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Araştırma Makalesi / Research Article

Auto-Bäcklund Transformation for Fifth Order Equation of the Burgers Hierarchy

İbrahim Enam İNAN

Fırat Üniversitesi, Eğitim Fakültesi, Elazığ

e posta: einan@firat.edu.tr, ORCID ID: https://orcid.org/0000-0003-3681-0497

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Keywords
Fifth order equation of the Burgers hierarchy; Auto-Bäcklund transformation; Solitary wave solution; Nonlinear partial differential equations

Abstract

In this paper, we implemented Auto-Bäcklund transformation for fifth order equation of the Burgers hierarchy. Auto-Bäcklund transformation was developed as a direct and simple method to obtain solutions of nonlinear partial differential equations by Fan.

Beşinci Mertebeden Burgers Hierarchy Denklemi için Auto-Bäcklund Dönüşümü

Anahtar kelimeler
Beşinci mertebeden Burgers hierarchy denklemi; Auto-Bäcklund dönüşümü; Solitary dalga çözümü; Llineer olmayan kısmi diferansiyel denklemler

Öz

Bu makalede beşinci mertebeden Burgers hierarchy denklemi için Auto-Bäcklund dönüşümü sunulmuştur. Auto-Bäcklund dönüşümü lineer olmayan kısmi diferansiyel denklemlerin çözümlerini elde etmek için doğrudan ve basit bir yöntem olarak Fan tarafından geliştirilmiştir.

1.Introduction

Nonlinear partial differential equations (NPDEs) have an important place in applied mathematics and physics (Debtnath 1997, Wazwaz 2002). Many analytical methods have been found in literature (Shang 2007, Bock and Kruskal 1997, Matveed and Salle 1991, Malfliet 1992, Chuntao 1996, Cariello and Tabor 1989, Fan 2000, Clarkson 1989). Besides these methods, there are many methods which reach to solution by using an auxiliary equation. Using these methods, partial differential equations are transformed into ordinary differential equations. These nonlinear partial differential equations are solved with the help of ordinary differential equations. Some of these methods are given in (Elwakil *et al*. 2002, Chen and Zhang 2004, Fu *et al.*2001, Shen and Pan 2003,Chen and Hong-

Qind 2004,Chen *et al*. 2004, Chen and Yan 2006, Wang and Zhang 2008, Li *et al*. 2010, Fan 2000, Manafian and Lakestain 2016). Many authors have applied these and similar methods to various equations (Don 2001, Wazwaz 2010, Manafian and Lakestain 2013, Manafian and Zamanpour 2013, Zhao e*t al*. 2006, Bekir 2008).

This article, we will obtain solitary wave solutions of fifth order equation of the Burgers hierarchy by using Auto-Bäcklund transformation.

2. Example. The fifth order equation of the Burgers hierarchy (Wazwaz,2010) is as follows,

$$u_t + u_{xxxxx} + 10u_{xx}^2 + 15u_xu_{xxx} + 5uu_{xxxx} + 15u_x^3 + 50uu_xu_{xx} + 10u^2u_{xxx} + 30u^2u_x^2 + 10u^3u_{xx} + 5u^4u_x = 0. \quad (1)$$

According to the idea of improved HB , (Fan,2000), we seek for Auto-Bäcklund transformation of Equation (1). When balancing u_{xxxxx} with uu_{xxxx} then given $m_1 = 1$.

We may choose

$$u = \frac{\partial}{\partial x} f(w) + u_0 = f'w_x + u_0. \quad (2)$$

Here $f = f(w)$, $w = w(x,t)$ and $u_0 = u_0(x,t)$. $f = f(w)$ and $w = w(x,t)$ are undetermined functions, also u and u_0 are two solutions of equation (1). Using transforms (2), we get the following derivatives,

$$u_t = f''w_tw_x + f'w_{xt} + (u_0)_t \ ,$$

$$u_{xxxxx} = f^{(6)}w_x^6 + 15f^{(5)}w_x^4w_{xx} + 45f^{(4)}w_x^2w_{xx}^2 + 20f^{(4)}w_x^3w_{xxx} + 15f'''w_{xx}^3 + 60f'''w_xw_{xx}w_{xxx} + 15f'''w_x^2w_{xxxx} + 10f''w_{xxx}^2 + 15f''w_{xx}w_{xxxx} + 6f''w_xw_{xxxxx} + f'w_{xxxxxx} + (u_0)_{xxxxx},$$

