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Research Article

Hopf algebra structure on superspace $\mathrm{SP}_q^{2|1}$

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Abstract

Super-Hopf algebra structure on the function algebra on the extended quantum symplectic superspace $\mathrm{SP}_q^{2|1}$, denoted by $\mathbb{F}(\mathrm{SP}_q^{2|1})$, is defined. A quantum Lie superalgebra derived from $\mathbb{F}(\mathrm{SP}_q^{2|1})$ is explicitly obtained.

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1. Introduction

Quantum supergroups and quantum superalgebras are even richer mathematical subjects as compared to Lie supergroups and Lie superalgebras. A quantum superspace is a space that quantum supergroup acts with linear transformations and whose coordinates belong to a noncommutative associative superalgebra [7].

Some algebras have been considered which are covariant with respect to the quantum supergroups in [4]. Using the corepresentation of the quantum supergroup $\mathrm{OSP}_q(1|2)$, some non-commutative spaces covariant under its coaction have been constructed [2]. In the present work, we set up a super-Hopf algebra structure on an algebra which appears in both paper. We denote this algebra by $\mathbb{O}(\mathrm{SP}_q^{2|1})$. As is known, the matrix elements of the quantum supergroups $\mathrm{OSP}_q(1|2)$ and $\mathrm{OSP}_q(2|1)$ are the same and they act both quantum superspaces $\mathrm{SP}_q^{1|2}$ and $\mathrm{SP}_q^{2|1}$. But these two quantum superspaces are not the same. A study on $\mathrm{SP}_q^{1|2}$ was made in [3]. Here we will work on the quantum symplectic superspace $\mathrm{SP}_q^{2|1}$.

2. Review of quantum symplectic group

In this section, we will give some information about the structures of quantum symplectic groups as much as needed.

The algebra $\mathbb{O}(\mathrm{OSP}_q(1|2))$ is generated by the *even* elements a, b, c, d and *odd* elements α, δ. Standard FRT construction [5] is obtained via the matrix R given in [6]. Using the RTT-relations and the q-orthosymplectic condition, all defining relations of $\mathbb{O}(\mathrm{OSP}_q(1|2))$ are explicitly obtained in [2]:

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Theorem 2.1. *The generators of* $\mathbb{O}(\mathrm{OSP}_q(1|2))$ *satisfy the relations*

$$\begin{aligned}
&ab = q^2ba, \quad ac = q^2ca, \quad a\alpha = q\alpha a,\\
&a\delta = q\delta a + (q-q^{-1})\alpha c, \quad ad = da + (q-q^{-1})[(1+q^{-1})bc + q^{-1/2}\alpha\delta],\\
&bc = cb, \quad bd = q^2db, \quad b\alpha = q^{-1}\alpha b, \quad b\delta = q\delta b,\\
&cd = q^2dc, \quad c\alpha = q^{-1}\alpha c, \quad c\delta = q\delta c,\\
&d\alpha = q^{-1}\alpha d + (q^{-1}-q)\delta b, \quad d\delta = q^{-1}\delta d,\\
&\alpha\delta = -q\delta\alpha + q^{-1/2}(q^2-1)bc, \quad \alpha^2 = q^{1/2}(q-1)ba, \quad \delta^2 = q^{1/2}(q-1)dc.
\end{aligned} \tag{2.1}$$

In (2.1), the relations involving the elements γ, e and β are not written. They can be found in [2]. Other relations that we need in this study are given below:

$$\begin{aligned}
&[e,\alpha]_q = q^{1/2}(q-1)(\gamma b + \beta a), \quad [e,\beta]_{q^{-1}} = q^{-1/2}(q^{-1}-1)(\delta b + \alpha d),\\
&[e,\gamma]_q = q^{1/2}(1-q)(\delta a + \alpha c), \quad [e,\delta]_{q^{-1}} = q^{-1/2}(1-q^{-1})(\gamma d + \beta c),\\
&\beta^2 = q^{1/2}(q-1)db, \quad \gamma^2 = q^{1/2}(q-1)ca,\\
&e^2 = 1 - q^{-1/2}[\alpha,\delta]_q = 1 + q^{1/2}[\beta,\gamma]_{q^{-1}}
\end{aligned} \tag{2.2}$$

where $[u,v]_Q = uv - Qvu$.

The quantum superdeterminant is defined by

$$\begin{aligned}
D_q &= ad - qbc - q^{1/2}\alpha\delta\\
&= da - q^{-1}bc + q^{-1/2}\delta\alpha.
\end{aligned}$$

The element D_q is a central element of $\mathbb{O}(\mathrm{OSP}_q(2|1))$.

