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Quality Properties of Connected Flow Model and Application for Traffic

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Abstract: Sequential moving of particles in one direction is considered. Model of totally-connected flow is introduced in [1] - [4] and concerned to the type of follow-the-leader in traffic flow theory.
Properties of traffic flow are significantly determinate by state-function. For describing of non-connected flow we introduce new model when acceleration of particles takes into consideration the dynamics of neighborhoods particles. For a chain of particles, the model is describing by differential equations of second degree. The function of communication for this model is defined. In partial case the sufficient conditions for convergence of solution the model to totally-connected state are obtained.
In the case of leader-follower pair of particles with linear state and communication functions the statements of belonging of solutions to some Sobolev classes of functions are proved.

1. Introduction

One of the basic models of traffic flow is a model of follow the leader [1]-[4]. This model reduces to study of next differential equations:

$$x_{n+1} - x_n = f(\dot{x}_n), \qquad (1)$$

where $x_n(t)$ is a vehicle coordinate, $f(\dot{x}_n)$ is a distance from a rear bumper of driven car to a rear bumper of leader car such that provided the possibility of emergency braking for driven car. The distance is called *dynamical dimension*. The coordinates of the chain particles are satisfying to

$$x_n(t) < x_{n+1}(t), n = 1, 2, \dots \qquad (2)$$

Flow satisfying (1) -(2) is called *totally connected.*
In classic case the function f in (1) is a parabola with positive coefficients,[1],

$$f(x) = a + bx + cx^2, \qquad (3)$$

where a is static distance, a > 0, b is driver reaction delay, b > 0, and c is braking distance coefficient, c > 0.

At considered interval $x \geq 0$ the function f is *continuous with p several successive derivatives, positive, monotone and convex.*

Let us denote the inverse of this function f by g and we obtain a system of differential equations

$$\dot{x}_n = g(x_{n+1} - x_n), \qquad n = 1, \dots, N-1. \qquad (4)$$

The function g is called a *state function.*

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2. Problem statement

We consider the system of differential equations

$$\ddot{x}_n = h(\dot{x}_n - g(x_{n+1} - x_n)) +$$

$$+ g'(x_{n+1} - x_n)(\dot{x}_{n+1} - \dot{x}_n), n = 1, 2, .., N-1, \qquad (5)$$

where the state function g increases strict monotonically,

$$\text{supp } g \in [1, \infty), \qquad g(1) = 0,$$

and g is a smooth function

$$g''(x) \leq 0, \quad x \geq 1. \qquad (6)$$

$$h: R_+ \rightarrow R,$$

h is strict monotonically decreasing smooth function such that $h(0) = 0$. We call the function h as *communication function.*
The equation (5) describes the case of non connected flows, when the equalities in equation (4) are not hold. The system of equations (4) is defined by the dependence between velocity and distance between neighboring particles, it is a model of dynamic dimensions, [1]. If equalities (4) do not hold, i.e. current distance between neighboring particles more or less than safety distance, then there appeares a force, acceleration, returning the flow characteristics to totally-connected state (4).
We suppose that following initial conditions

$$\begin{cases} x_1(0), \dots, x_{N-1}(0) \\ \dot{x}_1(0), \dots, \dot{x}_{N-1}(0) \end{cases} \qquad (7)$$

and boundary condition

$$x_N(t) = r(t) \qquad (8)$$

are given.

Function $r(t)$ is called leader moving law.

We suppose $\forall\ t \geq 0$ the derivative $\dot{r}$ is absolute continuous function

$$0 < \dot{r}(t) \leq M_1 \quad (9)$$

and almost everywhere the inequalities

$$-M_2 \leq \ddot{r}(t) \leq M_2 \quad (10)$$

hold.

Conditions (9) - (10) are equivalent to the statement, that the function $R(t) = \dot{r}(t) - M_1/2$ belongs to Sobolev class of functions $R \in W_\infty^1\ (R+)$,

$$\|R\|L_\infty(R_+) \leq M_1/2, \quad (11)$$

$$\|\dot{R}\|L_\infty(R_+) \leq M_2 \quad (12)$$

We study the quality properties of cluster $\{x_n\}_{n=1}^{N-1}$, where

$$x_1 < x_2 < x_3 < \ldots < x_{N-1} < x_N = r(t), \quad (13)$$

and $x_N(t)$ is the leader.
The physical sense of the problem gives the following conditions to particles velocities

$$\dot{x}_n(t) > 0, \qquad 1 \leq n \leq N, \qquad \forall t. \quad (14)$$

3. General properties of system (5) - (12)

The equation (5) can be transformed to the following

$$\begin{cases} y = \dot{x}_n - g(x_{n+1} - x_n) \\ \dot{y} = h(y) \end{cases} \quad (15)$$

We denote by $h(y)$ the solution of the following Cauchy problem

$$\begin{cases} \dot{y} = h(y) \\ y(0) = \dot{x}_n(0) - g(\dot{x}_{n+1}(0) - \dot{x}_n(0)) = y_{0n} \end{cases} \quad (16)$$

Lemma 1. Function $/y/$ monotonically decreases.

