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Banach Contraction Principle in Cone Modular Spaces with Banach Algebra

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Abstract

There are some mappings, which are not contraction mappings in the usual senses, such that they hold some contractive type conditions in the settings of some new abstract metric and modular spaces. In this paper, taking into account this fact, we introduce such a new type modular space by using the setting of cones in Banach algebras. In the first section, some basic notions and definitions are given. In the second part, it is shown that some result of Banach Contraction Principle in modular space with C^*-algebra is equal to the result of Banach Contraction Principle of the usual modular space. Then that new modular space mentioned above is introduced and some results are given. Finally the work is concluded with an example.

Banach Cebirli Koni Modüler Uzaylarda Banach Büzülme Prensibi

Anahtar kelimeler
Modüler uzay; Banach cebiri; sabit nokta teoremi; Δ_2-koşulu; F-normu; C^*-cebiri.

Öz

Bilinen anlamda büzülme dönüşümü olmayan öyle dönüşümler vardır ki bu dönüşümler bazı yeni metrik ve modüler uzay yapılarında bazı büzülme tipinde koşulları sağlarlar. Biz bu makalede bu durumu göz önünde bulundurarak Banach cebirlerdeki konilerin yardımıyla yeni bir modüler uzay kavramı sunduk. İlk kısımda temel tanım ve notasyonlar verildi. İkinci kısımda Banach Büzülme Prensibinin C^*-cebir değerli modüler uzaylardaki sonucuyla klasik modüler uzaylardaki sonucunun denkliği gösterildi. Sonra yukarıda bahsedilen o modüler uzaya giriş yapıldı ve bazı sonuçlar verildi. Son olarak çalışma bir örnekle desteklendi.

1. Introduction

Banach (1922) presented a fixed point theorem known as Banach Contraction Principle (BCP) that is one of the important mathematical tools in nonlinear analysis. Then many authors dealth with this theorem in different spaces. For example, Ma et al. (2014) presented this theorem in C^*-algebra-valued metric space and claimed that this is a generalization of BCP in the standart metric space. But later, Alsulami et al. (2016), Kadelburg and Radenovic' (2016) separately showed that BCP obtained in C^*-algebra-valued metric space is equivalent to the result of BCP in the classical metric space.

Nakano (1950) introduced the notion of modular space. Then Musielak and Orlicz (1959) generalized this space. By using the results of these works Khamsi and Kozlowski (1990) extended BCP to the frame of modular function space, an example of modular space, introduced by Kozlowski (1988). Inspired by the notion of C^*-algebra-valued metric space Ma et al. (2014), Shateri (2017) presented a generalization for modular space.

Now in this work, motivated by Alsulami et al. (2016) and Kadelburg and Radenovic' (2016) it is firstly shown that BCP in the setting of C^*-algebra-valued modular space does not provide a real extension for the usual one in the modular space. Secondly, introduced a new setting, namely, a cone modular space over Banach algebras, which enables one to obtain a proper generalization for BCP in the usual modular spaces. Finally, the work is concluded with an example.

2. Preliminaries

Modular functional is defined as follows:

Let V be a vector space and $\rho : V \to [0,\infty]$ be a functional for $x, y \in V$. θ_V represents the zero vector of V. ρ is called modular if the followings hold:

m1.) $\rho(x) = 0$ if and only if $x = \theta_V$.

m2.) $\rho(\mu x) = \rho(x)$ for each scalar with $|\mu| = 1$.

m3.) $\rho(\mu x + \alpha y) \leq \rho(x) + \rho(y)$ if $\mu = 1-\alpha$ for $\alpha, \mu \geq 0$.

It is clear that the set

$$V_\rho = \{x \in V: \rho(\lambda x) \to 0 \ as\ \lambda \to 0\}$$

is a vector subspace of V. V_ρ is called modular space.

In addition to the conditions above, if $\rho(\mu x + \alpha y) \leq \mu\rho(x) + \alpha\rho(y)$ for $\alpha, \mu \geq 0, \mu = 1-\alpha$, then the functional ρ is called convex.

Definition 2.1. The modular ρ satisfies the Δ_2-condition if $\lim_{n\to\infty} \rho(2x_n) = \theta_V$ whenever $\lim_{n\to\infty} \rho(x_n) = \theta_V$.

