $f : T \to R$ is rd-continuous if it is continuous at right-dense points in T and its left-sided limits exist at left-dense points in T. The class of real rd-continuous functions defined on a time scale T is denoted by $C_{rd}(T, R)$. If $f \in C_{rd}(T, R)$, then there exists a function $F(t)$ such that $F^{\Delta}(t) = f(t)$. The delta integral is defined by $\int_a^b f(x)\Delta x = F(b) - F(a)$.

Similarly, a function $g : T \to R$ is ld-continuous if it is continuous at left-dense points in T and its right-sided limits exist at right-dense points in T. The class of real ld-continuous functions defined on a time scale T is denoted by $C_{ld}(T, R)$. If $g \in C_{ld}(T, R)$then there exists a function $G(t)$ such that $G^{\nabla}(t) = g(t)$. The nabla integral is defined by $\int_a^b g(x)\nabla x = G(b) - G(a)$.

3. $L_{\nabla}^{p}(E)$ SPACES

Theorem 3. Suppose that $h(t): E \to R$ is a ∇ −measurable, this leads

$$\|h\|_{L_{\nabla}^{p}(E)} = \begin{cases} \left[\int_E |h|^p(s)\nabla s\right]^{\frac{1}{p}}, & if\ p \in [1, \infty) \\ inf\,\{K \in R: |h| \leq K\ \nabla - a.e. on\ E\}, & if\ p = \infty \end{cases} \tag{3}$$

is a norm.

Proof. 1. $\|h\|_{L_{\nabla}^{p}(E)} \geq 0$, since $|h| > 0$, therefore $\left[\int_E |h|^p(s)\nabla s\right]^{\frac{1}{p}} > 0$.

Additionally, it is apparent that if $h = 0$, then $\left[\int_E |h|^p(s)\nabla s\right]^{\frac{1}{p}} = 0$. If $\|h\|_{L_{\nabla}^{p}(E)} = 0$, in other words, $\left[\int_E |h|^p(s)\nabla s\right]^{\frac{1}{p}} = 0$, then $|h| = 0$ ∇-a.e. and $h = 0$ ∇ −a.e.

2. Let γ be a constant, then we can express the norm of γh as:

$$\|\gamma h\|_{L^p_\nabla(E)} = \left[\int_E |\gamma h|^p(s)\nabla s\right]^{\frac{1}{p}}$$

$$= \left[\int_E |\gamma|^p |h|^p(s)\nabla s\right]^{\frac{1}{p}}$$

$$= \left[|\gamma|^p \int_E |h|^p(s)\nabla s\right]^{\frac{1}{p}}$$

$$= |\gamma| \left[\int_E |h|^p(s)\nabla s\right]^{\frac{1}{p}}$$

$$= |\gamma| \|h\|_{L^p_\nabla(E)}.$$

3. $\|h+g\|_{L^p_\nabla(E)} \leq \|h\|_{L^p_\nabla(E)} + \|g\|_{L^p_\nabla(E)}$ is satisfied as an immidiate consequence of Minkowski inequality for ∇ −time scales calculus.

Theorem 4. Suppose that $p \in [1,\infty)$. Then $L^p_\nabla(E)$ is a Banach space together with the norm (3), for $E \subset T$.
Proof. Let $\{h_n\}$ be a Cauchy sequence in $L^p_\nabla(E)$ for $1 \leq p < \infty$.

Then $\forall \epsilon > 0\ \exists\ N(\epsilon) \in N$ such that for any $n, m \geq N(\epsilon)$, $\|h_n - h_m\|_{p,\nabla} < \epsilon$. Let us take $\epsilon = \frac{1}{2^k}$. Since $\{h_n\}$ is a Cauchy sequence, then it also has a Cauchy subsequence. For this reason, take the subsequence $\{h_{n_k}\}$. Then it satisfies the following:

$$\left\|h_{n_{k+1}} - h_{n_k}\right\|_{p,\nabla} < \frac{1}{2^k}.$$

Now, let us define a function $h(t)$ such that

$$h(t) = h_{n_1} + \textstyle\sum_{k=1}^{\infty}\left(h_{n_{k+1}} - h_{n_k}\right), t \in E. \tag{4}$$