Proof. As $y\dot{y} < 0$, then $(y^2)' = 2y\dot{y} < 0$. Therefore, y^2 monotonically decreases, hence $/y/$ monotonically decreases.

Suppose

$$h(y) = -y^\alpha = -/y/^\alpha sgn(y). \quad (17)$$

Then equation (15) has the solution

$$y = \begin{cases} y_{0n}\, e^{-t}, & \alpha = 1 \\ (1-\alpha)^{(1-\alpha)^{-1}}(C-t)^{(1-\alpha)^{-1}}, & \alpha \neq 1. \end{cases} \quad (18)$$

The constant C is defined from boundary condition (15).

If $0 < \alpha < 1$ then it is true that $y(t) = 0$ for $t > C$.

The following theorem is true.
Theorem 1. If communication function h is type (17), $0 < \alpha < 1$, then any finite cluster transforms to totally connected state for finite time.

Proof. Since

$$y(t) = \dot{x}_n(t) - g(x_{n+1} - x_n), \quad (19)$$

it is necessary to obtain that $x_n(t) > 0$, $t \in R+$. Really, if $y(0) > 0$, then $y(t) > 0\ \forall t \geq 0$. Hence $\dot{x}_n(t) - g(x_{n+1} - x_n) < 0$ $\forall t$. As well as g is positive, then $\dot{x}_n(t) > 0$.

If $y(0) < 0$, then $y(t) < 0\ \forall t$. So $\dot{y}(t) > 0\ \forall t$. Hence $\dot{y}(t) = \ddot{x}_n(t) - g'\ (x_{n+1}(t) - x_n(t))(\dot{x}_{n+1}(t) - \dot{x}_n(t)) > 0$.

If at some point $\dot{x}(T) = 0$, then

$$\ddot{x}_n(T) - g'(x_{n+1}(T) - x_n(T))\dot{x}_{n+1}(T) > 0,$$

from this $\ddot{x}_n(T) > 0$.

Consequently at any time T the velocity $\dot{x}_n(t)$ does't change a sign. The theorem is proved.

Theorem 2. If communication function h is type (17), $0 < \alpha \leqslant 1$, $\forall$ n, $1 \leqslant n < N$, there exist such constants $M_{n1}, M_{n2} > 0$ that

$$0 < \dot{x}_n(t) \leq M_{n1}, \quad (20)$$

and almost everywhere

$$-M_{n2} \leq \ddot{x}_n(t) \leq M_{n2}. \quad (21)$$

Proof.

The equation (19) considered transformed to the following form:

$$y(t) - \dot{x}_{n+1}(t) = \dot{x}_n(t) - \dot{x}_{n+1}(t) + g(x_{n+1} - x_n).$$

Denote by

$$u(x) = (x_{n+1} - x_n),\ F = -g,\ G = -y(t) + \dot{x}_{n+1}(t)$$

we obtain

$$\dot{u}(x) + F(u) = G(x). \quad (22)$$

1) At first we suppose that $\alpha = 1$ in (17). Then we can assume that $F(u) = u$. Using the method of variation of parameters for equation (22) we consider the solution in the following form

$$u(x) = C(x)e^{-x}.$$

Therefore we have $C(x)e^{-x} = G(x)$,

$$C(x) = u(0) + \int_0^x G(t)e^t dt$$

$$u(x) = C(x)e^{-x} = u(0)e^{-x} + G(x)e^{-x}. \quad (23)$$

If G is a bounded function, then

$$\|u\|_{C(R_+)} \leq \|G\|_{C(R_+)} \times \|e^{-x}\|_{L_1(R_+)}.$$

2) We suppose

$$F(u) = u^{\alpha}, \quad 0 < \alpha < 1.$$

We assume that $|G| \le M_G$. From the equation (22) it follows that if $|y| > M_G^{a-1}$, then $sgn(u) = -sgn(F(u))$, and hence the function $|u|$ increases. Thus,

$$|u| \le M_G^{a-1}. \tag{24}$$

From (22) we have, that

$$|\dot{u}| \le 2M_G. \tag{25}$$

The necessary estimates follow from (24), (25) and definitions.