That is seen from Khamsi and Kozlowski (1990) that the BCP is valid for a mapping $T: M \to M$ where M is a closed , bounded non-empty subset of the modular function space:

Theorem 2.1. Let ρ be a modular functional that satisfies the Δ_2-condition and M be a non-empty ρ-closed subset of the modular function space V_ρ. If $T: M \to M$ is Lipschitzian and M is ρ-bounded, then T has a unique fixed point.

Now it is recalled some basic definitions and results from Murphy (1990) and Ma et al. (2014).

An algebra is unital if it has the multiplicative unit. An involution on a unital algebra C is a conjugate-linear map $a \to a^*$ on C such that $aa^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in C$. $(C,*)$ is said to be a $*$-algebra. A Banach $*$-algebra is a $*$-algebra with a complete submultiplicative norm such that $\|a^*\|_C = \|a\|_C$ for each element a of it. A C^*-algebra is a Banach $*$-algebra such that $\|a^*a\|_C = \|a\|_C^2$ for every element a of it. In the rest of the the paper it is supposed that C is a unital C^*-algebra. $\sigma(x)$ stands for the spectrum of x. θ_C represents the zero element of C. The set $C^\# = \{x \in C: x^* = x\}$ denotes the hermitian or self-adjoint elements of C. If $x \in C^\#$ and $\sigma(x) \subseteq [0,\infty)$, then $x \in C$ is said to be a positive element of C. C^+ denotes the positive elements of C and $|x| = (x^*x)^{\frac{1}{2}}$. Thus a partial ordering $\preccurlyeq$ on $C^\#$ is defined as $x \preccurlyeq y$ iff $y - x \in C^+$.

Theorem 2.2. The following conditions are hold for C:

i) There is a unique element $b \in C^+$ such that $b^2 = a$ for $a \in C^+$.

ii) The set C^+ is equal to $\{aa^*: a \in C\}$.

iii) If $a, b \in C^\#$ and $\theta_C \preccurlyeq a \preccurlyeq b$, then $\|a\|_C \preccurlyeq \|b\|_C$.

iv) If $a, b \in C^\#$, $c \in C$ and $a \preccurlyeq b$, then $c^*ac \preccurlyeq c^*bc$.

Ma et al. (2014) introduced the notion of C^*-algebra-valued metric space and proved BCP in such spaces. Then motivated by the results obtained in Ma et al. (2014), Shateri (2017) presented the notion of C^*-algebra-valued modular space as follows:

Definition 2.2. Let V be a vector space over the field K. The functional $\rho: V \to C$ called C^*-algebra-valued modular if the followings hold:

cm1) $\rho(x) \succcurlyeq \theta_C$ and $\rho(x) = \theta_C$ if and only if $x = \theta_V$.

cm2) $\rho(\alpha x) = \rho(x)$ for each $\alpha \in K$ with $|\alpha| = 1$.

cm3) $\rho(\alpha x + \beta y) \preccurlyeq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$ and $\alpha = 1 - \beta$, for arbitrary $x, y \in V$.

Note that the subset

$$V_\rho = \left\{x \in V: \lim_{\lambda\to 0}\rho(\lambda x) = \theta_C\right\}$$

is a subspace of V and V_ρ is called C^*-algebra-valued modular space.

Definition 2.3. Let V_ρ be a C^*-algebra-valued modular space. Then a mapping $T: V_\rho \to V_\rho$ is called a C^*-algebra-valued contractive mapping on V_ρ if there is $k \in C$ with $\|k\| < 1$ and $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$ such that

$$\rho\big(\alpha(Tx - Ty)\big) \preccurlyeq k^*\rho\big(\beta(x-y)\big)k$$

for all $x, y \in V$.

Shateri (2017) gives definitions of ρ-convergence, Δ_2-condition, ρ-Cauchy and ρ-completeness in accordance with the literature and introduces the following theorem:

Theorem 2.3. Suppose that V_ρ is a ρ-complete modular space with the Δ_2-condition and T is a C^*-algebra-valued contractive mapping on V_ρ. Then T has a unique fixed point in V_ρ.

In the following some necessary definitions and properties are recalled. (Rudin 1991; Liu and Xu 2013).

