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RESEARCH ARTICLE / ARAŞTIRMA MAKALESİ

Stress-Strength Reliability Estimation for the Type I Extreme-Value Distribution Based on Records

I.Tip Uçdeğer Dağılımından Gelen Rekor Değerler İçin Stres Dayanıklılık Modelinin Güvenilirliğinin Tahmini

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Abstract

In this paper, we consider the stress-strength reliability $R = P(X > Y)$ for record data when the distribution of random stress Y and strength X have the type I extreme-value distribution. First, classical inference methods, namely uniformly minimum variance unbiased estimate (UMVUE) and maximum likelihood estimate (MLE), are used for R. Second, Bayesian inference of R are considered for gamma priors assumption. When the common parameter of stress and strength variables is known, the exact Bayes estimate and Bayesian credible interval of R are obtained. Markov Chain Monte Carlo (MCMC) method are used to derive the Bayes estimate and highest probability density (HPD) credible interval of R when the common parameter is unknown. Finally, Monte Carlo simulations are performed to compare the performance of the obtained estimates. A real data set about the weather temperature is analyzed to illustrate the performances of the derived estimators in the paper.

Keywords: Stress-strength model, Record values, Extreme-value distribution, Bayesian estimation.

Öz

Bu çalışmada, stres Y ve dayanıklılık X rastgele değişkenleri I. Tip uçdeğer dağılımına sahip olduğunda rekor değerler için stres dayanıklılık modelinin güvenilirliği $R = P(X > Y)$ ele alınmıştır. İlk olarak R için klasik yaklaşım yani değişmez en küçük varyanslı yansz minimum varyans tahmin edici ve en çok olabilirlik tahmin edicisi kullanılmıştır. Sonra, önsellerin gamma dağılımına sahip olması varsayımı altın R için Bayes yaklaşımı ele alınmıştır. Stres ve dayanıklılık değişkenlerinin ortak parametresi biliniyorken, R nin kesin Bayes tahmin edicisi ve Bayes güven aralığı elde edilmiştir. Stres ve dayanıklılık değişkenlerinin ortak parametresi bilinmiyorken, R'nin Bayes tahmin edicisi ve en yüksek olasılık yoğunluklu Bayes güven aralığı Markov Zinciri Monte Carlo (MCMC) metodu ile elde edilmiştir. Son olarak elde edilen tahmin edicilerin performanslarını karşılaştırmak için Monte Carlo simülasyonu gerçekleştirildi. Elde edilen tahmin edicilerin performanslarını göstermek için hava sıcaklıkları ile ilgili gerçek veri seti analiz edilmiştir.

Anahtar Kelimeler: Stres dayanıklılık modeli, Rekor değerler, Uçdeğer dağılımı, Bayes tahmini.

I. INTRODUCTION

In the literature, there are many lifetime distributions exist. It is known that to introduce new distributions or distribution families are also popular topic in recent years. The exponential and Weibull distributions are commonly used in many different areas and applications, see Murthy et al. [1]. The hazard rate function of the exponential distribution is constant and it is increasing or decreasing or constant for Weibull distribution. Hence, these distributions cannot be used to modelling for some data. That is why some extension and modified versions of Weibull distribution are proposed. Lai and Xie [2] introduced the new modified Weibull distribution (NMWD) with cumulative distribution function (cdf) and probability density function (pdf) are given by, respectively,

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$$F(x) = 1 - \exp(-\alpha x^b e^{\lambda x}), \quad (1)$$

$$f(x) = \alpha(b + \lambda x)x^{b-1} e^{\lambda x} \exp(-\alpha x^b e^{\lambda x}) \quad (2)$$

with parameters $\alpha > 0$, $b \geq 0$ and $\lambda > 0$. When $b = 0$ in (1), the NVWD reduces to

$$F(x) = 1 - \exp(-\alpha e^{\lambda x}), -\infty < x < \infty \quad (3)$$

which is a type I extreme-value distribution and is also known as a log-gamma distribution. The pdf of type I extreme-value distribution is

$$f(x) = \alpha\lambda e^{\lambda x} \exp(-\alpha e^{\lambda x}), -\infty < x < \infty, \quad (4)$$

and it is denoted by $EV(\alpha, \lambda)$.

Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables. An observation X_k is called an upper record value if its value exceeds all previous observations, i.e. $X_k > X_i$ for $k > i$. Using this analogous, the definition of lower record values can be given similarly. People are interested records such as weather records, sports records etc. in the real life. Also, records can be seen in life testing if one wants to observe only the minimum or maximum value of testing materials. The main concept of the record values was first introduced by Chandler [3]. Since then the statistical inferences of the records are considered by many researchers, for detailed references see Arnold et al. [4] and Ahsanullah and Nevzorov [5].

In the reliability literature, the probability of the random strength X of a component exceeds the random stress Y experienced by the system is called stress – strength reliability and defined as $R = P(X > Y)$. This problem was first introduced by Birnabum [6]. Since then statistical inference of R has been considerably studied by many researchers under different distributional assumptions and data types. Kotz et al. [7] present a great review for the development of the stress-strength reliability. Some recent contributions about the statistical inferences of reliability can be found the following papers Tavirdizade and Gharehchobogh [8], Basirat et al. [9], Kızılaslan and Nadar [10], Rasethuntsa and Nadar [11] and Çetinkaya and Genç [12].

In this paper, the statistical inference of reliability is considered in stress-strength setup when the underlying random variables are independent and follow the type I extreme-value distribution with parameters (β, λ) and (α, λ), respectively. When the common parameter λ is unknown, Bayes estimate and HPD credible interval of R have been developed by using MCMC method. When the common parameter λ is known, the MLE, UMVUE and exact Bayes estimates, as well as exact confidence and Bayesian credible intervals of R are derived. In this case, we also obtain Bayes estimates using MCMC to see the performance of the exact Bayes estimate.