4. Case of leader – follower with linear functions of state and communication

Suppose N=1, $x_1 = x$,

$$\ddot{x} = h(\dot{x} - g(r(t) - x)). \tag{26}$$

We suppose that h is a linear function,

$$h(x) = -kx, \qquad k > 0, \tag{27}$$

and f is a linear function

$$g(x) = l(x - 1), \quad l > 0. \tag{28}$$

Then

$$\ddot{x} = -k\dot{x} + kl(r(t) - x - 1),$$

i.e.

$$\ddot{x} + k\dot{x} + klx = kl(r(t) - 1). \tag{29}$$

Characteristic equation (26)

$$\alpha^2 + k\alpha + kl = 0$$

has either two negative real roots, or adjoint complex roots with negative real part. If $k = 4l$ then there is multiply real roots.
We suppose that $r(t)$ is a linear function

$$r(t) = 1 + At + B. \tag{30}$$

Then the equation (26) has a solution

$$x^*(t) = Ct + D,$$

where $kC + klCt + klD = Aklt + Bkl$, i.e.

$$A = C, D = B - A/l. \tag{31}$$

Thus, if $x(t)$ is arbitrary solution of (26) with $r(t) = 1+At+B$,
Then function $y(t) = x(t) - x^*(t)$ satisfies the homogeneous equation such that

$$\ddot{y} + k\dot{y} + kly = 0,$$

all solutions converge exponentially to zero. Therefore at in case (30) every solution of (26) converges to functions

$$x^* = At + (B - A/l).$$

Obviously, from some $t > t^*$ any solution x shall monotonically increase. Denote by x^* the solution of (29) with zero initial conditions. If the parameter A is equal to $M_1/2$, then, we obtain

$$\ddot{y} + k\dot{y} + kly = kl(r(t) - 1) - \frac{M_1}{2}t - B, \tag{32}$$

where right part belongs to $B_\infty^1\left(\frac{M_1}{2}\right) \cap B_\infty^1(M_2)$

Consequently, the problem is the following: *whether the solution of (32) belongs to the same class of function as the right part of this equation?*

If we differentiate the equation (32) and denote $z = \dot{y}$, then we obtain the follow

$$\ddot{z} + k\dot{z} + klz = h(t),$$

where $h \in W_\infty^1(R_+)$, i.e. h is bounded function on semi-line with its derivative. Does the solution belong to this class? From condition $h \in W_\infty^1(R_+)$ it follows $|\hat{h}| \le \frac{C_0}{p} + \frac{C_1}{p^2}$, where $\hat{h}$ is Laplace transform. Hence,

$$\hat{z} = \frac{h}{p^2 + kp + kl} + \hat{z}_0 \tag{33}$$

and it follows that $z \in W_\infty^1(R_+)$.

We define *the problem on exact estimations:*

$$\ddot{z} + k\dot{z} + klz = h(t), h \in W_\infty^1(R_+), \tag{34}$$

$$||\dot{z}||_{L_\infty(R_+)} \to \max, \tag{35}$$

or

$$||z||_{L_\infty(R_+)} \to \max. \tag{36}$$

If we denote $P(z) = \ddot{z} + k\dot{z} + klz$, then (34) - (36) transforms to

$$||P(z)||_{L_\infty(R_+)} \le 1, \tag{37}$$

$$||\dot{P}(z)||_{L_\infty(R_+)} \le 1, \tag{38}$$

$$||Q(z)||_{L_\infty(R_+)} \to \max, \tag{39}$$

where $Q(z) = z$ or $Q(z) = \dot{z}$.

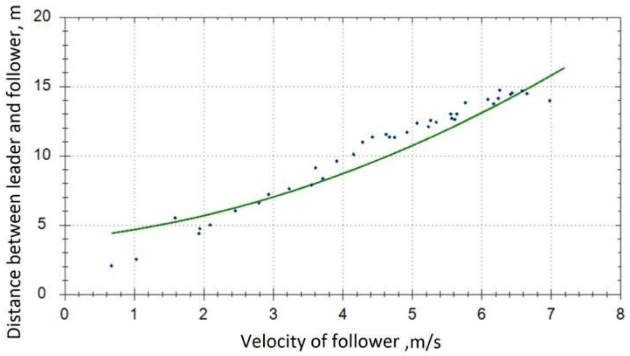

Figure 1. Distance-velocity data and function (1), (3)

Problems of such type belong to inequalities Kolmogorov kind for norms of differential operators, [5], and they have very meaningful history.

5. Real-time experiment using SSSR-Traffic infocommunication system

We have developed infocommunicational system *SSSR - Traffic* for receiving and processing of GPS-data in real-time mode, [6]-[9]. The leading car (leader) and the following car (follower) with smartphone client applications onboard drove one following the other. *SSSR-Traffic* system checked their characteristics. In Fig.1 it is shown experimental data using *SSSR-Traffic* system and plot of function f $(\dot{x})$, (1), (3). The experimental data and model have close results.

6. Conclusion and acknowledgments

Quality properties of chain behavior, described by dynamical system with functions of state and communication, are studied. The experimental research was provided with good results.

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