Definition 2.4. Let $\mathcal{A}$ be a Banach space over $K \in \{\mathbb{R}, \mathbb{C}\}$ and $\|.\|_{\mathcal{A}}$ be a norm on $\mathcal{A}$. $\mathcal{A}$ is said to be a Banach algebra if there is an operation of multiplication satisfying the following conditions:

i) $(u+v)w = uw + vw$ and $u(v+w) = uv + uw$.

ii) $(uv)w = u(vw)$.

iii) $\beta(uv) = (\beta u)v = u(\beta v)$.

iv) $\|uv\|_{\mathcal{A}} \leq \|u\|_{\mathcal{A}}\|v\|_{\mathcal{A}}$.

for all $u, v, w \in \mathcal{A}$ and $\beta \in K$. If there is an element $e \in \mathcal{A}$ such that $ea = ae = a$ for all $a \in \mathcal{A}$, then e is called the multiplicative unit of the Banach algebra $\mathcal{A}$. An element $a \in \mathcal{A}$ is called invertible if there is $a^{-1} \in \mathcal{A}$ such that $aa^{-1} = a^{-1}a = e$. In the rest of the paper $\mathcal{A}$ is supposed to be a Banach algebra with the multiplicative unit e and zero vector $\theta_{\mathcal{A}}$.

Definition 2.5. Let $P \subseteq \mathcal{A}$, then P is called a cone if the followings hold:

i) $\{e, \theta_{\mathcal{A}}\} \subset P$.

ii) $\mu P + \beta P \subset P$ where all μ, β are non-negative real numbers.

iii) $PP = P^2 \subset P$.

iv) $P \cap (-P) = \{\theta_{\mathcal{A}}\}$.

A partial ordering $\preccurlyeq$ on $\mathcal{A}$ is defined as $u \preccurlyeq v$ iff $v - u \in P$. $u \prec v$ stands for $u \preccurlyeq v$ and $u \neq v$. $intP$ denotes the interior of P. $u \ll v$ represents $v - u \in intP$. P is said to be a solid cone if $intP \neq \emptyset$. The cone P is said to be normal if there exists $L > 0$ such that for all $x, y \in \mathcal{A}$, $\theta_{\mathcal{A}} \preccurlyeq x \preccurlyeq y$ implies $\|x\|_{\mathcal{A}} \leq L\|y\|_{\mathcal{A}}$. From now on P denotes a normal solid cone of $\mathcal{A}$ unless otherwise stated.

Definition 2.6. Let X be a non-empty set and $d: X \times X \to \mathcal{A}$ be amapping holding the following conditions:

i) $\theta_{\mathcal{A}} \preccurlyeq d(u,v)$ for all $u, v \in X$ and $d(u,v) = \theta_{\mathcal{A}}$ if and only if $u = v$.

ii) $d(u,v) = d(v,u)$ for all $u, v \in X$.

iii) $d(u,w) \preccurlyeq d(u,v) + d(v,w)$ for all $u, v, w \in X$.

Then (X, d) is said to be a cone metric space over $\mathcal{A}$.

BCP in such spaces is introduced by Liu and Xu (2013) as follows:

Theorem 2.4. Let (X, d) be a cone metric space over $\mathcal{A}$ and P be a normal solid cone of $\mathcal{A}$ where $a \in P$ with $r(a) < 1$. If the mapping $T: X \to X$ holds following condition for all $x, y \in X$, then it has a unique fixed point in X:

$$d(Tx, Ty) \preccurlyeq ad(x,y).$$

After the announcement of this theorem, Xu and Radenovic' (2014) showed that there is no need to normality condition to prove BCP mentioned above. However, it must be noted that as a generalization of the usual modular space, a cone modular space in this paper can be defined if P holds the normality condition.

Lemma 2.1. The spectral radius $r(a)$ of $a \in \mathcal{A}$ holds $r(a) = \lim_{n\to\infty}\|a^n\|_{\mathcal{A}}^{\frac{1}{n}}$.

If $r(a) < 1$, then $e - a$ is invertible in $\mathcal{A}$. Furthermore

$$(e-a)^{-1} = \sum_{i=0}^{\infty} a^i.$$

3. Main Results

In the sequel it is first shown that BCP in C^*-algebra-valued modular spaces is equivalent to BCP in the usual modular spaces:

Theorem 3.1. BCP in the sense of Theorem 2.3. is equivalent to one in the usual modular space.