Similar to the function $h(t)$, the function $g(t)$ is defined as:

$$g(t) = |h_{n_1}| + \sum_{k=1}^{\infty}\left(|h_{n_{k+1}}| - |h_{n_k}|\right), t \in E.$$

Firstly, consider the partial sum of equality (4), then

$$S_N(h) = h_{n_1} + \sum_{k=1}^{N-1}\left(h_{n_{k+1}} - h_{n_k}\right)$$

is obtained. By using Nabla Minkowski inequality for nabla time scales calculus,

$$||S_N(g)||_p \leq \|h_{n_1}\| + \left\|\sum_{k=1}^{N-1}\left(h_{n_{k+1}} - h_{n_k}\right)\right\| \leq \|h_{n_1}\| + \sum_{k=1}^{N-1}\frac{1}{2^k} \leq \|h_{n_1}\| + 1.$$

Here, $\left(\|S_N(g)\|_p\right)_{N\in N}$ is bounded from above and an increasing sequence, therefore $\int_E g^p \,\nabla s < \infty$.

It is obvious that $|h(t)| \leq g$. Thus,

$$\int_E |h|^p \;\nabla s \leq \int_E g^p \;\nabla s$$

and this means h^p is $\nabla-$integrable.

Here, since $S_N(h)$ converges to h as n tends to infinity, by using Lebesgue dominated converges theorem for $\nabla-$time scales calculus, the following is obtained:

$$\lim_{n\to\infty}\left\|h - h_{n_N}\right\|^p = \int_E \lim_{n\to\infty} |S_N(h)(s) - h(s)|^p \nabla s = \int_E 0 \;\nabla s = 0.$$

In a Cauchy sequence if any of its subsequence is convergent then also the sequence is convergent. Therefore any Cauchy sequence $h_n \in L^p_\nabla(E)$ converges to $h(t) \in L^p_\nabla(E)$. This ends the proof.

Corollary 1. By using Equation (1), the extension of the function $h(t)$ is obtained. For this function, the following expression also holds:

$h(t) \in L^p_\nabla((a,b] \cap T)$, then $\tilde{h}(t) \in L^p([a,b])$. Therefore;

$$\|h\|_{L^p_\nabla} = \|\tilde{h}\|_{L^p}.$$

Proof. The proof is an immediate consequence of Theorem 1.

Theorem 5. If $p \in [1,\infty)$, then $C_{\mathrm{c,ld}}(E)$ which is the space of all ld-continuous functions on E with compact support in E is dense in $L^p_\nabla(E)$.

Proof. The desired result is obtained by using Theorem 2 and Proposition 1.

Definition 3. [10] A function $h: E \to R$ is called nabla absolutely continuous on E, if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\{(a_k, b_k] \cap T\}_{k=1}^n$ is a finite pairwise disjoint family of subintervals of E for $a_k, b_k \in E$ and satisfies if $\sum_{k=1}^n (b_k - a_k) < \delta$, then $\sum_{k=1}^n (h(b_k) - h(a_k)) < \epsilon$. And this type of functions are showed as $h \in AC_{ld}(E)$.

Theorem 6. [10] Fundamental Theorem of Calculus for Lebesgue $\nabla-$Integrals: A function $h: T \to R$ is ld-absolutely continuous on $[a,b] \cap T$ if and only if h is $\nabla-$differentiable $\nabla-$a.e. on $(a,b] \cap T$, $h^\nabla \in L^1_\nabla$ and

$$h(t) = h(a) + \int_{(a,t]\cap T} h^\nabla(s)\nabla s$$

is satisfied for every $t \in [a,b] \cap T$.