$$10u_{xx}^2 = 10(f''')^2w_x^6 + 60f''f'''w_x^4w_{xx} + 20f'f'''w_x^3w_{xxx} + 20f'''w_x^3(u_0)_{xx} + 90(f'')^2w_x^2w_{xx}^2 + 60f'f''w_xw_{xx}w_{xxx} + 60f''w_xw_{xx}(u_0)_{xx} + 10(f')^2w_{xxx}^2 + 20f'w_{xxx}(u_0)_{xx} + 10(u_0)_{xx}^2,$$

$$15u_xu_{xxx} = 15f''f^{(4)}w_x^6 + 15f'f^{(4)}w_x^4w_{xx} + 15f^{(4)}w_x^4(u_0)_x + 90f''f'''w_x^4w_{xx} + 90f'f'''w_x^2w_{xx}^2 + 90f'''w_x^2w_{xx}(u_0)_x + 60(f'')^2w_x^3w_{xxx} + 45(f'')^2w_x^2w_{xx}^2 + 15f'f''w_x^2w_{xxxx} + 60f'f''w_xw_{xx}w_{xxx} + 45f'f''w_{xx}^3 + 15f''w_x^2(u_0)_{xxx} + 60f''w_xw_{xxx}(u_0)_x + 45f''w_{xx}^2(u_0)_x + 15(f')^2w_{xx}w_{xxxx} + 15f'w_{xx}(u_0)_{xxx} + {+15}f'w_{xxxx}(u_0)_x + 15(u_0)_x(u_0)_{xxx},$$

$$5uu_{xxxx} = 5f'f^{(5)}w_x^6 + 5f^{(5)}w_x^5(u_0) + 50f'f^{(4)}w_x^4w_{xx} + 50f^{(4)}w_x^3w_{xx}(u_0) + 50f'f'''w_x^3w_{xxx} + 75f'f'''w_x^2w_{xx}^2 + 50f'''w_x^2w_{xxx}(u_0) + 75f'''w_xw_{xx}^2(u_0) + 25f'f''w_x^2w_{xxxx} + 50f'f''w_xw_{xx}w_{xxx} + 25f''w_xw_{xxxx}(u_0) + 50f''w_{xx}w_{xxx}(u_0) + 5(f')^2w_xw_{xxxxx} + 5f'w_x(u_0)_{xxxx} + 5f'w_{xxxxx}(u_0) + 5(u_0)(u_0)_{xxxx},$$

$$15u_x^3 = 15(f'')^3w_x^6 + 45f'(f'')^2w_x^4w_{xx} + 45(f'')^2w_x^4(u_0)_x + 45(f')^2f''w_x^2w_{xx}^2 + 90f'f''w_x^2w_{xx}(u_0)_x + 45f''w_x^2(u_0)_x^2 + 15(f')^3w_{xx}^3 + 45(f')^2w_{xx}^2(u_0)_x + 45f'w_{xx}(u_0)_x^2 + 15(u_0)_x^3,$$

$$50uu_xu_{xx} = 50f'f''f'''w_x^6 + 50f''f'''w_x^5(u_0) + 50(f')^2f'''w_x^4w_{xx} + 50f'f'''w_x^4(u_0)_x + 50f'f'''w_x^3w_{xx}(u_0) + 50f'''w_x^3(u_0)(u_0)_x + 150f'(f'')^2w_x^4w_{xx} + 150(f'')^2w_x^3w_{xx}(u_0) + 50(f')^2f''w_x^3w_{xxx} + 150(f')^2f''w_x^2w_{xx}^2 +$$

$50f'f''w_x^3(u_0)_{xx} + 150f'f''w_x^2w_{xx}(u_0)_x + 50f'f''w_x^2w_{xxx}(u_0) + 150f'f''w_xw_{xx}^2(u_0) + 50f''w_x^2(u_0)(u_0)_{xx} + 150f''w_xw_{xx}(u_0)(u_0)_x + 50(f')^3w_xw_{xx}w_{xxx} + 50(f')^2w_xw_{xx}(u_0)_{xx} + 50(f')^2w_xw_{xxx}(u_0)_x + 50(f')^2w_{xx}w_{xxx}(u_0) + 50f'w_x(u_0)_x(u_0)_{xx} + 50f'w_{xx}(u_0)(u_0)_{xx} + 50f'w_{xxx}(u_0)(u_0)_x + 50(u_0)(u_0)_x(u_0)_{xx},$