Proof. From the Definition 2.3. it is known that there is $a \in C$ with $\|a\|_C < 1$ and $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$ such that $\rho(\alpha(Tx - Ty)) \preccurlyeq a^*\rho(\beta(x-y))a$ for all $x, y \in V$. Moreover, by ii) in Theorem 2.2. it is seen that there exists $u_f \in C$ such that $\rho(\beta(x-y)) = {u_f}^* u_f$. Hence $\|\rho(\beta(x-y))\|_C = \|{u_f}^* u_f\|_C = \|u_f\|_C^2$. On the other hand, since $\rho(\alpha(Tx - Ty)) \preccurlyeq a^*\rho(\beta(x-y))a = a^*{u_f}^* u_f a = (u_f a)^* u_f a$, then by using iii) in Theorem 2.2. the following is obtained:

$$\begin{aligned}\|\rho(\alpha(Tx-Ty))\|_C &\preccurlyeq \left\|(u_f a)^* u_f a\right\|_C = \|u_f a\|_C^2 \\ &\preccurlyeq \|a\|_C^2 \|u_f\|_C^2 \\ &= \|a\|_C^2 \|\rho(\beta(x-y))\|_C. \quad (3.1)\end{aligned}$$

Now consider a mapping $F: V_\rho \to [0,\infty]$ such as $F(x) = \|\rho(x)\|_C$. Then F is a usual modular. Indeed,

i) Let $F(x) = 0$. Then $\|\rho(x)\|_C = 0$. Thus by the property of norm $\rho(x) = 0$. Since ρ is a modular, then $x = \theta_V$.

ii) Let μ be a scalar with $|\mu| = 1$. Then $F(\mu x) = \|\rho(\mu x)\|_C = \|\rho(x)\|_C = F(x)$.

iii) Let $\mu = 1 - \lambda$ for $\mu, \lambda \geq 0$. Then by using iii) in Theorem 2.2. and triangle inequality of the norm,

$$\begin{aligned}F(\mu x + \lambda y) = \|\rho(\mu x + \lambda y)\|_C &\leq \|\rho(x) + \rho(y)\|_C \\ &\leq \|\rho(x)\|_C + \|\rho(y)\|_C \\ &= F(x) + F(y).\end{aligned}$$

By letting $\|a\|_C^2$, $k < 1$. Thus by (3.1)

$$F(\alpha(Tx - Ty)) \preccurlyeq kF(\beta(x-y)).$$

Hence, BCP in C^*-algebra valued modular spaces is equivalent to one in the usual modular spaces.

Now introduced a proper space where a proper generalization for BCP in classical modular space could be obtained.

Definition 3.1. Let V be a vector space over K. A mapping $\rho: V \to \mathcal{A}$ is called a cone modular functional if it satisfies the followings:

cmf1 $\rho(u) \succcurlyeq \theta_{\mathcal{A}}$ and $\rho(u) = \theta_{\mathcal{A}}$ iff $u = \theta_V$.

cmf2) $\rho(\alpha u) = \rho(u)$ for each $\alpha \in K$ with $|\alpha| = 1$.

cmf3) $\rho(\alpha u + \beta v) \preccurlyeq \rho(u) + \rho(v)$ if $\alpha, \beta \geq 0$ and $\alpha = 1 - \beta$, for arbitrary $u, v \in V$.

In addition to the conditions above, if ρ satisfies) $\rho(\alpha u + \beta v) \preccurlyeq \alpha\rho(u) + \beta\rho(v)$ whenever $\alpha, \beta \geq 0$ and $\alpha = 1 - \beta$, then ρ is called convex.

It is clear that

$$V_\rho = \left\{x \in V: \lim_{\lambda \to 0} \rho(\lambda x) = \theta_{\mathcal{A}}\right\}$$

is a subspace of V. Indeed,

i) Let $x, y \in V_\rho$. Then $\lim_{\lambda \to 0} \rho(\lambda x) = \theta_{\mathcal{A}}$ and $\lim_{\lambda \to 0} \rho(\lambda y) = \theta_{\mathcal{A}}$. By using cmf3, $\rho(\lambda(x+y)) = \rho\left(\frac{1}{2}(2\lambda x + 2\lambda y)\right) \preccurlyeq \rho(2\lambda x) + \rho(2\lambda y)$. Taking $t = 2\lambda$, it is seen that $t \to 0$ as $\lambda \to 0$. So $\theta_{\mathcal{A}} \preccurlyeq \lim_{\lambda \to 0} \rho(\lambda x + \lambda y) \preccurlyeq \theta_{\mathcal{A}}$. Thus by the normaity of the cone, the Sandwich Theorem can be used. Therefore $\lim_{\lambda \to 0} \rho(\lambda(x+y)) = \theta_{\mathcal{A}}$, implying $x + y \in V_\rho$.

