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ON TIMES SCALE FRACTIONAL ORDER DIFFERENTIAL EQUATION INVOLVING RANDOM VARIABLE

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Abstract

In this paper, nonlocal and boundary value problems (BVP) of fractional differential equations involv- ing random walk on times scale is discussed. The sufficient conditions for existence and uniqueness of dynamical systems are obtained using standard fixed point methods. The stability of solutions is made sure by Ulam-Hyers stability method.

Keywords: Dynamical equations, Fractional calculus, Existence, Stability.

1. Introduction

The theory of time scales calculus allows us to study the dynamic equations, which include both difference and differential equations. Since the study on dynamic equations on time scales has received much attention of many researchers in recent days, see [1, 2, 3, 4, 5] and the references therein.

Randomness of the FDEs which arises in uncertainties and complexities. Such deterministic equations are hardly called as Random differential equations (RDEs). The recent development of RDEs of fractional order can be seen in [15, 18, 25].

Ever since the birth of Fractional differential equations (FDEs) in sixteenth century only in past few decades it received tremendous development in describe the real-life phenomena more accurately than integer order derivative. The main aspect of FDEs is to prove existence, uniqueness and stability of solutions. For the detailed study of FDEs one can refer to the books [11, 16, 17] and the papers [7, 9, 14, 19, 24]. The literature provides numerous numbers of fractional derivatives with singular kernals. Here in this article we use a special kind of fractional derivative called ψ-Hilfer fractional derivative integrate several classical derivative, detailed in [20]. For the recent works on ψ-Hilfer fractional derivative we refer the readers to [6, 10, 22, 23]

On the other hand, the stability investigation of differential and integral equations is important in applications. Here we extend the results of Ulam Hyers stability and Ulam Hyers Rassias(U-H-R) stability to fractional RDEs on times scale. The stability check of FDEs and theoretical analysis of Ulam type stability can be seen in [12, 21, 26].

From the above discussion and motivation in this work we study ψ-Hilfer fractional RDEs on times scale with boundary and nonlocal conditions. The existence, uniqueness and stability solutions are obtained by fixed point methods. First consider the BVP for ψ-Hilfer fractional RDEs on times scale of the form

$$\begin{cases} {}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(t,\omega) = \mathfrak{g}(t,\mathfrak{u}(t,\omega),\omega), \quad t \in J \subseteq \mathbb{T}, \\ a\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(t,\omega)|_{t=0} + b\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(t,\omega)|_{t=T} = c, \end{cases} \tag{1}$$

where (Ω, F, p) is a complete probability space, $\omega \in \Omega$, ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ is the ψ-HFD defined on $\mathbb{T}$, $0 < \alpha < 1$, $0 \le \beta \le 1$ and ${}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}$ is ψ-fractional integral of order $1-\gamma(\gamma = \alpha+\beta-\alpha\beta)$. Let $\mathbb{T}$ be a time scale, that is nonempty subset of Banach space. The function $\mathfrak{g} : J := [0,b] \times R \times \Omega \to R$ is a right-dense continuous function. Here, the Eq. (1) satisfies the random integral equation of the form

$$\mathfrak{u}(t,\omega) = \quad \left(c - b\ {}^{\mathbb{T}}\mathfrak{I}^{1-\beta+\alpha\beta;\psi}\mathfrak{g}(T,\mathfrak{u}(T,\omega),\omega)\right) \tfrac{(\psi(t)-\psi(0))^{\gamma-1}}{(a+b)\Gamma(\gamma)} +^{\mathbb{T}} \mathfrak{I}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{u}(t,\omega),\omega)\Delta s. \tag{2}$$

In the next section, we consider the nonlocal fractional random differential equation on times scale

$$\begin{cases} {}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{u}(t,\omega) = \mathfrak{g}(t,\mathfrak{u}(t,\omega),\omega) \quad t \in J, \\ {}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(t,\omega)|_{t=0} = \sum_{i=1}^{m} c_i\mathfrak{u}(\tau_i,\omega), \quad \tau_i \in J, \end{cases} \tag{3}$$

where $\tau_i, i = 0, 1, ..., m$ are prefixed points satisfying $0 < \tau_1 \le ... \le \tau_m < b$ and c_i is real numbers. Here, nonlocal condition $\mathfrak{u}(0,\omega) = \sum_{i=1}^{m} c_i\mathfrak{u}(\tau_i,\omega)$ can be applied in physical problems yields better effect than the initial conditions ${}^{\mathbb{T}}\mathfrak{I}^{1-\gamma;\psi}\mathfrak{u}(t,\omega)|_{t=0} = \mathfrak{u}_0$. Further (3) is equivalent to mixed integral type of the form