Theorem 7. If $h, g\colon E \to R$ are ld-absolutely continuous functions on E, where $E = [a, b] \cap T$, then $h.g$ is absolutely continuous on E and the following holds:

$$\int_E h^\nabla g + h^\rho g^\nabla = h(b)g(b) - h(a)g(b) = \int_E hg^\nabla + h^\nabla g^\rho \tag{5}$$

Proof. An ld-absolutely continuous function on a set $[a, b] \cap T$ is regulated, therefore this function is bounded from above and below by [6], see Theorem 1.65. Thus h, g are bounded functions on $[a, b] \cap T$ and one can say that $|h(t)| \leq M/2$ and $|g(t)| \leq M/2$. Let us take $f = h.g$. We know that $h(t)$ is ld-absolutely continuous then there exits δ_1 such that

$$\sum_{k=1}^{n} |h(b_k) - h(a_k)| < \frac{\epsilon}{M},$$

and there exits δ_2 such that

$$\sum_{k=1}^{n} |g(b_k) - g(a_k)| < \frac{\epsilon}{M}.$$

Take $\delta = min\{\delta_1, \delta_2\}$ and assume that $\sum_{\mathrm{k}=1}^{\mathrm{n}}(\mathrm{b_k} - \mathrm{a_k}) < \delta$, then

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{n} |h(b_k)g(b_k) - h(a_k)g(a_k)|$$

$$= \sum_{k=1}^{n} |h(b_k)g(b_k) - h(b_k)g(a_k) + h(b_k)g(a_k) - h(a_k)g(a_k)|$$

$$= \sum_{k=1}^{n} |h(b_k)||g(b_k) - g(a_k)| + \sum_{k=1}^{n} |g(a_k)||h(b_k) - h(a_k)|$$

$$= \frac{M}{2}\sum_{k=1}^{n} |g(b_k) - g(a_k)| + \frac{M}{2}\sum_{k=1}^{n} |h(b_k) - h(a_k)|$$

$$< \frac{M}{2}\frac{\epsilon}{M} + \frac{M}{2}\frac{\epsilon}{M} = \epsilon.$$

Hence, $h.g$ is ld-absolutely continuous. Additionally, f, g, h are ∇ −differentiable almost everywhere and

$$f^\nabla = h^\nabla g + h^\rho g^\nabla.$$

Thus, by using fundamental theorem of calculus for Lebesgue ∇ −integrals (5) is obtained.

Remark 1. Suppose that $E = [a, b] \cap T$, $h\colon E \to R$ and $\bar{h}(t)\colon [a, b] \to R$ is defined as:

$$\bar{h}(t) = \begin{cases} h(t)\,,\ if\ \ t \in E \\ h(\rho(t_i)) + \dfrac{h(t_i) - h(\rho(t_i))}{t_i - \rho(t_i)}(t - \rho(t_i)),\ if\ t \in (\rho(t_i), t_i), \qquad for\ \ i \in I_L \end{cases}$$

where I_L is the all left scattered points in $[a, b] \cap T$. Afterwards, h is ld-absolutely continuous on E if and only if $\bar{h}$ is absolutely continuous on $[a, b]$.

4. ∇ −SOBOLEV SPACES

Definition 4. Suppose $p \in [1, \infty]$ and $h: E \to R$ One can say that $h \in W_\nabla^{1,p}$ if and only if $h \in L_\nabla^p(E)$ and there exists $u \in L_\nabla^p(E)$ such that

$$\int_E (h.\Psi^\nabla)(\tau)\nabla\tau = -\int_E (u.\Psi^\rho)(\tau)\nabla\tau\ , \qquad \forall\, \Psi \in C_{0,ld}^1(E_\kappa). \tag{6}$$

Here $C_{0,ld}^1(E_\kappa) = \{\Psi: E \to R: \Psi \in C_{ld}^1(E_\kappa), \Psi(a) = \Psi(b) = 0\}$ and $C_{ld}^1(E_\kappa)$ is the set of all left dense continuous functions on E such that they are ∇ −differentiable on E_κ and their ∇ −derivatives are ld-continuous on E_κ.