$10u^2u_{xxx} = 10(f')^2f^{(4)}w_x^6 + 20f'f^{(4)}w_x^5(u_0) + 10f^{(4)}w_x^4(u_0)^2 + 60(f')^2f'''w_x^4w_{xx} + 120f'f'''w_x^3w_{xx}(u_0) + 60f'''w_x^2w_{xx}(u_0)^2 + 40(f')^2f''w_x^3w_{xxx} + 30(f')^2f''w_x^2w_{xx}^2 + 80f'f''w_x^2w_{xxx}(u_0) + 60f'f''w_xw_{xx}^2(u_0) + 40f''w_xw_{xxx}(u_0)^2 + 30f''w_{xx}^2(u_0)^2 + 10(f')^3w_x^2w_{xxxx} + 10(f')^2w_x^2(u_0)_{xxx} + 20(f')^2w_xw_{xxxx}(u_0) + 20f'w_x(u_0)(u_0)_{xxx} + 10f'w_{xxxx}(u_0)^2 + 10(u_0)^2(u_0)_{xxx},$

$30u^2u_x^2 = 30(f')^2(f'')^2w_x^6 + 60f'(f'')^2w_x^5(u_0) + 30(f'')^2w_x^4(u_0)^2 + 60(f')^3f''w_x^4w_{xx} + 60(f')^2f''w_x^4(u_0)_x + 120(f')^2f''w_x^3w_{xx}(u_0) + 120f'f''w_x^3(u_0)(u_0)_x + 60f'f''w_x^2w_{xx}(u_0)^2 + 60f''w_x^2(u_0)^2(u_0)_x + 30(f')^4w_x^2w_{xx}^2 + 60(f')^3w_x^2w_{xx}(u_0)_x + 60(f')^3w_xw_{xx}^2(u_0) + 30(f')^2w_x^2(u_0)_x^2 + 120(f')^2w_xw_{xx}(u_0)(u_0)_x + 30(f')^2(u_0)^2w_{xx}^2 + 60f'w_x(u_0)(u_0)_x^2 + 60f'w_{xx}(u_0)^2(u_0)_x + 30(u_0)^2(u_0)_x^2,$

$10u^3u_{xx} = 10(f')^3f'''w_x^6 + 30(f')^2f'''w_x^5(u_0) + 30f'f'''w_x^4(u_0)^2 + 10f'''w_x^3(u_0)^3 + 30(f')^3f''w_x^4w_{xx} + 90(f')^2f''w_x^3w_{xx}(u_0) + 90f'f''w_x^2w_{xx}(u_0)^2 + 30f''w_xw_{xx}(u_0)^3 + 10(f')^4w_x^3w_{xxx} + 10(f')^3w_x^3(u_0)_{xx} + 30(f')^3w_x^2w_{xxx}(u_0) + 30(f')^2w_x^2(u_0)(u_0)_{xx} + 30(f')^2w_xw_{xxx}(u_0)^2 + 30f'w_x(u_0)^2(u_0)_{xx} + 10f'w_{xxx}(u_0)^3 + 10(u_0)^3(u_0)_{xx},$

$5u^4u_x = 5(f')^4f''w_x^6 + 20(f')^3f''w_x^5(u_0) + 30(f')^2f''w_x^4(u_0)^2 + 20f'f''w_x^3(u_0)^3 + 5f''w_x^2(u_0)^4 + 5(f')^5w_x^4w_{xx} + 5(f')^4w_x^4(u_0)_x + 20(f')^4w_x^3w_{xx}(u_0) + 20(f')^3w_x^3(u_0)(u_0)_x + 30(f')^3w_x^2w_{xx}(u_0)^2 + 30(f')^2w_x^2(u_0)^2(u_0)_x + 20(f')^2w_xw_{xx}(u_0)^3 + 20f'w_x(u_0)^3(u_0)_x + 5f'w_{xx}(u_0)^4 + 5(u_0)^4(u_0)_x. \quad (3)$

If the derivatives obtained by (3) are written in place of equation (1) and the same-order derivatives of f are arranged:

$w_x^6\big[f^{(6)} + 5(f')^4f'' + 10(f')^3f''' + 50f'f''f''' + 10(f')^2f^{(4)} + 30(f')^2(f'')^2 + 10(f''')^2 + 15f''f^{(4)} + 5f'f^{(5)} + 15(f'')^3\big]$