$$\mathfrak{u}(t,\omega) = \begin{cases} \frac{T}{\Gamma(\alpha)}\left(\psi(t)-\psi(0)\right)^{\gamma-1} \sum_{i=1}^{m} c_i \displaystyle\int_0^{\tau_i} \psi'(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s \\ +\frac{1}{\Gamma(\alpha)} \displaystyle\int_0^{t} \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s \end{cases} \tag{4}$$

where

$$T = \frac{1}{\Gamma(\gamma) - \sum_{i=1}^{m} c_i(\psi(\tau_i)-\psi(0))^{\gamma-1}}.$$

The novelty of paper is given as follows: In Section 2, basic definitions and preliminary are discussed. Existence, uniqueness and stability with random walk for BVP and nonlocal problems are discussed in Section 3 and Section 4 respectively.

2. Preliminaries

Definition 2.1. *Let $C(J)$ be continuous function endowed with the norm*

$$\|\mathfrak{u}\|_C = \max\left\{|\mathfrak{u}(t,\omega)| : t \in J\right\}.$$

We denote the $C_{1-\gamma,\psi}(J)$ as follows

$$C_{1-\gamma,\psi}(J) := \left\{\mathfrak{g}(t,\omega) : J \times \Omega \to R|\left(\psi(t)-\psi(0)\right)^{1-\gamma}\mathfrak{g}(t,\omega) \in C(J)\right\}, 0 \le \gamma < 1$$

where $C_{1-\gamma,\psi}(J)$ *is the weighted space of the continuous functions* $\mathfrak{g}$ *on the finite interval* J.

Obviously, $C_{1-\gamma,\psi}(J)$ *is the Banach space with the norm*

$$\|\mathfrak{g}\|_{C_{1-\gamma,\psi}} = \left\|(\psi(t)-\psi(0))^{1-\gamma}\,\mathfrak{g}(t,\omega)\right\|_{C}.$$

Definition 2.2. *Let time scale be* $\mathbb{T}$. *The forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ *is defined by* $\sigma(t) := \inf\{s\in\mathbb{T}: s>t\}$, *while the backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ *is defined by* $\rho(t) := \sup\{s\in\mathbb{T}: s<t\}$.

Proposition 2.3. *Suppose* $\mathbb{T}$ *is a time scale and* $[a,b]\subset\mathbb{T}$, $\mathfrak{g}$ *is increasing continuous function on* $[a,b]$. *If the extension of* $\mathfrak{g}$ *is given in the following form:*

$$F(s) = \begin{cases} \mathfrak{g}(s); & s\in\mathbb{T} \\ \mathfrak{g}(t); & s\in(t,\sigma(t))\notin\mathbb{T}. \end{cases}$$

Then we have

$$\int_a^b \mathfrak{g}(t)\Delta t \le \int_a^b F(t)dt.$$

Definition 2.4. *Let* $\mathbb{T}$ *be a time scale,* $J\in\mathbb{T}$. *The left-sided R-L fractional integral of order* $\alpha\in R^+$ *of function* $\mathfrak{f}(t)$ *is defined by*

$$\left({}^{\mathbb{T}}\mathfrak{J}^{\alpha}\mathfrak{g}\right)(t) = \int_0^t \psi^{'}(s)\frac{(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\mathfrak{g}(s)\Delta s, \quad (t>0),$$

where $\Gamma(\cdot)$ *is the Gamma function.*

Definition 2.5. *Suppose* $\mathbb{T}$ *is a time scale,* $[0,b]$ *is an interval of* $\mathbb{T}$. *The left-sided R-L fractional derivative of order* $\alpha\in[n-1,n)$, $n\in\mathbb{Z}^+$ *of function* $f(t)$ *is defined by*

$$\left({}^{\mathbb{T}}\Delta^{\alpha}\mathfrak{g}\right)(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_0^t \psi^{'}(s)\frac{(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\mathfrak{g}(s)\Delta s, \quad (t>0).$$

Definition 2.6. *[9] The left-sided* ψ*-HFD of function* $\mathfrak{f}(t)$ *is defined by*

$${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}\mathfrak{g}(t) = \left({}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta\left({}^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi}\mathfrak{g}\right)\right)(t),$$

where ${}^{\mathbb{T}}\Delta := \frac{d}{dt}$.