The paper is organized as follows. In Section 2, classical inference of R is considered for both λ is known and unknown cases. In Section 3, Bayes estimate and HPD credible interval of R are developed in exactly and approximately when the parameters have independent gamma priors. In Section 4, the performance of the obtained point estimates and intervals of R are compared by using Monte Carlo simulations. Some plots are presented to see the difference of estimates performance. Furthermore, a temperature data set is used to illustrate the findings. Finally, the paper is concluded in Section 5.

II. CLASSICAL INFERENCE OF *R*

When the common parameter λ is known and unknown, the ML and UMVU estimates of $R = P(X > Y)$ are obtained. Also, the exact confidence interval of R is constructed for λ is known case.

Let the strength X and stress Y be independent random variables from the type I extreme-value distribution with parameters (β, λ) and (α, λ), respectively. Then, the stress-strength reliability is

$$R = P(X > Y) = \int_{-\infty}^{\infty} P(X > Y | Y = y) f_Y(y) dy = \frac{\beta}{\alpha + \beta}. \quad (5)$$

In this study, we assume that the stress and strength random variables follow from the type I extreme value distribution. Let $R_1, \ldots, R_n$ and $S_1, \ldots, S_m$ are independent set of upper records from $EV(\alpha, \lambda)$ and $EV(\beta, \lambda)$, respectively. Then, joint pdf of based on $(R_1, \ldots, R_n, S_1, \ldots, S_m)$ is obtained as using Arnold et el. [4]

$$L(\alpha, \beta, \lambda | \underline{r}, \underline{s}) = f(r_n) \prod_{i=1}^{n-1} \frac{f(r_i)}{1 - F(r_i)} g(s_m) \prod_{j=1}^{m-1} \frac{g(s_j)}{1 - G(s_j)}, -\infty < r_1 < \cdots < r_n < \infty, -\infty < s_1 < .. < s_m < \infty$$

where $\underline{r} = (r_1, \ldots, r_n)$ and $\underline{s} = (s_1, \ldots, s_m)$, f and F are the pdf and cdf of stress variables from $EV(\alpha, \lambda)$ and g and G are the pdf and cdf of strength variables from $EV(\beta, \lambda)$. Then, we have

$$L(\alpha, \beta, \lambda | \underline{r}, \underline{s}) = \alpha^n \beta^m \lambda^{n+m} e^{\lambda(\sum_{i=1}^n r_i + \sum_{j=1}^m s_j)} \exp(-\alpha e^{\lambda r_n}) \exp(-\beta e^{\lambda s_m}) \quad (6)$$

The ML estimates of α, β and λ have a closed forms and are given by

$$\hat{\alpha}_{MLE} = \frac{n}{e^{\hat{\lambda}_{MLE} R_n}}, \hat{\beta}_{MLE} = \frac{m}{e^{\hat{\lambda}_{MLE} S_m}}, \hat{\lambda}_{MLE} = \frac{n+m}{\sum_{i=1}^n (R_n - R_i) + \sum_{j=1}^m (S_m - S_j)}. \quad (7)$$

Then, the MLE of R, $\hat{R}_{MLE}$, is given by $\hat{R}_{MLE} = \hat{\beta}_{MLE} / (\hat{\beta}_{MLE} + \hat{\alpha}_{MLE})$.

If the common parameter λ is known i.e. $\lambda = \lambda_0$, we can find the distribution of $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$. It is readily obtained that $2\alpha e^{\lambda_0 R_n} \sim \chi^2_{2n}$ and $2\beta e^{\lambda_0 S_m} \sim \chi^2_{2m}$. Using simple transformations, the pdf of $\hat{R}_{MLE}$ is derived as

$$f_{\hat{R}_{MLE}}(r) = \frac{1}{B(n,m)}\left(\frac{m\beta}{n\alpha}\right)^{m} \frac{\left(\frac{1-r}{r}\right)^{m-1}}{\left(1+\frac{m\beta(1-r)}{n\alpha r}\right)^{n+m}}, 0 < r < 1.$$

Since $F = \frac{R}{1-R}\frac{1-\hat{R}_{MLE}}{\hat{R}_{MLE}} \sim F_{2m,2n}$, the $100(1-\gamma)\%$ exact confidence interval of R is derived as

$$\left(\left[1 + F_{2n,2m}(\gamma/2)\left(\frac{1-\hat{R}_{MLE}}{\hat{R}_{MLE}}\right)\right]^{-1}, \left[1 + F_{2n,2m}(1-\gamma/2)\left(\frac{1-\hat{R}_{MLE}}{\hat{R}_{MLE}}\right)\right]^{-1}\right)$$
(8)

where $P\left(F \geq F_{2n,2m}\left(\frac{\gamma}{2}\right) = \frac{\gamma}{2}\right.$ is the $\left(1-\frac{\gamma}{2}\right)$th percentile points of a F distribution with $(2n, 2m)$ degrees of freedom.

Moreover, the UMVUE of R can be derived. In this case, the joint likelihood function is

$$L(\alpha,\beta|\lambda_0,\underline{r},\underline{s}) = \lambda_0^{\,n+m} e^{\lambda_0(\sum_{i=1}^{n} r_i + \sum_{j=1}^{m} s_j)} \alpha^n \beta^m \exp(-\alpha e^{\lambda_0 r_n}) \exp(-\beta e^{\lambda_0 s_m})$$

and $(T_1, T_2) = (e^{\lambda_0 R_n}, e^{\lambda_0 S_m})$ are the complete sufficient statistics for (α,β) and follow Gamma distributions with parameters (n,α) and (m,β), respectively. Let

$$\phi(R_1, S_1) = \begin{cases} 1 \ if \ R_1 > S_1 \\ 0 \ if \ R_1 \leq S_1 \end{cases}$$

where R_1 and S_1 are the first record values. Since $P_1 = e^{\lambda_0 R_1}$ and $P_2 = e^{\lambda_0 S_1}$ follow exponential distributions with means $1/\alpha$ and $1/\beta$, we can obtain that $E(\phi(R_1,S_1)) = R$. It is easily seen that the conditional distributions are derived by using Lemma 1 in Basirat et al. [13]

$$f_{P_1|T_1=t_1}(p_1|t_1) = (n-1)\frac{1}{t_1}\left(1-\frac{p_1}{t_1}\right)^{n-2}, 0 < p_1 < t_1$$

$$f_{P_2|T_2=t_2}(p_2|t_2) = (m-1)\frac{1}{t_2}\left(1-\frac{p_2}{t_2}\right)^{m-2}, 0 < p_2 < t_2$$