Remark 2. By using Theorem 7 for ld-absolutely continuous functions on E one has the following relation

$$V_\nabla^{1,p} := \{x \in AC_{ld}(E): h^\nabla \in L_\nabla^p(E)\} \subset W_\nabla^{1,p}, \qquad for\ \ p \in [1, \infty].$$

Lemma 2. Suppose $h \in L_\nabla^1$ if

$$\int_E (h\,u)(\tau)\nabla\tau = 0\ \ \forall\, u \in C_{c,ld}(E), \tag{7}$$

then

$$h = 0\ \ \nabla - a.e. on\, E,$$

which guarantees that there exists a function $h_1 \in C_{c,ld}(E)$ such that $\|h - h_1\|_{L_\nabla^1} < \epsilon$.

Proof. Let us fix an $\epsilon > 0$ and use the density of $C_{c,ld}(E)$ in L_∇^p.

$$\left|\int_E (h_1 u)(s)\nabla s\right| = \left|\int_E (h_1.u)(s)\nabla s - \int_E (h.u)\,(s)\,\nabla s\right|$$
$$\leq \|u\|\|h - h_1\| \leq \epsilon\, ||u||.$$

Consider the following sets:

$$E_1 = \{s \in E: h_1(s) \geq \epsilon\}, \qquad E_2 = \{s \in E: h_1(s) \leq -\epsilon\}.$$

These are the compact and disjoint subsets of E, then by using Urysohn's Lemma, we can define the following function:

$$u_0 \equiv \begin{cases} 1, & on\ E_1 \\ -1, & on\ E_2 \end{cases}$$

$|u_0| \leq 1\ on\ E$. Let us define $\tilde{E} := E_1 \cup E_2$, then the following is obtained for any arbitrary ϵ:

$$\int_E |h_1|(s)\nabla s = \int_E (h_1 u_0)(s)\nabla s - \int_{E/\tilde{E}} (h_1 u_0)(s)\nabla s + \int_{E/\tilde{E}} |h_1|(s)\nabla s \leq \epsilon + 2\epsilon\,(b-a).$$

Thus, the desired result is obtained.

Lemma 3. Assume that $h \in L^1_\nabla(E)$. Then

$$\int_E \left(h\Psi^\nabla\right)(s)\nabla s = 0 \ \forall\, \Psi \in C^1_{0,ld}(E_\kappa) \tag{8}$$

if and only if

$$h \equiv c\ \nabla - a.e\ on\ E, \tag{9}$$

where $c \in R$.

Proof. If (8) is true, then by using (6), (7) and Lemma 2, we obtain Equation (9). If $h = c\ \nabla - a.e\ on\ E$, then by using (6) and fundamental theorem for nabla time scales calculus, we get the desired result.

Theorem 8. Assume that $u \in W^{1,p}_\nabla(E)$ for some $p \in [1,\infty]$. Equation (6) is satisfied for $g \in L^p_\nabla(E)$. Then there exists a unique $x \in V^{1,p}_\nabla(E)$ such that the followings are satisfied:

$$x = u, \qquad x^\nabla = g\ \nabla - a.e.\ on\ E.$$

Proof. First, define $v: E \to R$:

$$v(t) := \int_{[a,t)\cap T} g(s)\nabla s \ \ \forall\, t \in E.$$

Then the fundamental theorem of calculus for Lebesgue ∇ −integrals guarantees that $v \in V^{1,p}_\nabla(E)$. Then by using Equation (6) and Lemma 3, one can obtain that:

$$\int_E [(v-u)\Psi^\nabla](s)\nabla s = -\int_E \left[\left(v^\nabla - g\right)\Psi^\rho\right](s)\nabla s = 0;\ \ \Psi \in C^1_{0,ld}(E_\kappa)$$

Then by using Lemma (3), $v - u \equiv c$ is obtained almost everywhere on E. Since $x = u\,\nabla - a.e. on\ E$, for all $t \in E$. Then, by using fundamental theorem of calculus for Lebesgue ∇ −integrals, we get that $x(t) = v(t) - c$ is the unique function in $V^{1,p}_\nabla(E)$.