If $\mathbb{A}$ and $\mathbb{B}$ are $\mathbb{Z}_2$-graded algebras, then their tensor product $\mathbb{A}\otimes\mathbb{B}$ is the $\mathbb{Z}_2$-graded algebra whose underlying space is $\mathbb{Z}_2$-graded tensor product of $\mathbb{A}$ and $\mathbb{B}$. The following definition gives the product rule for tensor product of algebras. Let us denote by $\tau(a)$ the *grade* (or *degree*) of an element $a \in \mathbb{A}$.

Definition 2.2. If $\mathbb{A}$ is a $\mathbb{Z}_2$-graded algebra, then the product rule in the $\mathbb{Z}_2$-graded algebra $\mathbb{A}\otimes\mathbb{A}$ is defined by

$$(a_1\otimes a_2)(a_3\otimes a_4) = (-1)^{\tau(a_2)\tau(a_3)}a_1a_3\otimes a_2a_4$$

where a_i's are homogeneous elements in the algebra $\mathbb{A}$.

Definition 2.3. A super-Hopf algebra is a vector space $\mathbb{A}$ over $\mathbb{K}$ with three linear maps $\Delta : \mathbb{A} \longrightarrow \mathbb{A}\otimes\mathbb{A}$, called the coproduct, $\epsilon : \mathbb{A} \longrightarrow \mathbb{K}$, called the counit, and $S : \mathbb{A} \longrightarrow \mathbb{A}$, called the coinverse, such that

$$(\Delta\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta, \tag{2.3}$$

$$m\circ(\epsilon\otimes\mathrm{id})\circ\Delta = \mathrm{id} = m\circ(\mathrm{id}\otimes\epsilon)\circ\Delta, \tag{2.4}$$

$$m\circ(S\otimes\mathrm{id})\circ\Delta = \eta\circ\epsilon = m\circ(\mathrm{id}\otimes S)\circ\Delta, \tag{2.5}$$

together with $\Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}$, $\epsilon(\mathbf{1}) = 1$, $S(\mathbf{1}) = \mathbf{1}$ and for any $a, b \in \mathbb{A}$

$$\Delta(ab) = \Delta(a)\Delta(b), \epsilon(ab) = \epsilon(a)\epsilon(b), S(ab) = (-1)^{\tau(a)\tau(b)}S(b)S(a) \tag{2.6}$$

where $m : \mathbb{A}\otimes\mathbb{A} \longrightarrow \mathbb{A}$ is the product map, $\mathrm{id} : \mathbb{A} \longrightarrow \mathbb{A}$ is the identity map and $\eta : \mathbb{K} \longrightarrow \mathbb{A}$.

3. Quantum symplectic superspace $\mathrm{SP}_q^{2|1}$

In this section, we define a super-Hopf algebra structure on the extended function algebra of the quantum superspace $\mathrm{SP}_q^{2|1}$.

3.1. The algebra of polynomials on the superspace $\mathrm{SP}_q^{2|1}$

The elements of the symplectic superspace are supervectors generated by two even and an odd components. We define a $\mathbb{Z}_2$-graded symplectic space $\mathrm{SP}^{2|1}$ by dividing the superspace $\mathrm{SP}^{2|1}$ of 3x1 matrices into two parts $\mathrm{SP}^{2|1} = V_0 \oplus V_1$. A vector is an element of V_0 (resp. V_1) and is of grade 0 (resp. 1) if it has the form

$$\begin{pmatrix} x \\ 0 \\ y \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} 0 \\ \theta \\ 0 \end{pmatrix}.$$

While the even elements commute to everyone, the odd element satisfies the relation $\theta^2 = 0$.

In [3], the quantum superspace $\mathrm{SP}_q^{2|1}$ is considered as the dual space of quantum superspace $\mathrm{SP}_q^{1|2}$ and then relations (3.1) below are obtained by interpreting the coordinates as differentiations.

Definition 3.1. Let $\mathbb{K}\langle x, \theta, y\rangle$ be a free associative algebra generated by x, θ, y and I_q be a two-sided ideal generated by $x\theta - q\theta x$, $xy - q^2 yx$, $y\theta - q^{-1}\theta y$, $\theta^2 - q^{1/2}(q-1)yx$. The quantum superspace $\mathrm{SP}_q^{2|1}$ with the function algebra

$$\mathbb{O}(\mathrm{SP}_q^{2|1}) = \mathbb{K}\langle x, \theta, y\rangle / I_q$$

is called $\mathbb{Z}_2$-graded quantum symplectic space (or quantum symplectic superspace).

Here the coordinates x and y with respect to the $\mathbb{Z}_2$-grading are of grade 0 (or even), the coordinate θ with respect to the $\mathbb{Z}_2$-grading is of grade 1 (or odd).