Then, the UMVUE of R, $\hat{R}_{UMVUE}$, is obtained by using Lehmann-Scheffe's Therorem

$$\hat{R}_{UMVUE} = E(\phi(R_1,S_1)|T_1 = t_1, T_2 = t_2) = \iint_{P_1>P_2} f_{P_1|T_1=t_1}(p_1|t_1) f_{P_2|T_2=t_2}(p_2|t_2)dp_1\,dp_2 = \begin{cases} \sum_{i=0}^{n-1}(-1)^i\left(\frac{t_2}{t_1}\right)^i \frac{\binom{n-1}{i}}{\binom{m+i-1}{i}} \ if \ t_2 < t_1 \\ 1 - \sum_{j=0}^{m-1}(-1)^i\left(\frac{t_1}{t_2}\right)^i \frac{\binom{m-1}{i}}{\binom{n+i-1}{i}} \ \ if \ t_1 \geq t_2 \end{cases} \quad (9)$$

III. BAYESIAN INFERENCE OF R

In this section, it is assumed that the parameters α, β and λ are statistically independent random variables and follow gamma priors with parameters $(a_i, b_i), i = 1,2,3$, respectively. If the random variable X follows gamma distribution (a, b), i.e. by $X \sim Gamma(a,b)$, then its pdf is given as

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, x > 0, a, b > 0$$

We obtain the joint posterior density of α, β and λ given data $(\underline{r}, \underline{s})$ as follows

$$\pi(\alpha,\beta,\lambda|\underline{r},\underline{s}) = I(\underline{r},\underline{s})\, \alpha^{n+a_1-1} e^{-\alpha(b_1+e^{\lambda_0 r_n})}\, \beta^{m+a_2-1}\, e^{-\beta(b_2+e^{\lambda_0 s_m})}\, \lambda^{n+m+a_3-1} e^{-\lambda(b_3-\sum_{i=1}^{n} r_i - \sum_{j=1}^{m} s_j)} \quad (10)$$

where $I(\underline{r},\underline{s})$ is the normalizing constant. Then, under the squared error loss (SEL) function, the Bayes estimate of R, $\hat{R}_{Bayes}$, is given by

$$\hat{R}_{Bayes} = \int_0^\infty \int_0^\infty \int_0^\infty R\ \pi(\alpha,\beta,\lambda|\underline{r},\underline{s})\ d\alpha\, d\beta\, d\lambda. \quad (11)$$

Since the integral in Equation (11) cannot be obtained explicitly, we use the MCMC method to obtain the point estimate and HPD credible interval of R. In the MCMC method, samples are generated from the posterior distributions and then Bayes estimates are computed by using these samples. The marginal posterior density functions of α, β and λ given data $(\underline{r}, \underline{s})$ are obtained as

$$\alpha|\lambda,\underline{r},\underline{s} \sim \text{Gamma}(\text{n} + a_1, b_1 + e^{\lambda r_n}),\ \beta|\lambda,\underline{r},\underline{s} \sim \text{Gamma}(\text{m} + a_2, b_2 + e^{\lambda s_m})$$
(12)

and

$$\pi(\lambda|\alpha,\beta,\underline{r},\underline{s}) \propto \lambda^{n+m+a_3-1} e^{-\lambda(b_3-\sum_{i=1}^{n} r_i - \sum_{j=1}^{m} s_j)} e^{-\alpha e^{\lambda r_n}} e^{-\beta e^{\lambda s_m}}$$
(13)

It is clear that samples from α and β are generated easily from the Gamma distributions. However, the posterior distribution of λ is not well known distribution. Normal distribution can be used to approximate the posterior density function, when it is unimodal and roughly symmetric (see Gelman et al., [14]). Since $\pi(\lambda|\alpha,\beta,\underline{r},\underline{s})$ is log-concave function of λ, the hybrid Metropolis-Hastings and Gibbs sampling algorithm can be used in our case. In this algorithm, the Metropolis-Hastings scheme is combined with the Gibbs sampling scheme under the Gaussian proposal distribution. The following algorithm is used

Step 1. Start with initial point $\lambda_{(0)}$.

Step 2. Set $i = 1$.

Step 3. ith value of α, i.e. $\alpha_{(i)}$, is generated from $\mathrm{Gamma}\left(n + a_1, b_1 + e^{\lambda_{(i-1)} r_n}\right)$.

Step 4. ith value of β, i.e. $\beta_{(i)}$, is generated from $\mathrm{Gamma}\left(m + a_2, b_2 + e^{\lambda_{(i-1)} s_m}\right)$.

Step 5. $\lambda_{(i)}$ is generated from $\pi\left(\lambda | \alpha, \beta, \underline{r}, \underline{s}\right)$ using the Metropolis-Hastings algorithm under the proposal distribution $q(\lambda)$ follows $N(\lambda_{(i-1)}, 1)$. It is given as follows

(a) Let $v = \lambda_{(i-1)}$.

(b) w is generated from the proposal distribution q.

(c) Let $p(v, w) = min\left\{1, \frac{\pi(w|\alpha_{(i)}, \beta_{(i)}, \underline{r}, \underline{s})\, q(v)}{\pi(v|\alpha_{(i)}, \beta_{(i)}, \underline{r}, \underline{s})\, q(w)}\right\}$.

(d) Generate u from $Uniform(0,1)$. If $u \le p(v, w)$, then accept the proposal and set $\lambda_{(i)} = w$, otherwise set $\lambda_{(i)} = v$.

Step 6. The stress-strength reliability is computed as $R_{(i)} = \beta_{(i)}/(\beta_{(i)} + \alpha_{(i)})$.

Step 7. Set $i = i + 1$.