Lemma 4. Assume that $p \in [1,\infty]$). $W^{1,p}_\nabla(E)$ is a Banach space with the norm:

$$\|x\|_{W^{1,p}_\nabla} := \|x\|_{L^p_\nabla} + \left\|x^\nabla\right\|_{L^p_\nabla}.$$

Proof. Assume that $\{h_n\}_{n\in N} \subset W_\nabla^{1,p}(E)$ is a Cauchy sequence, then Theorem 4 guarantees that there exist f, g such that $\{h_n\}_{n\in N}$ and $\{h_n^\nabla\}_{n\in N}$ converge strongly in $L_\nabla^p(E)$ to f and g respectively. Then, the following is obtained:

$$\int_E \left(f\Psi^\nabla\right)(s)\nabla s = lim_{n\to\infty}\int_E \left(h_n\Psi^\nabla\right)(s)\nabla s = -lim_{n\to\infty}\int_E \left(h_n^\nabla\Psi^\rho\right)(s)\nabla s$$

$$= -\int_E (g\Psi^\rho)(s)\nabla s \ , \Psi \in C_{0,ld}^1(E_\kappa).$$

Then, we have that $f \in W_\nabla^{1,p}(E)$. Then, by using Theorem 8, there exists $x \in W_\nabla^{1,p}(E)$ such that h_n strongly converges to x in $W_\nabla^{1,p}(E)$ and h_n^∇ strongly converges to x^Δ in $W_\nabla^{1,p}(E)$.

Theorem 9. Let $p \in [1,\infty]$, there will a constant $M > 0$ such that

$$\|h\|_{C_{ld}(E)} \leq M\|h\|_{\mathrm{W}_\nabla^{1,p}(E)}$$

is satisfied for all $h \in \mathrm{W}_\nabla^{1,p}(E)$, where the norm $\|.\|_{C_{ld}(E)}$is the supremum norm and this means that $\mathrm{W}_\nabla^{1,p}(E)$ is continuously immersed into the space $C_{ld}(E)$.

Proof. First, let us fix an element h of $\mathrm{W}_\nabla^{1,p}(E)$ Let $t, T \in E$ and $t < T$. Then we get the following if we use fundamental theorem of calculus for Lebesgue ∇ −integrals:

$$h(T) = h(t) + \int_{(t,T]_T} h^\nabla(s)\nabla s.$$

Now, if we take the absolute value of both sides, we have

$$|h(T)| \leq |h(t)| + \int_E \left|h^\nabla(s)\right|\nabla s \leq \int_E |h(t)|\nabla s + \int_E |h^\nabla(s)|\nabla s.$$

Then by taking supremum of both sides and using nabla Minkowski inequality, the desired result

$$\|h\|_{C_{ld}(E)} \leq M\ \|h\|_{\mathrm{W}_\nabla^{1,p}(E)}$$

is obtained.

Proposition 2. Let $p \in (1,\infty]$, $\{h_n\}_{n\in N} \subset W_\nabla^{1,p}(E)$ and also $h \in W_\nabla^{1,p}(E)$. If $\{h_n\}_{n\in N}$ weakly converges to h in $W_\nabla^{1,p}(E)$ then $\{h_n\}_{n\in N}$ strongly converges to h in $C_{ld}(E)$.

Proof. Assume that $\{h_n\}_{n\in N}$ weakly converges to $h \in W_\nabla^{1,p}(E)$. Then by using Theorem 9, we can say that $\{h_n\}_{n\in N}$ weakly converges to h in $C_{ld}(E)$. Since $\{h_n\}_{n\in N}$ is equicontinuous, the desired result is obtained.

Definition 5. Let $p \in [1,\infty)$, then $W_{0,\nabla}^{1,p}(E)$ is defined as the closure of the $C_{0,ld}(E_\kappa)$ in $W_{0,\nabla}^{1,p}(E)$.