Remark 2.7. 1. *The operator* ${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi}$ *also can be written as*

$${}^{\mathbb{T}}\Delta^{\alpha,\beta;\psi} = {}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta{}^{\mathbb{T}}\mathfrak{J}^{(1-\beta)(1-\alpha);\psi} = {}^{\mathbb{T}}\mathfrak{J}^{\beta(1-\alpha);\psi}\ {}^{\mathbb{T}}\Delta^{\gamma;\psi},\ \gamma=\alpha+\beta-\alpha\beta.$$

2. *Let* $\beta=0$, *the left-sided R-L derivative can be presented as* ${}^{\mathbb{T}}\Delta^{\alpha} := {}^{\mathbb{T}}\Delta^{\alpha,0}$.
3. *Let* $\beta=0$, *left-sided Caputo fractional derivative can be presented as* ${}^{\mathbb{T}}_{c}\Delta^{\alpha} := {}^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\ {}^{\mathbb{T}}\Delta$.

Next, we review some lemmas which will be used to extabilish our existence results.

Lemma 2.8. *If* $\alpha>0$ *and* $\beta>0$, *there exist*

$$\left[{}^{\mathbb{T}}\mathfrak{J}^{\alpha}\left(\psi(s)-\psi(0)\right)^{\beta-1}\right](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\psi(t)-\psi(0)\right)^{\beta+\alpha-1}.$$

Lemma 2.9. *Let $\alpha \geq 0$, $\beta \geq 0$ and $\mathfrak{g} \in L^1(J)$. Then*

$$^{\mathbb{T}}\mathfrak{J}^{\alpha}\ ^{\mathbb{T}}\mathfrak{J}^{\beta}\mathfrak{g}(t) \overset{a.e}{=}\ ^{\mathbb{T}}\mathfrak{J}^{\alpha+\beta}\mathfrak{g}(t).$$

Lemma 2.10. *Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $\mathfrak{g} \in C_{\gamma}(J)$ and $^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g} \in C^1_{\gamma}(J)$, then*

$$^{\mathbb{T}}\mathfrak{J}^{\alpha}\ ^{\mathbb{T}}\Delta^{\alpha}\mathfrak{g}(t) = \mathfrak{g}(t) - \frac{\left(^{\mathbb{T}}\mathfrak{J}^{1-\alpha}\mathfrak{g}\right)(0)}{\Gamma(\alpha)}\left(\psi(t) - \psi(0)\right)^{\alpha-1}.$$

Lemma 2.11. *Suppose $\alpha > 0$, $a(t,\omega)$ is a nonnegative function locally integrable on $0 \leq t < b$ (some $b \leq \infty$), and let $g(t,\omega)$ be a nonnegative, nondecreasing continuous function defined on $0 \leq t < b$, such that $g(t,\omega) \leq K$ for some constant K. Further let $\mathfrak{u}(t,\omega)$ be a nonnegative locally integrable on $a \leq t < b$ function with*

$$|\mathfrak{u}(t,\omega)| \leq a(t,\omega) + g(t,\omega)\int_0^t \psi'(s)\left(\psi(t) - \psi(s)\right)^{\alpha-1}\mathfrak{u}(s,\omega)\Delta s,$$

with some $\alpha > 0$. Then

$$|\mathfrak{u}(t,\omega)| \leq a(t,\omega) + \int_0^t \left[\sum_{n=1}^{\infty}\frac{(g(t,\omega)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi'(s)\left(\psi(t) - \psi(s)\right)^{n\alpha-1}\right]\mathfrak{u}(s,\omega)\Delta s, \quad 0 \leq t < b.$$

Theorem 2.12. *[8](Schauder's Fixed Point Theorem) Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E and $N : Q \to Q$ is compact, and continuous map. Then N has at least one fixed point in Q.*

Theorem 2.13. *[8](Krasnoselskii's fixed point theorem) Let X be a Banach space, let Ω be a bounded closed convex subset of X and let T_1, T_2 be mapping from Ω into X such that $T_1x + T_2y, \in \Omega$ for every pair $x, y \in \Omega$. If T_1 is contraction and T_2 is completely continuous, then the equation $T_1x + T_2x = x$ has a solution on Ω.*