Step 8. Repeat Steps 2 through – 7, T times and obtain the posterior sample $R_{(i)}, i = 1, \ldots, T$.

Then, the Bayes estimate of R under the SEL function is given by

$$\hat{R}_{MCMC} = \frac{1}{T-M} \sum_{i=M+1}^{T-M} R_{(i)} \quad (14)$$

where M is the burn-in period. Using the method in Chen and Shao [15], the HPD credible interval of R is constructed by using these samples.

If the common parameter λ is known, i.e. $\lambda = \lambda_0$, then the Bayes estimate of R is derived explicitly in terms of Gauss hypergeometric function. In this case, it is assumed that α and β are statistically independent random variables and follow gamma priors with parameters $(a_i, b_i), i = 1,2$, respectively. Then, the joint posterior density of (α, β) given data $(\underline{r}, \underline{s})$ and posterior density of R are derived as

$$\pi\left(\alpha, \beta | \lambda_0, \underline{r}, \underline{s}\right) = \frac{{\varphi_1}^{\delta_1} {\varphi_2}^{\delta_2}}{\Gamma(\delta_1)\Gamma(\delta_2)} \alpha^{\delta_1 - 1} \beta^{\delta_2 - 1} e^{-\alpha\varphi_1} e^{-\beta\varphi_2}$$

and

$$f_R(r) = \frac{{\varphi_1}^{\delta_1} {\varphi_2}^{\delta_2}}{Beta(\delta_1, \delta_2)} \frac{r^{\delta_2 - 1}(1-r)^{\delta_1 - 1}}{(r\varphi_2 + (1-r)\varphi_1)^{\delta_1 + \delta_2}}, 0 < r < 1$$

where $\delta_1 = n + a_1$, $\delta_2 = m + a_2$, $\varphi_1 = b_1 + e^{\lambda_0 r_n}$ and $\varphi_2 = b_2 + e^{\lambda_0 s_m}$. Under the SEL function, the exact Bayes estimate of R, $\hat{R}_{Exact,B}$, is obtained as

$$\hat{R}_{Exact,B} = \int_0^1 r f_R(r) dr = \begin{cases} \frac{\delta_2}{\delta_1 + \delta_2} \left(\frac{\varphi_2}{\varphi_1}\right)^{\delta_2} F_{2,1}\left(\delta_1 + \delta_2, \delta_2 + 1; \delta_1 + \delta_2 + 1, 1 - \frac{\varphi_2}{\varphi_1}\right) & if\ \varphi_2 < \varphi_1 \\ \frac{\delta_2}{\delta_1 + \delta_2} \left(\frac{\varphi_1}{\varphi_2}\right)^{\delta_2} F_{2,1}\left(\delta_1 + \delta_2, \delta_1; \delta_1 + \delta_2 + 1, 1 - \frac{\varphi_1}{\varphi_2}\right) & if\ \varphi_1 \le \varphi_2 \end{cases} \quad (15)$$

where $F_{2,1}(.,.;.,.)$ is the Gauss hypergeometric function and

$F_{21}(\alpha, \beta; \gamma, w) = \int_0^\infty t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-ct)^{-\alpha}\, dt, |c| <$

We also use MCMC method to evaluate the Bayes estimate of R R. Hence, we can compare the alternative method results with the exact results.

For the MCMC case, using the Gibbs sampling algorithm, we generate the samples of α and β from $\mathrm{Gamma}\left(n + a_1, b_1 + e^{\lambda_0 r_n}\right)$ and $\mathrm{Gamma}\left(m + a_2, b_2 + e^{\lambda_0 s_m}\right)$. Then, Bayes estimate and HPD credible interval of R are computed similar to λ is unknown case.

Moreover, we can easily obtain the Bayesian credible interval of R using the relations $2\varphi_1\alpha | \lambda_0, \underline{r} \sim \chi^2_{2(n+a_1)}$ and $2\varphi_2\beta | \lambda_0, \underline{s} \sim \chi^2_{2(m+a_2)}$. Then, we have $\frac{2\varphi_1\alpha|\lambda_0, \underline{r}\ /2(n+a_1)}{2\varphi_2\beta|\lambda_0, \underline{s}\ /2(m+a_2)} \sim F_{2(n+a_1), 2(m+a_2)}$. Hence, the $100(1-\gamma)\%$ Bayesian credible interval for R is obtained as

$$\left(\left[1 + F_{2(n+a_1), 2(m+a_2)}(\gamma/2)\left(\frac{\varphi_2(n+a_1)}{\varphi_1(m+a_2)}\right)\right]^{-1}, \left[1 + F_{2(n+a_1), 2(m+a_2)}(1-\gamma/2)\left(\frac{\varphi_2(n+a_1)}{\varphi_1(m+a_2)}\right)\right]^{-1}\right) \quad (16)$$

where $F_{2(n+a_1), 2(m+a_2)}(\gamma/2)$ and $F_{2(n+a_1), 2(m+a_2)}(1-\gamma/2)$ are the $(1-\frac{\gamma}{2})$th and $\frac{\gamma}{2}th$ percentile points of a F distribution with $(2(n+a_1), 2(m+a_2))$ degrees of freedom.

IV. SIMULATION STUDY

In this section, some numerical results are presented for the obtained estimates of type I extreme-value distribution based on upper records. The MSEs of the classic estimates (i.e. MLE and UMVUE) and estimated risks (ERs) of Bayesian estimate are listed in tables. The performance of the point estimates is compared by using MSE and ER values. The confidence and credible intervals and their corresponding coverage probabilities (cps) are also listed in tables. The performance of the interval estimates is compared by using average lengths and cps. When θ is estimated by $\hat{\theta}$, the ER of θ under the SEL function is given by $ER(\theta) = \sum_{i=1}^{N}(\hat{\theta}_i - \theta_i)^2/N$. All the simulations results are based 2500 replications.