4.1. Application: Hardy-Sobolev-Mazya Inequality on Nabla Time Scales Calculus

Theorem 10. Let $q \geq 2$. If $\frac{\nu(t)}{b-t}$ is a non-increasing on $(a,b]_T$, constant C_q that is dependent on q satisfies

$$\int_a^b \left|g^\nabla(t)\right|^2 \nabla t \geq \frac{1}{4}\int_a^b \frac{|g^\rho(t)|^2}{4\big(b-\rho(t)\big)^2}\nabla t + C_q\left[\left(\int_a^b |g(t)|^q\right)\left(\int_a^b |G(t)|^q\right)^{\frac{q}{q+2}}\right]^{\frac{q+2}{q}},$$

$\forall\, g(t) \in W_{0,\nabla}^{1,p}((a,b]_T)$, where $G(t) = max\{g(t), g^\rho(t)\}$.

Proof. Suppose that k is a function such that

$$g(t) = \eta(t)k(t), \qquad t \in [a,b)_T,$$

where $\eta(t) = \sqrt{b-t}$, $\forall$ t $\in [a,b)_T$. Then $\eta \in C_{ld}^1([a,b)_T)$ and $\eta^\nabla(t) = -\frac{1}{\eta(t)+\eta^\rho(t)}$. Then by taking the nabla differentiation of g then

$$g^\nabla(t) = k^\nabla(t)\eta(t) + \eta^\nabla(t)k^\rho(t)$$

$$= k^\nabla(t)\eta(t) - \frac{k^\rho(t)}{\eta(t)+\eta^\rho(t)}.$$

After this, one can obtain the following

$$k^\nabla(t)\eta(t) = g^\nabla(t) + \frac{g^\rho(t)}{\eta\eta^\rho + (\eta^\rho)^2}.$$

$$\left|k^\nabla(t)\eta(t)\right|^2 = \left(g^\nabla(t)\right)^2 + \frac{2g^\rho(t)g^\nabla(t)}{\eta\eta^\rho + (\eta^\rho)^2} + \frac{\left(g^\rho(t)\right)^2}{(\eta\eta^\rho + (\eta^\rho)^2)^2}$$

$$\leq \left(g^\nabla(t)\right)^2 + \frac{2g^\rho(t)}{\eta\eta^\rho + (\eta^\rho)^2}\left[g^\nabla(t) + \frac{g^\rho(t)}{\eta\eta^\rho + (\eta^\rho)^2}\right] - \frac{\left(g^\rho(t)\right)^2}{4\big(b-\rho(t)\big)^2}$$

$$\leq \left(g^\nabla(t)\right)^2 + \Psi^\rho(t)k^\rho(t)k^\nabla(t) - \frac{\left(g^\rho(t)\right)^2}{4\big(b-\rho(t)\big)^2},$$

where $\Psi^\rho(t) = -2\eta(t)\eta^\nabla(t)$ for all $t \in [\sigma(a), b]_T$. Then $\Psi^\rho(t)$ is $\nabla-$differentiable for all left scattered points. Suppose that $t \in [a,b]_T$ such that t is a left dense point, then it means that t is an accumulation point. So, there are two cases:

1. Let $\alpha, \beta \in [a,b]_T$ and $t \in [\alpha,\beta] \subset [a,b]_T$, then $\Psi^\rho(t)$ is $\nabla-$differentiable and $[\Psi^\rho(t)]^\nabla = 0$, since for $t \in [\alpha,\beta], \eta^\rho(t) = \eta(t)$.