3. BVP for fractional RDEs on times scale

Here we list the following assumptions which are going to be useful in proving the results:

(H1) Let ℓ_g be a positive constant satisfies

$$|g(t, u, \omega) - g(t, v, \omega)| \leq \ell_g |u - v|.$$

(H2) Let m, n be a positive constants and $M(\omega) = \sup m(t, \omega)$, $N(\omega) = \sup n(t, \omega)$, such that

$$|g(t, u, \omega) - g(t, v, \omega)| \leq m(t, \omega) + n(t, \omega)\,|u(t, \omega)|.$$

(H3) For the increasing function $\phi \in C_{1-\gamma,\psi}(J)$, there exists $\lambda_{\phi} > 0$ such that

$$^{\mathbb{T}}\mathfrak{J}^{\alpha}\varphi(t) \leq \lambda_{\varphi}\varphi(t,\omega).$$

Theorem 3.1. *Assume (H2) hold. Then, Eq. (1) has at least one solution.*

Proof. Consider the operator $\mathscr{P}: C_{1-\gamma,\psi}(J) \to C_{1-\gamma,\psi}(J)$. The equivalent integral of (2) is of the operator form

$$(\mathscr{P}\mathfrak{u})(t,\omega) = \left(c - b\ ^{\mathbb{T}}\mathfrak{J}^{1-\beta+\alpha\beta;\psi}\mathfrak{g}(T,\mathfrak{u}(T,\omega),\omega)\right)\frac{(\psi(t)-\psi(0))^{\gamma-1}}{(a+b)\Gamma(\gamma)} +^{\mathbb{T}} \mathfrak{J}^{\alpha;\psi}\mathfrak{g}(t,\mathfrak{u}(t,\omega),\omega) \tag{5}$$

Define $B_r = \left\{\mathfrak{u} \in C_{1-\gamma,\psi}(J) : \|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \le r\right\}$. In order to prove the fixed point here we utilize Theorem 2.12. We prove the result in the following steps

Step 1: We check that $\mathscr{P}(B_r) \subset B_r$.

$$\begin{aligned}
&\left|(\psi(t)-\psi(0))^{1-\gamma}(\mathscr{P}\mathfrak{u})(t,\omega)\right| \\
&\le \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b}{(a+b)\Gamma(\gamma)}\frac{1}{\Gamma(1-\beta+\alpha\beta)}\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)^{1-\beta+\alpha\beta-1}|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,\Delta s \\
&\quad + \frac{(\psi(t)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,\Delta s \\
&\le \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b}{(a+b)\Gamma(\gamma)}\frac{1}{\Gamma(1-\beta+\alpha\beta)}\int_0^T \psi'(s)\left(\psi(T)-\psi(s)\right)^{1-\beta+\alpha\beta-1}\left(m(s,\omega)+n(s,\omega)|\mathfrak{u}(s,\omega)|\right)ds \\
&\quad + \frac{(\psi(t)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,ds, \quad \text{(by Proposition 2.3)} \\
&\le \frac{c}{(a+b)\Gamma(\gamma)} + \frac{b}{(a+b)\Gamma(\gamma)}\frac{M(\omega)}{\Gamma(2-\beta+\alpha\beta)}\left(\psi(T)-\psi(0)\right)^{1-\beta+\alpha\beta} + \frac{M(\omega)}{\Gamma(\alpha+1)}\left(\psi(T)-\psi(0)\right)^{\alpha+\gamma-1} \\
&\quad + \left(\frac{b}{(a+b)\Gamma(\gamma)}\frac{N(\omega)}{\Gamma(1-\beta+\alpha\beta)}B(\gamma,1-\beta+\alpha\beta)\left(\psi(T)-\psi(0)\right)^{\alpha} + \frac{N(\omega)}{\Gamma(\alpha)}B(\gamma,\alpha)\left(\psi(T)-\psi(0)\right)^{\alpha}\right)r \\
&\le r.
\end{aligned}$$

Which yields that $\mathscr{P}(B_r) \subset B_r$.

Next we prove that the operator $\mathscr{P}$ is completely continuous.

Step 2: The operator $\mathscr{P}$ is continuous.

Let $\mathfrak{u}_n$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in $C_{1-\gamma,\psi}(J)$. Then for each $t \in J$,

$$\|\mathscr{P}\mathfrak{u}_n - \mathscr{P}\mathfrak{u}\|_{C_{1-\gamma,\psi}} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 3: $\mathscr{P}(B_r)$ is relatively compact.