$+\big[15f^{(5)}w_x^4w_{xx} + 20(f')^3f''w_x^5(u_0) + 5(f')^5w_x^4w_{xx} + 30(f')^2f'''w_x^5(u_0) + 50f''f'''w_x^5(u_0) + 50(f')^2f'''w_x^4w_{xx} + 150f'(f'')^2w_x^4w_{xx} + 20f'f^{(4)}w_x^5(u_0) + 60(f')^2f'''w_x^4w_{xx} + 60f'(f'')^2w_x^5(u_0) + 60(f')^3f''w_x^4w_{xx} + 60f''f'''w_x^4w_{xx} + 15f'f^{(4)}w_x^4w_{xx} + 90f''f'''w_x^4w_{xx} + 5f^{(5)}w_x^5(u_0) + 50f'f^{(4)}w_x^4w_{xx} + 45f'(f'')^2w_x^4w_{xx} + 30(f')^3f''w_x^4w_{xx}\big]$

$+\big[45f^{(4)}w_x^2w_{xx}^2 + 20f^{(4)}w_x^3w_{xxx} + 30(f')^2f''w_x^4(u_0)^2 + 5(f')^4w_x^4(u_0)_x + 20(f')^4w_x^3w_{xx}(u_0) + 30f'f'''w_x^4(u_0)^2 + 90(f')^2f''w_x^3w_{xx}(u_0) + 10(f')^4w_x^3w_{xxx} + 50f'f'''w_x^4(u_0)_x + 50f'f'''w_x^3w_{xx}(u_0) + 150(f'')^2w_x^3w_{xx}(u_0) + 50(f')^2f''w_x^3w_{xxx} + 150(f')^2f''w_x^2w_{xx}^2 + 10f^{(4)}w_x^4(u_0)^2 + 120f'f'''w_x^3w_{xx}(u_0) + 40(f')^2f''w_x^3w_{xxx} + 30(f')^2f''w_x^2w_{xx}^2 + 30(f'')^2w_x^4(u_0)^2 + 60(f')^2f''w_x^4(u_0)_x + 120(f')^2f''w_x^3w_{xx}(u_0) + 30(f')^4w_x^2w_{xx}^2 + 20f'f'''w_x^3w_{xxx} +$

$90(f'')^2 w_x^2 w_{xx}^2 + 15f^{(4)} w_x^4 (u_0)_x +$
$90f'f''' w_x^2 w_{xx}^2 + 60(f'')^2 w_x^3 w_{xxx} +$
$45(f'')^2 w_x^2 w_{xx}^2 + 50f^{(4)} w_x^3 w_{xx}(u_0) +$
$50f'f''' w_x^3 w_{xxx} + 75f'f''' w_x^2 w_{xx}^2 +$
$45(f'')^2 w_x^4 (u_0)_x + 45(f')^2 f'' w_x^2 w_{xx}^2]$

$+[15f''' w_{xx}^3 + 60f''' w_x w_{xx} w_{xxx} +$
$15f''' w_x^2 w_{xxxx} + 20f'f'' w_x^3 (u_0)^3 +$
$20(f')^3 w_x^3 (u_0)(u_0)_x + 30(f')^3 w_x^2 w_{xx} (u_0)^2 +$
$10f''' w_x^3 (u_0)^3 + 90f'f'' w_x^2 w_{xx} (u_0)^2 +$
$10(f')^3 w_x^3 (u_0)_{xx} + 30(f')^3 w_x^2 w_{xxx} (u_0) +$
$50f''' w_x^3 (u_0)(u_0)_x + 50f'f'' w_x^3 (u_0)_{xx} +$
$150f'f'' w_x^2 w_{xx} (u_0)_x + 50f'f'' w_x^2 w_{xxx} (u_0) +$
$150f'f'' w_x w_{xx}^2 (u_0) + 50(f')^3 w_x w_{xx} w_{xxx} +$
$60f''' w_x^2 w_{xx} (u_0)^2 + 80f'f'' w_x^2 w_{xxx} (u_0) +$
$60f'f'' w_x w_{xx}^2 (u_0) + 10(f')^3 w_x^2 w_{xxxx} +$
$120f'f'' w_x^3 (u_0)(u_0)_x + 60f'f'' w_x^2 w_{xx} (u_0)^2 +$
$60(f')^3 w_x^2 w_{xx} (u_0)_x + 60(f')^3 w_x w_{xx}^2 (u_0) +$
$20f''' w_x^3 (u_0)_{xx} + 60f'f'' w_x w_{xx} w_{xxx} +$
$15f'f'' w_x^2 w_{xxxx} + 90f''' w_x^2 w_{xx} (u_0)_x +$
$60f'f'' w_x w_{xx} w_{xxx} + 45f'f'' w_{xx}^3 +$
$50f''' w_x^2 w_{xxx} (u_0) + 75f''' w_x w_{xx}^2 (u_0) +$
$25f'f'' w_x^2 w_{xxxx} + 50f'f'' w_x w_{xx} w_{xxx} +$
$90f'f'' w_x^2 w_{xx} (u_0)_x + 15(f')^3 w_{xx}^3]$