2. Assume that $L := \{t \in T : \rho(t) < t\} = \{t_j\}_{j\in N}$ from Lemma 1. For $(t_n)_{n\in N} \in L \cap [a,b]_T, t_n$'s are the isolated points and as n tends to infinity t_n tends to t. In this case, $\Psi^\nabla(t)$ do not exists. Nevertheless,

$$\mu_\nabla(\{t \in [a,b]_T : \rho(t) = t \text{ and} \lim_{n\to\infty} t_n = t, \qquad (t_n)_{n\in N} \in L\}) = 0.$$

Hence, $\Psi^{\rho}(t)$ is ∇ −a.e differentiable on $[a,b]_T$. Let $t,s \in [\sigma(a),b]_T$ and $t > s$, then

$$\begin{aligned}
\Psi^{\rho}(t) - \Psi^{\rho}(s) &= \frac{1}{2}\Psi^{\rho}(t)\Psi^{\rho}(s)\left[\frac{2}{\Psi^{\rho}(s)} - \frac{2}{\Psi^{\rho}(t)}\right] \\
&= \frac{1}{2}\Psi^{\rho}(t)\Psi^{\rho}(s)\left[\frac{2}{-2\eta^{\nabla}(s)\eta(s)} - \frac{2}{-2\eta^{\nabla}(t)\eta(t)}\right] \\
&= \frac{1}{2}\Psi^{\rho}(t)\Psi^{\rho}(s)\left[\frac{\eta(s)\eta^{\rho}(s)}{\eta(s)} - \frac{\eta(t)+\eta^{\rho}(t)}{\eta(t)}\right] \\
&= \frac{1}{2}\Psi^{\rho}(t)\Psi^{\rho}(s)\left[\frac{\eta^{\rho}(s)}{\eta(s)} - \frac{\eta^{\rho}(t)}{\eta(t)}\right] \\
&= \frac{1}{2}\Psi^{\rho}(t)\Psi^{\rho}(s)\left[\sqrt{\frac{b-\rho(s)}{b-s}} - \sqrt{\frac{b-\rho(t)}{b-t}}\right] \\
&= \frac{1}{2}\Psi^{\rho}(t)\Psi^{\rho}(s)\left[\sqrt{1+\frac{\nu(s)}{b-s}} - \sqrt{1+\frac{\nu(t)}{b-t}}\right] > 0.
\end{aligned}$$

Therefore $\Psi^{\rho}(t)$ is an increasing function. The following is obtained by integration by parts formula for nabla time scale calculus:

$$\int_a^b \Psi^{\rho}(t)k^{\rho}(t)k^{\nabla}(t)\nabla t = \Psi(b)k^2(b) - \Psi(a)k^2(a) - \int_a^b [\Psi(t)k(t)]^{\nabla}k(t)\nabla t$$

$\Psi(b) = \Psi^{\rho}(\sigma(b))$, since $b \in [a,b]_T$, then $\sigma(b) = b$. Therefore, $\Psi(b) = \Psi^{\rho}(b)$ and $\Psi^{\rho}(b) = 0$, then $\Psi(b) = 0$. Thus, one obtains

$$\begin{aligned}
\int_a^b \psi^{\rho}(t)k^{\rho}(t)k^{\nabla}(t)\nabla t &\le -\int_a^b [\Psi(t)k(t)]^{\nabla}k(t)\nabla t \\
&= -\left[\int_a^b \Psi^{\nabla}(t)k^2(t)\nabla t + \int_a^b \Psi^{\rho}(t)k^{\nabla}(t)k(t)\nabla t\right] \\
&\le -\int_a^b \Psi^{\rho}(t)k^{\nabla}(t)[k^{\rho}(t) + k^{\nabla}(t)\nu(t)]\nabla t
\end{aligned}$$

$$\begin{aligned}
&\le -\int_a^b \Psi^{\rho}(t)\left(k^{\nabla}(t)\right)^2\nu(t)\nabla t - \int_a^b \Psi^{\rho}(t)k^{\nabla}(t)k^{\rho}(t)\nabla t \\
&\le -\int_a^b \Psi^{\rho}(t)k^{\nabla}(t)k^{\rho}(t)\nabla t.
\end{aligned}$$

Thus, we obtain that $\int_a^b \Psi^{\rho}(t)k^{\nabla}(t)k^{\rho}(t)\nabla t = 0$ and

$$\int_a^b |\eta(t)k^{\nabla}(t)|^2\nabla t \le \int_a^b \left(\left(g^{\nabla}(t)\right)^2 - \frac{g^{\rho}(t)}{4\left(b-\rho(t)\right)^2}\right)\nabla t.$$