Thus $\mathscr{P}(B_r)$ is uniformly bounded. Let $t_1, t_2 \in J$, $t_1 < t_2$, then

$$\begin{aligned}
&\left|(\mathscr{P}\mathfrak{u})(t_2,\omega)\left(\psi(t_2)-\psi(0)\right)^{1-\gamma} - (\mathscr{P}\mathfrak{u})(t_1,\omega)\left(\psi(t_1)-\psi(0)\right)^{1-\gamma}\right| \\
&\le \left|\frac{(\psi(t_2)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_2}\psi'(s)\left(\psi(t_2)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s\right. \\
&\quad \left. - \frac{(\psi(t_1)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)\left(\psi(t_1)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)\left|\left(\psi(t_2)-\psi(0)\right)^{1-\gamma}\left(\psi(t_2)-\psi(s)\right)^{\alpha-1}\right. \\
&\quad \left. - \left(\psi(t_1)-\psi(0)\right)^{1-\gamma}\left(\psi(t_1)-\psi(s)\right)^{\alpha-1}\right| |\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,\Delta s \\
&\quad + \frac{(\psi(t_2)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\left(\psi(t_2)-\psi(s)\right)^{\alpha-1}|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)|\,\Delta s
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s) \Big| (\psi(t_2)-\psi(0))^{1-\gamma} (\psi(t_2)-\psi(s))^{\alpha-1} \\
&\quad - (\psi(t_2)-\psi(0))^{1-\gamma} (\psi(t_1)-\psi(s))^{\alpha-1} \Big| \left|\mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\right| ds \\
&\quad + \frac{(\psi(t_2)-\psi(0))^{1-\gamma}}{\Gamma(\alpha)} (\psi(t_2)-\psi(t_1))^{\alpha+\gamma-1} B(\gamma,\alpha) \left\|\mathfrak{g}\right\|_{C_{1-\gamma,\psi}}.
\end{aligned}
$$

Thus, right-hand side of the above inequality tends to zero. Hence along with the Arzëla-Ascoli theorem and from Step 1-3, it is concluded that $\mathscr{P}$ is completely continuous. Thus the proposed problem has at least one solution. □

Lemma 3.2. *Assume that (H1) is fulfilled. If*

$$
\left(\frac{b}{(a+b)\Gamma(\gamma)} \frac{B(\gamma, 1-\beta+\alpha\beta)}{\Gamma(1-\beta+\alpha\beta)} + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} \right) \ell_{\mathfrak{g}} \left(\psi(T)-\psi(0)\right)^{\alpha} < 1, \tag{6}
$$

then the problem (1) *has a unique solution.*

Next, we shall give the definitions and the criteria generalized U-H-R stability for ψ-HFD of dynamic equations on time scales.

Definition 3.3. *Eq.* (1) *is generalized U-H-R stable with respect to $\varphi \in C_{1-\gamma,\psi}(J)$ if there exists a real number $c_{\mathfrak{g},\varphi} > 0$ such that for each solution $\mathfrak{v} \in C_{1-\gamma,\psi}(J)$ of the inequality*

$$
\left| {}^{\mathbb{T}}\Delta^{\alpha,\beta}\mathfrak{v}(t,\omega) - \mathfrak{g}(t,\mathfrak{v}(t,\omega),\omega) \right| \leq \varphi(t), \tag{7}
$$

there exists a solution $\mathfrak{u} \in C_{1-\gamma,\psi}(J)$ of equation (1) *with*

$$
|\mathfrak{v}(t,\omega) - \mathfrak{u}(t,\omega)| \leq c_{\mathfrak{g},\varphi}\varphi(t,\omega), \quad t \in J.
$$

Theorem 3.4. *Assume that (H1), (H3) and* (6) *are satisfied. Then, the problem* (1) *is generalized U-H-R stable.*

4. Nonlocal fractional RDEs on times scale

Theorem 4.1. *Assume that [H1] and [H2] are satisfied. Then, Eq.*(3) *has at least one solution.*