ii) Take an arbitrary $\alpha \in K$ and $x \in V_\rho$. Then $\lim_{\lambda \to 0} \rho(\lambda x) = \theta_{\mathcal{A}}$. Letting $\alpha\lambda = t$, $t \to 0$ as $\lambda \to 0$. Hence $\lim_{\lambda \to 0} \rho(\lambda \alpha x) = \theta_{\mathcal{A}}$. So $\alpha x \in V_\rho$.

In the following V_ρ denotes a cone modular space over Banach algebra $\mathcal{A}$.

Note that the cone modular space over $\mathcal{A}$ is a generalization of the usual modular space.

Let a functional on V_ρ be defined as $\|x\|_F = inf\left\{\delta > 0: \left\|\rho\left(\frac{x}{\delta}\right)\right\|_{\mathcal{A}} \leq \delta\right\}$. Note that $\|.\|_F$ is an F-norm, that is, it satisfies the following conditions:

i) $\|x\|_F = 0$ iff $x = \theta_V$.

ii) $\|x + y\|_F \leq \|x\|_F + \|y\|_F$.

iii) $\|-x\|_F = \|x\|_F$.

iv) $\alpha_n \to \alpha$ and $\|x_n - x\|_F \to 0$ imply $\|\alpha x_n - \alpha x\|_F \to 0$.

Definition 3.2. Let $\{x_n\}$ be in V_ρ.

i) $\{x_n\}$ is called ρ-convergent to $x \in V_\rho$ if for each $\varepsilon > 0$ there is a natural number N and $\mu > 0$ such that $\|\rho(\mu(x_n - x))\|_{\mathcal{A}} < \varepsilon$ for all $n \geq N$.

ii) $\{x_n\}$ is a ρ-Cauchy if for each $\varepsilon > 0$ there is a natural number N and $\mu > 0$ such that $\|\rho(\mu(x_n - x_m))\|_{\mathcal{A}} < \varepsilon$ for all $n, m \geq N$.

iii) V_ρ is ρ-complete if each ρ-Cauchy sequence with respect to $\mathcal{A}$ is ρ-convergent.

iv) ρ satisfies Δ_2-condition if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\|\rho(2x_n)\|_{\mathcal{A}} < \varepsilon$ whenever $\|\rho(x_n)\|_{\mathcal{A}} < \varepsilon$ for $n \geq n_0$.

Remark 3.1. Since $\|\rho(x)\|_{\mathcal{A}} \leq \|x\|_F$, then the norm convergence implies modular convergence to the same limit.

Remark 3.2. If $0 < \alpha < \beta$, then from the Definition 3.1., $\rho(\alpha x) = \rho\left(\frac{\alpha}{\beta}\beta x\right) \preccurlyeq \rho(\beta x)$ for all $x \in V$ with $y = 0$. Furthermore, if ρ is a convex cone modular on V and $|\alpha| \leq 1$, then $\rho(\alpha x) \preccurlyeq \alpha\rho(x)$ for all $x \in V$.

Definition 3.3. A mapping $T: V_\rho \to V_\rho$ is called a cone contractive mapping on V_ρ if there exists a scalar vector $k \in P$ with $r(k) < 1$ and $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$ such that for all $x, y \in V_\rho$

$$\rho(\alpha(Tx - Ty)) \preccurlyeq k\rho(\beta(x - y)). \qquad (3.2)$$

Theorem 3.2. Let V_ρ be a ρ-complete modular space with Δ_2-condition and T be a cone contractive mapping on V_ρ. Then T has a unique fixed point in V_ρ.