For the common parameter λ is known ($\lambda = 3$), the ML, UMVU and Bayes estimates of R and their MSEs and ERs are given by using Equations (5), (7), (9) and (15) in Table 1. The point and interval estimates are evaluated for $R = 0.2308, 0.6000$ and 0.9231 when $(\alpha, \beta) = (5,1.5), (8,12)$ and $(2,24)$, respectively. In the Bayesian case, Prior 1: $(a_1, b_1) = (5,1), (a_2, b_2) = (3,2)$, Prior 2: $(a_1, b_1) = (5,1), (a_2, b_2) = (3,2)$ and Prior 3: $(a_1, b_1) = (1,2), (a_2, b_2) = (6,1/4)$ are used for $R = 0.2308, 0.6000$ and 0.9231, respectively. In the MCMC case, 5000 samples are generated for each step and using these samples Bayes estimate and HPD credible interval of R are computed. The Bayesian credible interval of R is also computed by using Equation (16). From Table 1, we observe that when R approaches to tails, the MLE and UMVUE have similar performance. When R is around 0.5, the MLE has good performance with respect to UMVUE. The ERs of Bayes estimates has smaller than that of MLE and UMVUE in all cases. The estimate and ER of Bayes estimate which is obtained by using MCMC method are very close to the exact Bayes estimate. The average lengths of the HPD credible intervals are smaller than other intervals but its cp values are close to nominal value as the sample size increases. However, the exact confidence intervals can be preferable to the other intervals with respect to the cp values.

Moreover, some graphs of R vs MSEs and ERs (for exact Bayes estimate) and R vs Biases are presented in Figures (1)-(6) to see the performance of the obtained estimates when $\lambda = 2$. These graphs are plotted based on the ML, UMVU and exact Bayes estimates of R for $(n, m) = (5,5), (5,8), (8,8), (10,12), (12,8)$ and $(15,12)$. In these figures, the true values of R are taken from 0.0476 to 0.9921 and Monte Carlo simulation is carried for each R value based on 2500 replications. As the sample size increases, the MSEs, ERs and Biases of the estimates decrease. When R is near to tails, the MSEs, ERs and Biases of the estimates are small. However, these values are large, when R is near to 0.5. The ERs of the Bayes estimates are smaller than that of MLE and UMVUE in all cases. In addition, the MLE has good performance with respect to UMVUE when R is around 0.5 and their performances are similar when R is near to tails. The similar outcomes are also observed in Table 1.

For the common parameter λ is unknown, the ML and Bayes estimates using MCMC method and their MSE and ERs are tabulated in Table 2. The point and interval estimates are evaluated for $R = 0.2500, 0.4444, 0.6250$ and 0.9000 when $(\alpha, \beta, \lambda) = (12,4,4), (5,4,3), (3,5,2)$ and $(0.5,4.5,2.5)$, respectively. In the Bayesian case, Prior 4:

Table 1. Estimates of R when $\lambda = 3$ (Note: 1st row estimates (interval), 2nd row MSE or ER (length/cp))

(n, m)	R	$\hat{R}_{MLE}$	$\hat{R}_{UMVUE}$	$\hat{R}_{Exact,B}$	$\hat{R}_{MCMC}$	$Exact\ C.I.$	$Bayesian\ C.I.$	$HPD\ C.I.$
(5,5)	0.2308	0.2481	0.2294	0.2456	0.2457	(0.0875,0.5229)	(0.1058,0.4371)	(0.0933,0.4151)
		0.0141	0.0147	0.0039	0.0039	0.4357/0.9528	0.3313/0.9964	0.3218/0.9928
(8,8)		0.2419	0.2301	0.2433	0.2433	(0.1071,0.4550)	(0.1200,0.4064)	(0.1101,0.3897)
		0.0085	0.0087	0.0036	0.0036	0.3479/0.9508	0.2864/0.9876	0.2796/0.9828
(10,10)		0.2400	0.2305	0.2422	0.2423	(0.1162,0.4284)	(0.1269,0.3920)	(0.1182,0.3777)
		0.0066	0.0067	0.0033	0.0033	0.3122/0.9540	0.2651/0.9828	0.2595/0.9740
(12,12)		0.2371	0.2291	0.2400	0.2399	(0.1226,0.4068)	(0.1318,0.3789)	(0.1237,0.3659)
		0.0054	0.0055	0.0030	0.0030	0.2843/0.9544	0.2471/0.9764	0.2422/0.9700
(15,15)		0.2353	0.2289	0.2384	0.2384	(0.1309,0.3846)	(0.1384,0.3646)	(0.1315,0.3537)
		0.0045	0.0045	0.0027	0.0027	0.2537/0.9504	0.2263/0.9764	0.2223/0.9700
(5,5)	0.6000	0.5884	0.5970	0.5919	0.5919	(0.302,0.8273)	(0.3807,0.7840)	(0.3892,0.7888)
		0.0221	0.0262	0.0050	0.0050	0.5253/0.9424	0.4032/0.9960	0.3996/0.9932
(8,8)		0.5913	0.5968	0.5923	0.5923	(0.3572,0.7915)	(0.4066,0.7624)	(0.4133,0.7663)
		0.0139	0.0155	0.0049	0.0049	0.4343/0.9496	0.3558/0.9912	0.3529/0.9856
(10,10)		0.6003	0.6051	0.5979	0.5978	(0.3887,0.7810)	(0.4249,0.7560)	(0.4313,0.7598)
		0.0111	0.0121	0.0046	0.0046	0.3923/0.9480	0.3311/0.9884	0.3285/0.9820
(12,12)		0.5939	0.5977	0.5936	0.5936	(0.3996,0.7633)	(0.4307,0.7438)	(0.4362,0.7469)
		0.0094	0.0101	0.0044	0.0044	0.3637/0.9524	0.3130/0.9844	0.3107/0.9788
(15,15)		0.5945	0.5976	0.5941	0.5942	(0.4196,0.7488)	(0.4437,0.7334)	(0.4486,0.7863)