Proof. Consider the operator $P : C_{1-\gamma,\psi}(J) \to C_{1-\gamma,\psi}(J)$, it is well defined and given by

$$
P\mathfrak{u}(t,\omega) = \begin{cases} \frac{T}{\Gamma(\alpha)} \left(\psi(t)-\psi(0)\right)^{\gamma-1} \sum_{i=1}^{m} c_i \displaystyle\int_0^{\tau_i} \psi'(s) \left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1} \mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s \\ +\frac{1}{\Gamma(\alpha)} \displaystyle\int_0^{t} \psi'(s) \left(\psi(t)-\psi(s)\right)^{\alpha-1} \mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s. \end{cases} \tag{8}
$$

Set $\widetilde{\mathfrak{g}}(s) = \mathfrak{g}(s,0,\omega)$. Consider the ball $B_r = \left\{ \mathfrak{u} \in C_{1-\gamma,\psi}(J) : \|\mathfrak{u}\|_{C_{1-\gamma,\psi}} \leq r \right\}$.
Now we subdivide the operator P into two operator P_1 and P_2 on B_r as follows

$$
P_1\mathfrak{u}(t,\omega) = \frac{T}{\Gamma(\alpha)} \left(\psi(t)-\psi(0)\right)^{\gamma-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi'(s) \left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1} \mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s
$$

and

$$
P_2\mathfrak{u}(t,\omega) = \frac{1}{\Gamma(\alpha)} \int_0^{t} \psi'(s) \left(\psi(t)-\psi(s)\right)^{\alpha-1} \mathfrak{g}(s,\mathfrak{u}(s,\omega),\omega)\Delta s.
$$

The proof is divided into several steps.

Step.1 $P_1\mathfrak{u}+P_2\mathfrak{y}\in B_r$ for every $\mathfrak{u},\mathfrak{y}\in B_r$. By direct computation and utilizing condition and with proposition 2.3 we obtain

$$\|P_1\mathfrak{u}+P_2\mathfrak{y}\|_{C_{1-\gamma,\psi}}\le\|P_1\mathfrak{u}\|_{C_{1-\gamma,\psi}}+\|P_2\mathfrak{y}\|_{C_{1-\gamma,\psi}}\le r.$$

where

$$\|P_1\mathfrak{u}\|_{C_{1-\gamma,\psi}}\le\frac{B(\gamma,\alpha)T}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha+\gamma-1}\left(\ell_{\mathfrak{g}}\|\mathfrak{u}\|_{C_{1-\gamma,\psi}}+\|\widetilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}\right)$$

and

$$\|P_2\mathfrak{u}\|_{C_{1-\gamma,\psi}}\le\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\psi(t)-\psi(0)\right)^{\alpha}\left(\ell_{\mathfrak{g}}\|\mathfrak{u}\|_{C_{1-\gamma,\psi}}+\|\widetilde{\mathfrak{g}}\|_{C_{1-\gamma,\psi}}\right).$$

Step.2 P_1 is a contration mapping.
For any $\mathfrak{u},\mathfrak{y}\in B_r$

$$\|P_1\mathfrak{u}-P_1\mathfrak{y}\|_{C_{1-\gamma,\psi}}\le\frac{\ell_{\mathfrak{g}}T}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha+\gamma-1}B(\gamma,\alpha)\|\mathfrak{u}-\mathfrak{y}\|_{C_{1-\gamma,\psi}}.$$

The operator P_1 is contraction.

Step.3 The operator P_2 is compact and continuous.
According to Step 1, we know that operator P_2 is uniformly bounded.
Now we prove the compactness of operator B.
For $0<t_1<t_2<T$, we have

$$|P_2\mathfrak{u}(t_1,\omega)-P_2\mathfrak{u}(t_2,\omega)|\le\|\mathfrak{g}\|_{C_{1-\gamma,\psi}}B(\gamma,\alpha)\left|(\psi(t_1)-\psi(0))^{\alpha+\gamma-1}-(\psi(t_2)-\psi(0))^{\alpha+\gamma-1}\right|$$

tending to zero as $t_1\to t_2$. Thus P_2 is equicontinuous. Hence, the operator P_2 is compact on B_r by the Arzela-Ascoli Theorem. We now conclude the result of the theorem based on the Theorem 2.13. □

Theorem 4.2. *If hypothesis (H1) and the constant*

$$\delta=\frac{\ell_{\mathfrak{g}}B(\gamma,\alpha)}{\Gamma(\alpha)}\left(T\sum_{i=1}^{m}c_i\left(\psi(\tau_i)-\psi(0)\right)^{\alpha+\gamma-1}+\left(\psi(T)-\psi(0)\right)^{\alpha}\right)<1$$

holds. Then, Eq. (3) *has unique solution.*

Theorem 4.3. *Let hypotheses (H1) and (H3) are fullfilled. Then Eq.*(3) *is generalized-U-H-R stable.*

[1] A. Ahmadkhanlu, M. Jahanshahi, On the existence and uniqueness of solution of initial value problem for fractional order differential equations on time scales, *B. Iran. Math. Soc.*, **38**, (2012), 241-252.