Proof. If $k = \theta_{\mathcal{A}}$, then the proof is clear. Thus, assume that $k \neq \theta_{\mathcal{A}}$. Let $\alpha_0 \in \mathbb{R}^+$ be with $\frac{\beta}{\alpha} + \frac{1}{\alpha_0} = 1$. For an arbitrary $x \in V_\rho$ and $n \in \mathbb{N}$, set $x_{n+1} = Tx_n = T^{n+1}x$. Since $\alpha > \beta$, then using Remark 3.2. and Definition 3.2.

$$\begin{aligned}
\rho(\beta(x_{n+1} - x_n)) &= \rho(\beta(Tx_n - Tx_{n-1})) \\
&\preccurlyeq \rho(\alpha(Tx_n - Tx_{n-1})) \\
&\preccurlyeq k\rho(\beta(x_n - x_{n-1})) \\
&= k\rho(\beta(Tx_{n-1} - Tx_{n-2})) \\
&\preccurlyeq k\rho(\alpha(Tx_{n-1} - Tx_{n-2})) \\
&\preccurlyeq k^2\rho(\beta(x_{n-1} - x_{n-2})) \ldots \\
&\preccurlyeq k^n\rho(\beta(x_1 - x_0)).
\end{aligned}$$

Since $\frac{\beta}{\alpha} + \frac{1}{\alpha_0} = 1$, then using cmf3

$$\begin{aligned}
&\rho(\beta(x_{n+1} - x_{n-1})) \\
&= \rho(\beta(x_{n+1} + x_n - x_n - x_{n-1})) \\
&= \rho(\beta(x_{n+1} - x_n) + \beta(x_n - x_{n-1})) \\
&= \rho\left(\beta\frac{\alpha}{\alpha}(x_{n+1} - x_n) + \beta\frac{\alpha_0}{\alpha_0}(x_n - x_{n-1})\right) \\
&\preccurlyeq \rho(\alpha(x_{n+1} - x_n)) + \rho(\beta\alpha_0(x_n - x_{n-1})).
\end{aligned}$$

Since $\alpha > \beta$, then by using 3.2 the following is obtained,

$$\rho(\beta(x_{n+1} - x_{n-1})) \preccurlyeq k\rho(\beta(x_n - x_{n-1})) + \rho(\beta\alpha_0(x_n - x_{n-1})).$$

By applying recursively the approach used above, the following inequality is obtained

$$\rho(\beta(x_{n+1} - x_{n-1})) \preccurlyeq k^n\rho(\beta\alpha_0(x_1 - x_0)) + k^{n-1}\rho(\beta\alpha_0(x_1 - x_0)).$$

Thus for $n + 1 > m$

$$\begin{aligned}
\rho(\beta(x_{n+1}-x_m)) &\preccurlyeq \rho(\alpha(x_{n+1}-x_{m+1})) + \rho(\beta\alpha_0(x_{m+1}-x_m)) \\
&\preccurlyeq \rho(\alpha(x_{n+1}-x_{m+1})) + k^m\rho(\beta\alpha_0(x_1-x_0)) \\
&= \rho(\alpha(T_n-T_m)) + k^m\rho(\beta\alpha_0(x_1-x_0)) \\
&\preccurlyeq k\rho(\beta(x_n-x_m)) + k^m\rho(\beta\alpha_0(x_1-x_0)) \\
&\preccurlyeq k\big[\rho\big(\alpha(x_n-x_{m+1})\big) + \rho\big(\beta\alpha_0(x_{m+1}-x_m)\big)\big] + k^m\rho(\beta\alpha_0(x_1-x_0)) \\
&\preccurlyeq k\rho\big(\alpha(x_n-x_{m+1})\big) + kk^m\rho(\beta\alpha_0(x_1-x_0)) + k^m\rho(\beta\alpha_0(x_1-x_0)) \\
&\preccurlyeq k^2\rho\big(\beta(x_{n-1}-x_m)\big) + \{k^{m+1}+k^m\}\rho(\beta\alpha_0(x_1-x_0)) \\
&\preccurlyeq k^3\rho\big(\beta(x_{n-2}-x_m)\big) + \{k^{m+2}+k^{m+1}+k^m\}\rho(\beta\alpha_0(x_1-x_0)).
\end{aligned}$$

By induction,

$$\begin{aligned}
\rho(\beta(x_{n+1}-x_m)) &\preccurlyeq k^{n-m+1}\rho(\beta(x_m-x_m)) + \{k^{m+n-m}+\cdots+k^{m+1}+k^m\}\rho(\beta\alpha_0(x_1-x_0)) \\
&= k^m(e+k+k^2+\cdots+k^{n-m})\rho(\beta\alpha_0(x_1-x_0)).
\end{aligned}$$