		0.0073	0.0077	0.0039	0.0039	0.3292/0.9576	0.2898/0.9836	0.2877/0.9744
(5,5)	0.9231	0.9078	0.9214	0.9400	0.9400	(0.7410,0.9725)	(0.8675,0.9809)	(0.8817,0.9864)
		0.0036	0.0028	0.0006	0.0006	0.2316/0.9468	0.1133/0.9624	0.1047/0.9192
(8,8)		0.9159	0.9239	0.9353	0.9353	(0.8026,0.9673)	(0.8718,0.9741)	(0.8827,0.9791)
		0.0017	0.0015	0.0005	0.0005	0.1647/0.9568	0.1023/0.9648	0.0964/0.9196
(10,10)		0.9167	0.9230	0.9327	0.9327	(0.8201,0.9641)	(0.8734,0.9705)	(0.8829,0.9750)
		0.0013	0.0012	0.0004	0.0004	0.1440/0.9520	0.0971/0.9676	0.0922/0.9336
(12,12)		0.9184	0.9236	0.9316	0.9316	(0.8344,0.9620)	(0.8762,0.9678)	(0.8845,0.9720)
		0.0011	0.0010	0.0004	0.0004	0.1276/0.9468	0.0916/0.9644	0.0875/0.9372
(15,15)		0.9185	0.9227	0.9294	0.9294	(0.8460,0.9588)	(0.8782,0.9641)	(0.8853,0.9679)
		0.0008	0.0007	0.0004	0.0004	0.1128/0.9512	0.0859/0.9624	0.0826/0.9324

Table 2. Estimates of R when λ is unknown (Note: 1st row estimates (interval), 2nd row MSE or ER (length/cp))

(n,m)	R	$\hat{R}_{MLE}$	$\hat{R}_{MCMC}$	$HPD\ C.I.$	R	$\hat{R}_{MLE}$	$\hat{R}_{MCMC}$	$HPD\ C.I.$
(5,5)	0.2500	0.2226	0.2805	(0.1391,0.4335	0.6250	0.6478	0.6255	(0.4130,0.8273)
		0.0238	0.0025	0.2944/0.9988		0.0341	0.0069	0.4143/0.9916
(8,8)		0.2324	0.2744	(0.1437,0.4150)		0.6370	0.6245	(0.4381,0.8033)
		0.0140	0.0025	0.2714/0.9952		0.0187	0.0065	0.3652/0.9832
(10,10)		0.2340	0.2704	(0.1459,0.4038)		0.6361	0.6262	(0.4532,0.7922)
		0.0104	0.0023	0.2579/0.9940		0.0154	0.0065	0.3390/0.9684
(12,12)		0.2345	0.2671	(0.1483,0.3941)		0.6367	0.6279	(0.4656,0.7838)
		0.0086	0.0022	0.2458/0.9900		0.0118	0.0057	0.3181/0.9664
(15,15)		0.2353	0.2634	(0.1522,0.3822)		0.6332	0.6265	(0.4769,0.7705)
		0.0065	0.0020	0.2300/0.9896		0.0092	0.0051	0.2936/0.9568
(5,5)	0.4444	0.4502	0.4504	(0.2408,0.6635)	0.9000	0.9192	0.8927	(0.7737,0.9831)
		0.0407	0.0078	0.4227/0.9944		0.0064	0.0017	0.2093/0.9928
(8,8)		0.4405	0.4449	(0.2601,0.6329)		0.9167	0.8971	(0.7942,0.9779)
		0.0202	0.0067	0.3727/0.9820		0.0040	0.0016	0.1838/0.9776
(10,10)		0.4377	0.4425	(0.2704,0.6177)		0.9110	0.8959	(0.7995,0.9735)
		0.0160	0.0064	0.3472/0.9736		0.0034	0.0016	0.1740/0.9648
(12,12)		0.4434	0.4461	(0.2845,0.6103)		0.9110	0.8975	(0.8076,0.9709)
		0.0130	0.0060	0.3258/0.9648		0.0027	0.0013	0.1632/0.9636
(15,15)		0.4433	0.4457	(0.2965,0.5971)		0.9080	0.8974	(0.8150,0.9663)
		0.0096	0.0051	0.3006/0.9668		0.0023	0.0013	0.1512/0.9524

$(a_1,b_1)=(12,1),(a_2,b_2)=(8,2),(a_3,b_3)=(8,2)$, Prior 5: $(a_1,b_1)=(5,1),(a_2,b_2)=(4,1),(a_3,b_3)=(3,1)$, Prior 6: $(a_1,b_1)=(3,1),(a_2,b_2)=(5,1),(a_3,b_3)=(4,2)$ and Prior 7: $(a_1,b_1)=(1,2),(a_2,b_2)=(9,2),(a_3,b_3)=(5,2)$ are used for $R=0.2500, 0.4444, 0.6250$ and 0.9000, respectively. In the MCMC case, two MCMC chains are used with different initial points and 6000 iterations are generated for each chain. The first 1000 draws is discarded and focus on the other 5000 iterations for diminishing the effect of the starting distribution. In computing of Bayes estimates, we use only every 5th sample values after discarding the first 1000 iterations because of breaking the dependency in the Markov chains. Gelman et al. [12] proposed the scale reduction factor estimate for the convergence of MCMC simulations. This index is used in our MCMC part for more details see Gelman et al. [12]. The scale factor of the MCMC Bayes estimates are smaller than 1.1 in our simulation studies. It means the MCMC method is converged. From Table 2, it is observed that the Bayes estimate has good performance with respect to the MLE. The MSE and ER of estimates and average lengths decrease when the sample size increases.