[2] R.P. Agarwal, M. Bohner, Basic calculus on time scales and some of its applications, *Results Math.*, **35**, (1999), 3-22.

[3] N. Benkhettou, A. Hammoudi, D. F. M. Torres, Existence and uniqueness of solution for a fractional Riemann-lioville initial value problem on time scales, *Journal of King Saud University-Science*, **28**,(2016),87-92.

[4] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhãuser, Boston, 2003.

[5] M. Bohner, A. Peterson, Dtnamica equations on times scale, Birkhauser Boston, Boston, MA.

[6] E. Capelas de Oliveira, J. Vanterler da C. Sousa, UlamHyersRassias stability for a class of fractional integro-differential equations, *Results Math.*, (2018), 73:111.

[7] K.M. Furati, M.D. Kassim, N.e-. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, *Comput. Math. Appl.*, **64**, (2012), 1616-1626.

[8] A. Granas, and J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.

[9] H. Gu, J.J. Trujillo, Existence of mild solution for evolution equation with Hilfer fractional derivative, *Appl. Math. Comput.*, **15**(2015), 344-354.

[10] S. Harikrishnan, Kamal Shah, Dumitru Baleanu, K. Kanagarajan, Note on the solution of random differential equations via ψ-Hilfer fractional derivative, *Advances in Difference Equations*, 2018, 2018:224.

[11] R.Hilfer, Application of fractional Calculus in Physics, World Scientific, Singapore, 1999.

[12] R. W. Ibrahim, Generalized Ulam-Hyers stability for fractional differential equations, *Int. J. Math.*, **23**,(2012).

[13] K.M. Kolwankar, A.D. Gangal, Fractional differentiability of nowhere differentiable functions and dimension, *Chaos*, **6** (1996) 505-513.

[14] A. Keten, M. Yavuz, D. Baleanu, Nonlocal Cauchy Problem via a Fractional Operator Involving Power Kernel in Banach Spaces, *Fractal and Fractional*, **3(2)**, (2019).

[15] V. Lupulescu, S.K. Ntouyas, Random fractional differential equations *International electronic journal of pure and applied mathematics*, **4**(2), 2012, 119-136.

[16] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[17] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach Sci. Publishers, Yverdon, 1993.

[18] T. T. Soong, Random Differential Equations in Science and Engineering. Academic Press, New York, 1973.

[19] T.A. Sulaiman, M. Yavuz, H. Bulut, H. M. Baskonus, Investigation of the fractional coupled viscous Burgers' equation involving Mittag Leffler kernel, *Physica A: Statistical Mechanics and its Applications*, **527**, (2019) 121-126.

[20] J. Vanterler da C. Sousa, E. Capelas de Oliveira, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci. Numer. Simul.*, In Press, Accepted Manuscript-2018.

[21] J. Vanterler da C. Sousa, E. Capelas de Oliveira, Ulam-Hyers stability of a nonlinear fractional Volterra integro-differential equation, *Appl. Math. Lett.*, Accepted Manuscript-2018.

[22] J. Vanterler da C. Sousa, D. Santos Oliveira, E. Capelas de Oliveira, On the existence and stability for noninstantaneous impulsive fractional integrodifferential equation, *Math. Meth. Appl. Sci.*, (2018) 113.

[23] J. Vanterler da C. Sousa, Kishor D. Kucche, E. Capelas de Oliveira, Stability of ψ-Hilfer impulsive fractional differential equations, *Appl. Math. Lett.*, (2018).

[24] D. Vivek, K. Kanagarajan, E.M. Elsayed, Some existence and stability results for Hilfer-fractional implicit differential equations with nonlocal conditions, *Mediterr. J. Math.*, **15**, (2018), 1-15.

[25] H. Vu, Random fractional functional differential equations, *International Journal of Nonlinear Analysis and Applications*, **7**(2), (2016), 253-267.

[26] J. Wang, L. Lv, Y. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electron J. Qual. Theory Differ. Equ.*, **63**, (2011), 1-10.