Since $r(k)<1$, then by Lemma 2.1. it is known that $e-k$ is invertible and $(e-k)^{-1}=\sum_{i=0}^{\infty}k^i$. Thus

$$\begin{aligned}
\rho(\beta(x_{n+1}-x_m)) &\preccurlyeq k^m\left[\sum_{i=0}^{\infty}k^i\right]\rho(\beta\alpha_0(x_1-x_0)) \\
&= k^m(e-k\,)^{-1}\rho(\beta\alpha_0(x_1-x_0)).
\end{aligned}$$

Since P is a normal solid cone with a normal constant L and $\|k^m\|_{\mathcal{A}}\to 0$ $(m\to\infty)$. Thus for $(m\to\infty)$

$$\begin{aligned}
&\|\rho(\beta(x_{n+1}-x_m))\|_{\mathcal{A}} \\
&\leq L\|k^m\|_{\mathcal{A}}\|(e-k\,)^{-1}\|_{\mathcal{A}}\|\rho(\beta\alpha_0(x_1-x_0))\|_{\mathcal{A}} \\
&\to 0.
\end{aligned}$$

Thus $\{x_n\}$ is a ρ-Cauchy sequence. Since V_ρ is a ρ-complete cone modular space over the Banach algebra $\mathcal{A}$, there exists $x^*\in V_\rho$ and $\alpha>0$ such that

$$\|\rho(\alpha(x_n-x^*))\|_{\mathcal{A}} = \|\rho(\alpha(Tx_{n-1}-x^*))\|_{\mathcal{A}} < c.$$

Now it remains to show that x^* is a fixed point of T. Indeed,

$$\begin{aligned}
\rho\left(\frac{\alpha}{2}(Tx^*-x^*)\right) &= \rho\left(\frac{\alpha}{2}(Tx^*-T^{n+1}x) + \frac{\alpha}{2}(T^{n+1}x-x^*)\right) \\
&\preccurlyeq \rho\big(\alpha(Tx^*-T^{n+1}x)\big) + \rho\big(\alpha(T^{n+1}x-x^*)\big) \\
&\preccurlyeq k\rho\big(\beta(x^*-T^nx)\big) + \rho\big(\alpha(T^{n+1}x-x^*)\big) \\
&\preccurlyeq k\rho\big(\alpha(x^*-T^nx)\big) + \rho\big(\alpha(T^{n+1}x-x^*)\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|\rho\left(\frac{\alpha}{2}(Tx^*-x^*)\right)\right\|_{\mathcal{A}} &\leq L\Big(\|k\|_{\mathcal{A}}\big\|\rho\big(\alpha(x^*-T^nx)\big)\big\|_{\mathcal{A}} + \big\|\rho\big(\alpha(T^{n+1}x-x^*)\big)\big\|_{\mathcal{A}}\Big).
\end{aligned}$$

For $(n\to\infty)$, $L\Big(\|k\|_{\mathcal{A}}\big\|\rho\big(\alpha(x^*-T^nx)\big)\big\|_{\mathcal{A}} + \big\|\rho\big(\alpha(T^{n+1}x-x^*)\big)\big\|_{\mathcal{A}}\Big)\to 0$. Thus $\left\|\rho\left(\frac{\alpha}{2}(Tx^*-x^*)\right)\right\|_{\mathcal{A}}=0$. Therefore $Tx^*=x^*$. Now assume that $y^*\neq(x^*)$ be another fixed point of T. Then

$$\begin{aligned}
\rho\big(\beta(x^*-y^*)\big) &= \rho\big(\beta(Tx^*-Ty^*)\big) \\
&\preccurlyeq \rho\big(\alpha(Tx^*-Ty^*)\big) \\
&\preccurlyeq k\rho\big(\beta(x^*-y^*)\big) \\
&\preccurlyeq k^2\rho\big(\beta(x^*-y^*)\big)\ldots \\
&\preccurlyeq k^n\rho\big(\beta(x^*-y^*)\big).
\end{aligned}$$

Since

$$\big\|\rho\big(\beta(x^*-y^*)\big)\big\|_{\mathcal{A}} \leq L\|k^n\|_{\mathcal{A}}\big\|\rho\big(\beta(x^*-y^*)\big)\big\|_{\mathcal{A}} \to 0$$

while $n\to\infty$, then $\rho\big(\beta(x^*-y^*)\big)=\theta_{\mathcal{A}}$ and so $x^*=y^*$. Hence the fixed point is unique.