As a real data analysis, we use the monthly average temperatures (in Celsius) Reykjavik, Iceland which is located close to the North Pole. It is observed that the monthly average temperatures of February and March are – 0.3 and 0.4, respectively from 1870 to 2011. The data sets of February $(\underline{r})$ and March $(\underline{s})$ months from 1970 to 2011 are considered (it can be downloaded from https://crudata.uea.ac.uk/cru/data/temperature/) and their corresponding upper records data are listed in Table 3. We have checked to see whether type I extreme-value distribution is adequate to fit these two data sets or not. The Kolmogorov-Smirnov (K-S) distances between fitted and the empirical distribution functions and corresponding p-values, the estimates of the parameters and stress-

Table 3. Upper record values from February and March

i	1	2	3	4	5
$\underline{r}$ (February)	-2.2	0.6	2.4	2.9	3.3
$\underline{s}$ (March)	-1.7	1.3	2.1	3.7	3.9

Table 4. Real data analysis

Kolmogorov-Smirnov test results		Parameter and reliability estimates		
Data set	K-S (p-value)	Parameter	MLE	Bayes (MCMC)
$\underline{r}$	0.5578(>0.05)	(α, β, λ)	(0.9364,0.6906,0.5076)	-
$\underline{s}$	0.5371(>0.05)	R	0.4244	0.4301

strength reliability (R) are computed and listed in Table 4. It is observed that the type I extreme-value distribution provides an adequate fit for both data sets $\underline{r}$ and $\underline{s}$. The MLE of R is found as 0.4244. The Bayes estimate and HPD credible interval are found by using MCMC method of as 0.4301 and (0.1548, 0.6979) when all the prior parameters are $a_i = b_i = 0.001, i = 1,2,3$.

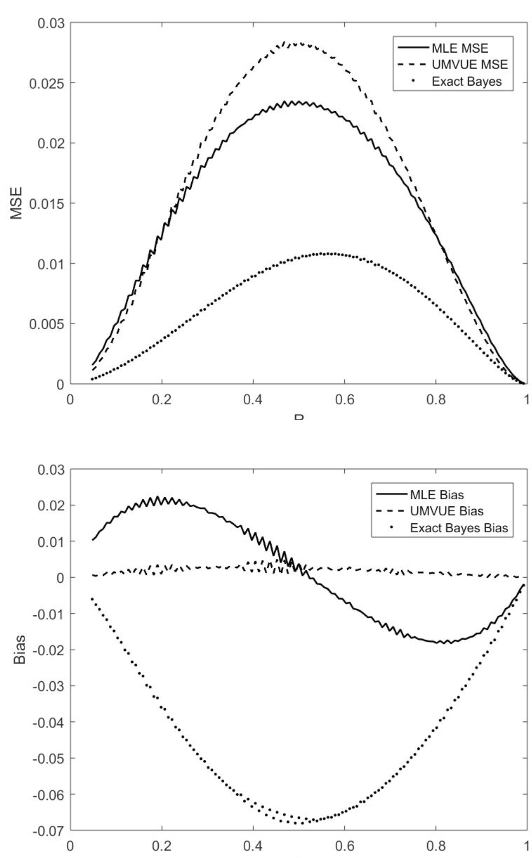


Figure 1. MSE and Bias against R for $\lambda = 2$ and $(n, m) = (5,5)$

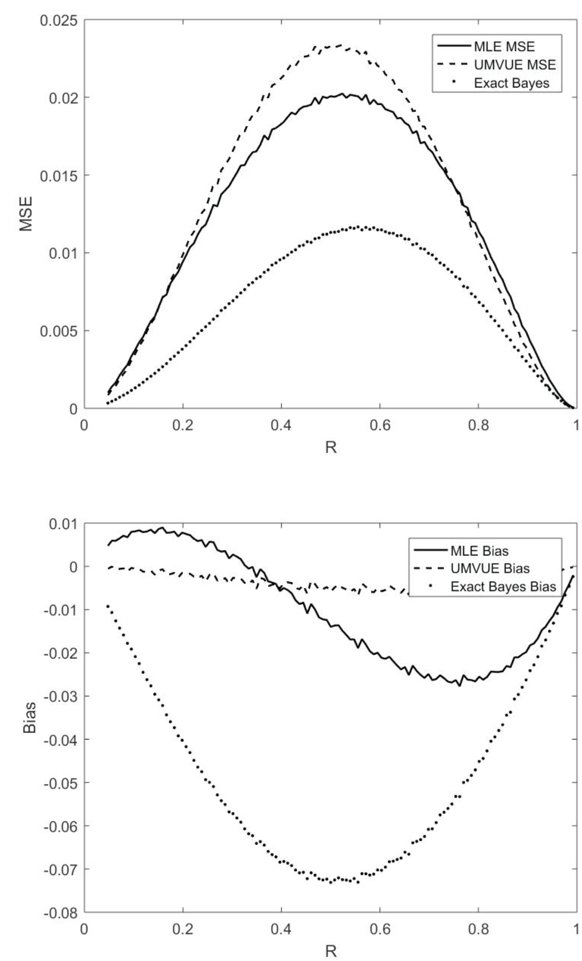
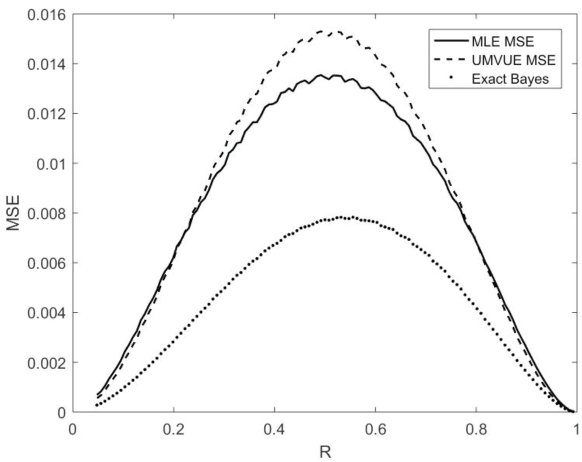