$+[f'' w_t w_x + 10f'' w_{xxx}^2 + 15f'' w_{xx} w_{xxxx} +$
$6f'' w_x w_{xxxxx} + 5f'' w_x^2 (u_0)^4 +$
$30(f')^2 w_x^2 (u_0)^2 (u_0)_x + 20(f')^2 w_x w_{xx} (u_0)^3 +$
$30f'' w_x w_{xx} (u_0)^3 + 30(f')^2 w_x^2 (u_0)(u_0)_{xx} +$
$30(f')^2 w_x w_{xxx} (u_0)^2 + 50f'' w_x^2 (u_0)(u_0)_{xx} +$
$150f'' w_x w_{xx} (u_0)(u_0)_x + 50(f')^2 w_x w_{xx} (u_0)_{xx} +$
$50(f')^2 w_x w_{xxx} (u_0)_x + 50(f')^2 w_{xx} w_{xxx} (u_0) +$
$40f'' w_x w_{xxx} (u_0)^2 + 30f'' w_{xx}^2 (u_0)^2 +$
$10(f')^2 w_x^2 (u_0)_{xxx} + 20(f')^2 w_x w_{xxxx} (u_0) +$
$60f'' w_x^2 (u_0)^2 (u_0)_x + 30(f')^2 w_x^2 (u_0)_x^2 +$
$120(f')^2 w_x w_{xx} (u_0)(u_0)_x + 30(f')^2 w_{xx}^2 (u_0)^2 +$
$60f'' w_x w_{xx} (u_0)_{xx} + 10(f')^2 w_{xxx}^2 +$
$15f'' w_x^2 (u_0)_{xxx} + 60f'' w_x w_{xxx} (u_0)_x +$
$45f'' w_{xx}^2 (u_0)_x + 15(f')^2 w_{xx} w_{xxxx} +$
$25f'' w_x w_{xxxx} (u_0) + 50f'' w_{xx} w_{xxx} (u_0) +$
$5(f')^2 w_x w_{xxxxx} + 45f'' w_x^2 (u_0)_x^2 +$
$45(f')^2 w_{xx}^2 (u_0)_x]$

$+[f' w_{xt} + f' w_{xxxxxx} + 20f' w_x (u_0)^3 (u_0)_x +$
$5f' (u_0)^4 w_{xx} + 30f' w_x (u_0)^2 (u_0)_{xx} +$
$10f' w_{xxx} (u_0)^3 + 50f' w_{xx} (u_0)(u_0)_{xx} +$
$50f' w_{xxx} (u_0)(u_0)_x + 20f' w_x (u_0)(u_0)_{xxx} +$
$10f' w_{xxxx} (u_0)^2 + 60f' w_x (u_0)(u_0)_x^2 +$
$60f' w_{xx} (u_0)^2 (u_0)_x + 20f' w_{xxx} (u_0)_{xx} +$
$15f' w_{xx} (u_0)_{xxx} + 15f' w_{xxxx} (u_0)_x +$
$5f' w_x (u_0)_{xxxx} + 5f' w_{xxxxx} (u_0) +$
$45f' w_{xx} (u_0)_x^2 + 50f' w_x (u_0)_x (u_0)_{xx}]. \quad (4)$

Setting the coefficients of w_x^6 in (4) to zero, we obtain a set of ordinary differential equations

$$f^{(6)} + 5(f')^4 f'' + 10(f')^3 f''' + 50f'f''f''' + 10(f')^2 f^{(4)} + 30(f')^2 (f'')^2 + 10(f''')^2 + 15f'' f^{(4)} + 5f' f^{(5)} + 15(f'')^3 = 0.$$

one of the solutions of this equation

$$f = lnw \quad (5)$$

there by from (5) it holds that

$(f')^3 f'' = -\frac{1}{24} f^{(5)}$, $(f')^2 f''' = \frac{1}{12} f^{(5)}$, $f''f''' = -\frac{1}{12} f^{(5)}$, $(f'')^2 f' = (f')^5 = \frac{1}{24} f^{(5)}$, $f'f^{(4)} = -\frac{1}{4} f^{(5)}$, $(f')^2 f'' = \frac{1}{6} f^{(4)}$, $(f')^4 = (f'')^2 = -\frac{1}{6} f^{(4)}$, $f'f''' = -\frac{1}{3} f^{(4)}$, $f'f'' = -\frac{1}{2} f'''$, $(f')^3 = \frac{1}{2} f'''$, $(f')^2 = -f''$ (6) By using (6), equation (4) can be written as the sum of some terms with f' and f'' setting their coefficients to zero will lead to