Now an example is presented to show that the main result of this work provides a real generalization for the fixed point theory in the modular spaces:

Example 3.1. Let $\mathcal{A} = \mathbb{R}^2$. For each $(b_1, b_2) \in \mathcal{A}$, $\|(b_1, b_2)\|_{\mathcal{A}} = |b_1| + |b_2|$. The multiplication is defined as $ba = (b_1, b_2)(a_1, a_2) = (b_1 a_1, b_1 a_2 + b_2 a_1)$. Then it is obvious that $\mathcal{A}$ is a Banach algebra with unit $e = (1,0)$. Let $P = \{(b_1, b_2) \in \mathbb{R}^2 : b_1, b_2 > 0\}$. Thus P is a normal solid cone with a constant $L = 1$. Let $V = \mathbb{R}^2$ and the cone modular ρ be defined by $\rho(b) = \rho\big((b_1, b_2)\big) = (|b_1|, |b_2|)$. So, $\rho(b) \in P$. Then $V_\rho = \left\{b \in V : \lim_{\lambda \to 0} \rho(\lambda b) = \theta_{\mathcal{A}}\right\}$ is a ρ-complete cone modular space over $\mathcal{A}$. The mapping $T: V_\rho \to V_\rho$ is defined by

$$T(b) = T\big((b_1, b_2)\big) = (\log(4 + |b_1|), arctan(3 + |b_2|) + \lambda b_1),$$

where λ can be any large positive real number. By Lagrange mean value theorem

$$\rho\left(\alpha\big(T(b_1, b_2) - T(a_1, a_2)\big)\right) \preccurlyeq \left(\frac{\alpha}{4}|b_1 - a_1|, \frac{\alpha}{10}|b_2 - a_2| + \lambda(b_1 - a_1)\right) \preccurlyeq \left(\frac{1}{2}, \lambda\right)\rho\left(\frac{\alpha}{2}\big((b_1, b_2) - (a_1, a_2)\big)\right).$$

Since $r\left(\left(\frac{1}{2}, \lambda\right)\right) = \lim_{n \to \infty} \left\|\left(\frac{1}{2}, \lambda\right)^n\right\|^{\frac{1}{n}} = \frac{1}{2} < 1$, then by Theorem 3.2., T has a unique fixed point theorem in $\mathcal{A}$. Now it is shown that T is not a contraction in the setting of usual modular spaces. Indeed, let $\rho^* = \xi_c \circ \rho$ where $c \in intP$ and $\xi_c: \mathcal{A} \to \mathbb{R}$ is the nonlinear scalarization function defined by $\xi_c(b) = inf\{t \in \mathbb{R}: b \in tc - P\} = inf\{t \in \mathbb{R}: b \leq tc\}$ (Gerstewitz, 1983) Therefore, since $intP = \{(c_1, c_2) \in \mathbb{R}^2 : c_1, c_2 > 0\}$, then

$$\xi_c(b) = \xi_c\big((b_1, b_2)\big) = inf\{t \in \mathbb{R}: (b_1, b_2) \leq t(c_1, c_2)\} = max\left\{\frac{b_1}{c_1}, \frac{b_2}{c_2}\right\}$$

for $c = (c_1, c_2) \in intP$ and $b = (b_1, b_2) \in \mathcal{A}$. Thus,

$$\rho^*(a) = (\xi_c \circ \rho)(a_1, a_2) = max\left\{\frac{|a_1|}{c_1}, \frac{|a_2|}{c_2}\right\}$$

for $a, b \in V$. Let $\alpha > \frac{c_2}{c_1}$ and consider $a = (1,0), b = (0,0)$. Thus

$$\rho^*(Ta - Tb) = max\left\{\frac{\log 5 - \log 4}{c_1}, \frac{\alpha}{c_2}\right\} \succcurlyeq \frac{\alpha}{c_2} \succ \frac{1}{c_1} = \rho^*(a - b)$$

implying that T is not a contraction in the setting of modular space V_{ρ^*}.

4. References

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