Figure 2. MSE and Bias against R for $\lambda = 2$ and $(n, m) = (5,8)$



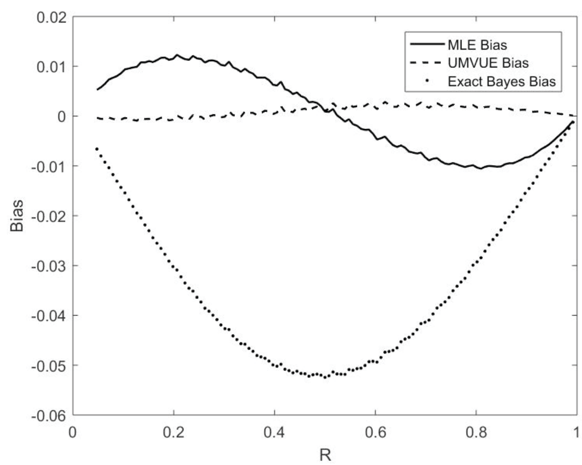
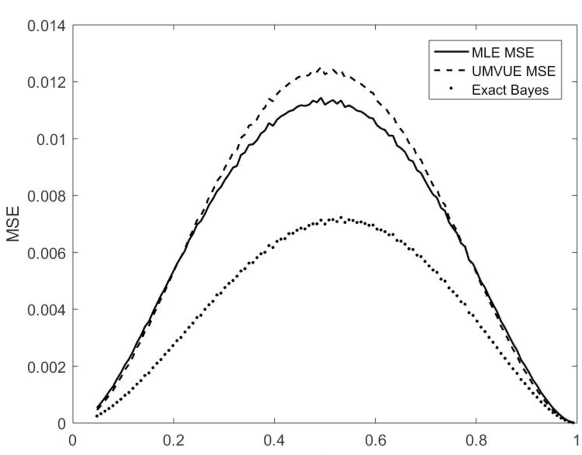


Figure 3. MSE and Bias against R for $\lambda = 2$ and $(n, m) = (8,8)$



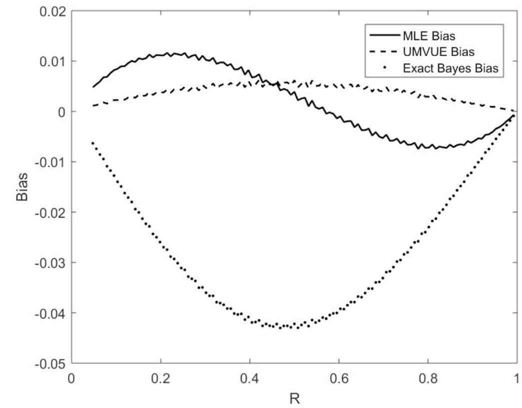
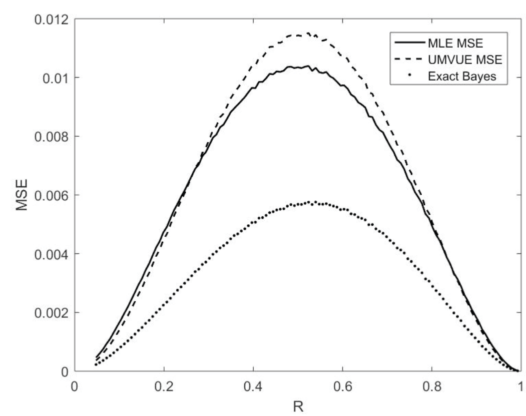


Figure 4. MSE and Bias against R for $\lambda = 2$ and $(n, m) = (10,12)$



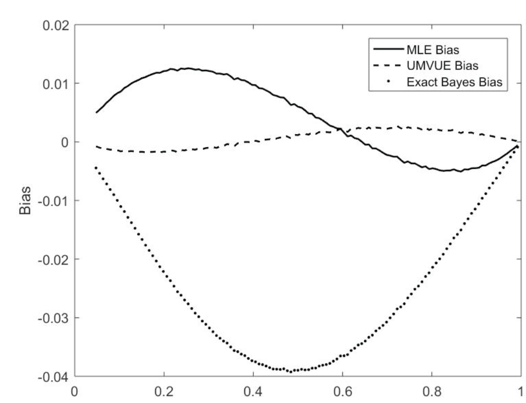


Figure 5. MSE and Bias against R for $\lambda = 2$ and $(n, m) = (12,8)$

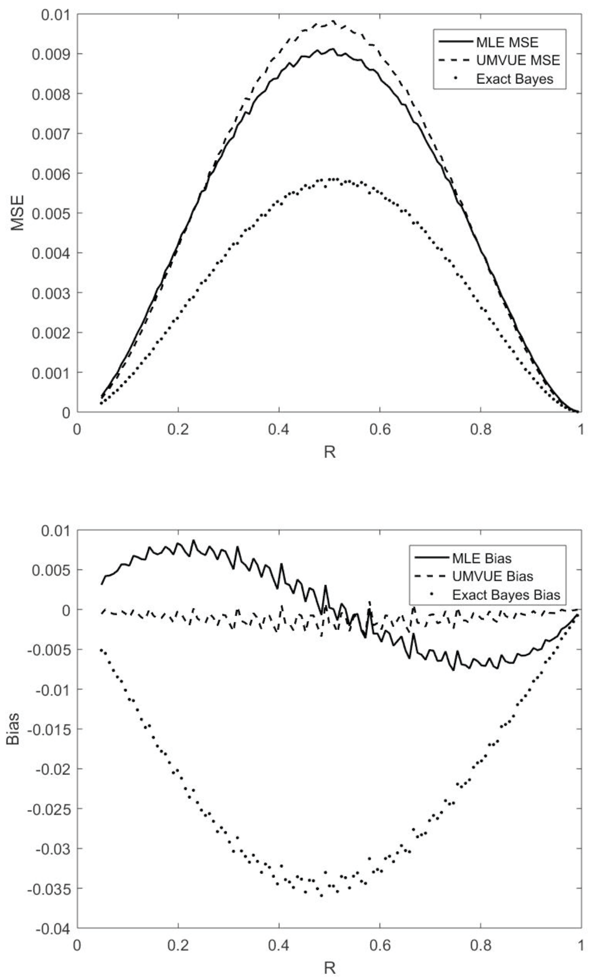


Figure 6. MSE and Bias against R for $\lambda = 2$ and $(n, m) = (15,12)$

V. CONCLUSION

In this study, the stress-strength reliability estimation for the type I extreme-value distribution is considered based on upper records. As expected, the MSEs and ERs of estimates and average length of the intervals decrease when the sample size increases. The performance of the Bayes estimates is superior to the ML and UMVU (when it is available) estimates in all cases. MCMC method is a good alternative to obtain the Bayes estimates when it cannot be obtained analytically.

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