$w_x[w_t + w_{xxxxx} + 5w_x (u_0)^4 +$
$30w_x (u_0)^2 (u_0)_x + 10w_{xx} (u_0)^3 +$

$20w_x(u_0)(u_0)_{xx} + 10w_{xxx}(u_0)^2 + 30w_{xx}(u_0)(u_0)_x + 10w_{xx}(u_0)_{xx} + 10w_{xxx}(u_0)_x + 5w_x(u_0)_{xxx} + 5w_{xxxx}(u_0) + 15w_x(u_0)_x^2] = 0,$

$$\frac{\partial}{\partial x}[w_t + w_{xxxxx} + 5w_x(u_0)^4 + 30w_x(u_0)^2(u_0)_x + 10w_{xx}(u_0)^3 + 20w_x(u_0)(u_0)_{xx} + 10w_{xxx}(u_0)^2 + 30w_{xx}(u_0)(u_0)_x + 10w_{xx}(u_0)_{xx} + 10w_{xxx}(u_0)_x + 5w_x(u_0)_{xxx} + 5w_{xxxx}(u_0) + 15w_x(u_0)_x^2] = 0.$$

Above equation is satisfied provided that

$$w_t + w_{xxxxx} + 5w_x(u_0)^4 + 30w_x(u_0)^2(u_0)_x + 10w_{xx}(u_0)^3 + 20w_x(u_0)(u_0)_{xx} + 10w_{xxx}(u_0)^2 + 30w_{xx}(u_0)(u_0)_x + 10w_{xx}(u_0)_{xx} + 10w_{xxx}(u_0)_x + 5w_x(u_0)_{xxx} + 5w_{xxxx}(u_0) + 15w_x(u_0)_x^2 = 0 \tag{7}$$

From (2) and (5), we obtain Auto-Bäcklund transformation of equation (1),

$$u = \frac{\partial}{\partial x} lnw + u_0 \tag{8}$$

where w satisfying (7). We take initial solutions of equation (1) as $u_0 = 0$, then (7) and (8) respectively reduce to

$$w_t + w_{xxxxx} = 0, \tag{9}$$

$$u = \frac{\partial}{\partial x} lnw. \tag{10}$$

Specially, we take a solution of (9)

$$w = 1 + exp[c(x - c^4 t)] \tag{11}$$

Then, the solitary wave solution of equation (1) can be written by using Eq. (10) as following

$$u(x,t) = \frac{c}{2}\left(1 + tanh\left[\frac{c}{2}(x - c^4 t)\right]\right) \tag{12}$$

where c is arbitrary constant.

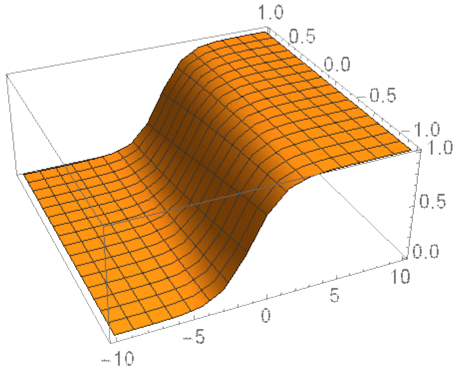

Fig.1 The 3 Dimensional surfaces of Eq. (12) for $c = 1$.

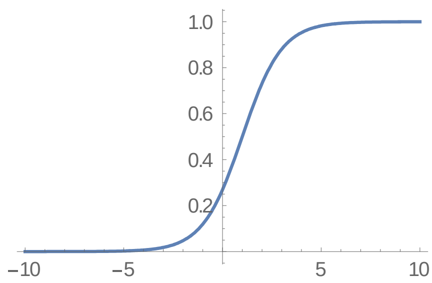

Fig.2 The 2 Dimensional surfaces of Eq. (12) for $c = 1$ and $t = 1$.

3. Conclusion

We used the Auto-Bäcklund transformation for find solitary wave solutions of fifth order equation of the Burgers hierarchy. This method has been successfully applied to solve some nonlinear wave equations and can be used to many other nonlinear equations or coupled ones.

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