By using nabla chain rule in [11],

$$\begin{aligned}
\left(|k(t)|^{\frac{q+2}{2}}\right)^{\nabla} &= \frac{q+2}{2}|k^{\nabla}(t)|\int_0^1 |hk(t)+(1-h)k^{\rho}(t)|^{\frac{q}{2}}\,dh \\
&\le \frac{q+2}{2}|k^{\nabla}(t)||K(t)|^{\frac{q}{2}},
\end{aligned}$$

where $K(t) = max\{k, k^{\rho}\}$. Thus, by using that η is non-increasing, the following is obtained

$$|g(t)|^{\frac{q+2}{2}} = |\eta(t)|^{\frac{q+2}{2}}.\int_a^t \left[|k(s)|^{\frac{q+2}{2}}\right]^{\nabla}\nabla s$$

$$\le |\eta(t)|^{\frac{q+2}{2}}.\int_a^t \frac{q+2}{2}|k^{\nabla}(s)||K(s)|^{\frac{q}{2}}\nabla s$$
$$= \frac{q+2}{2}\int_a^t |\eta(t)|^{\frac{q+2}{2}} |k^{\nabla}(s)||K(s)|^{\frac{q}{2}}\nabla s$$
$$\le \frac{q+2}{2}\int_a^t |\eta(s)|^{\frac{q+2}{2}} |k^{\nabla}(s)|\,|K(s)|^{\frac{q}{2}}\nabla s$$
$$\le \frac{q+2}{2}\int_a^b |\eta(t)|^{\frac{q+2}{2}} |k^{\nabla}(s)|\,|K(s)|^{\frac{q}{2}}\nabla s$$

By using Nabla Hölder inequality, the following is obtained:

$$|g(t)|^{q+2} \le m_q\left(\int_a^b |k^{\nabla}(t)|^2\eta^2(t)\nabla t\right)\left(\int_a^b |K(t)|^q\eta^q(t)\nabla t\right)$$
$$\le m_q\left(\int_a^b |g^{\nabla}(t)|^2 - \frac{|g^{\rho}(t)|^2}{4\big(b-\rho(t)\big)^2}\nabla t\right)\left(\int_a^b |G(t)|^q\nabla t\right), \tag{11}$$

where $G:=\{g, g^{\rho}\}$ and $m_q = \left(\frac{q+2}{2}\right)^2$.

Now, firstly if we take the $\frac{q}{q+2}$th power of inequality (11) and then its integral, the following form is got,

$$\int_a^b |g(t)|^q\nabla t \le \int_a^b [\, m_q^{\frac{q}{q+2}}\left(\int_a^b \left|g^{\nabla}(t)\right|^2 - \frac{|g^{\rho}(t)|^2}{4\big(b-\rho(t)\big)^2}\nabla t\right)^{\frac{q}{q+2}}.\ \left(\int_a^b |G(t)|^q\,\nabla t\right)^{\frac{q}{q+2}}]$$
$$\le m_q^{\frac{q}{q+2}}(b-a)\left(\int_a^b \left|g^{\nabla}(t)\right|^2 - \frac{|g^{\rho}(t)|^2}{4\big(b-\rho(t)\big)^2}\nabla t\right)^{\frac{q}{q+2}}\ \left(\int_a^b |G(t)|^q\,\nabla t\right)^{\frac{q}{q+2}}.$$

Therefore, one can also get

$$\left(\int_a^b |g(t)|^q\nabla t\right)\left(\int_a^b |G(t)|^q\nabla t\right)^{-\frac{q}{q+2}} \le m_q^{\frac{q}{q+2}}(b-a)\left(\int_a^b \left|g^{\nabla}(t)\right|^2 - \frac{|g^{\rho}(t)|^2}{4\big(b-\rho(t)\big)^2}\nabla t\right)^{\frac{q}{q+2}}$$

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This means that the desired result is got.

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