According to the above definition, if $(x, \theta, y)^t \in \mathrm{SP}_q^{2|1}$ then we have

$$x\theta = q\theta x, \quad \theta y = qy\theta, \quad yx = q^{-2}xy, \quad \theta^2 = q^{1/2}(q-1)yx \tag{3.1}$$

where q is a non-zero complex number. This associative algebra over the complex numbers is known as the algebra of polynomials over quantum (2+1)-superspace.

It is easy to see the existence of representations that satisfy (3.1); for instance, there exists a representation $\rho : \mathbb{O}(\mathrm{SP}_q^{2|1}) \to \mathrm{M}(3, \mathbb{C})$ such that matrices

$$\rho(x) = \begin{pmatrix} q & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \rho(\theta) = \begin{pmatrix} 0 & q-1 & 0 \\ 0 & 0 & 0 \\ q^{1/2} & 0 & 0 \end{pmatrix}, \ \rho(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

representing the coordinate functions satisfy the relations (3.1).

Note that the last two relations in (3.1) can be also written as a single relation. Therefore, we say that $\mathbb{O}(\mathrm{SP}_q^{2|1})$ is the superalgebra with generators $x_\pm$ and θ satisfying the relations [4]

$$x_\pm\theta = q^{\pm 1}\theta x_\pm, \quad [x_+, x_-] = q^{-1/2}(q+1)\theta^2. \tag{3.2}$$

where $x_+ = x$ and $x_- = y$.

Definition 3.2 ([4])**.** The quantum supersphere on the quantum symplectic superspace is defined by

$$r = q^{1/2}x_-x_+ + \theta^2 - q^{-1/2}x_+x_-.$$

3.2. A $\star$-structure on the algebra $\mathbb{O}(\mathrm{SP}_q^{2|1})$

Here we define a $\mathbb{Z}_2$-graded involution on the algebra $\mathbb{O}(\mathrm{SP}_q^{2|1})$.

Definition 3.3. Let $\mathbb{A}$ be an associative superalgebra. A $\mathbb{Z}_2$-graded linear map $\star : \mathbb{A} \longrightarrow \mathbb{A}$ is called a superinvolution (or $\mathbb{Z}_2$-graded involution) if

$$(ab)^\star = (-1)^{\tau(a)\tau(b)} b^\star a^\star, \qquad (a^\star)^\star = a$$

for any elements $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, \star)$ is called a $\mathbb{Z}_2$-graded $\star$-algebra.

If the parameter q is real, then the algebra $\mathbb{O}(\mathrm{SP}_q^{2|1})$ becomes a $\star$-algebra with involution determined by the following proposition.

Proposition 3.4. *If $q > 0$ then the algebra $\mathbb{O}(\mathrm{SP}_q^{2|1})$ supplied with the $\mathbb{Z}_2$-graded involution determined by*

$$x_+^\star = q^{1/2}\, x_-, \quad \theta^\star = \mathbf{i}\, \theta, \quad x_-^\star = q^{-1/2}\, x_+$$

becomes a super $\star$-algebra where $\mathbf{i} = \sqrt{-1}$.

Proof. We must show that the relations (3.2) are invariant under the star operation. If q is a positive number, we have

$$(x_\pm\theta - q^{\pm 1}\theta x_\pm)^\star = (\mathbf{i}\,\theta)(q^{\pm 1/2}x_\mp) - q^{\pm 1}(q^{\pm 1/2}x_\mp)(\mathbf{i}\,\theta) = q^{\pm 1/2}\mathbf{i}\,(\theta x_\mp - q^{\pm 1}\theta x_\mp)$$

and since $[x_+, x_-]^\star = [x_+, x_-]$

$$[x_+, x_-] = [x_+, x_-]^\star = q^{-1/2}(q+1)(-\theta^\star\theta^\star) = q^{-1/2}(q+1)\theta^2.$$

Hence the ideal $(x_\pm\theta - q^{\pm 1}\theta x_\pm,\, [x_+, x_-] - q^{-1/2}(q+1)\theta^2)$ is $\star$-invariant and the quotient algebra $\mathbb{K}\langle x_+, \theta, x_-\rangle/(x_\pm\theta - q^{\pm 1}\theta x_\pm,\, [x_+, x_-] - q^{-1/2}(q+1)\theta^2)$ becomes a $\star$-algebra. □

3.3. The super-Hopf algebra structure on $\mathrm{SP}_q^{2|1}$

We define the extended $\mathbb{Z}_2$-graded quantum symplectic space to be the algebra containing $\mathrm{SP}_q^{2|1}$, the unit and x_+^{-1}, the inverse of x_+, which obeys $x_+x_+^{-1} = \mathbf{1} = x_+^{-1}x_+$. We will denote the unital extension of $\mathbb{O}(\mathrm{SP}_q^{2|1})$ by $\mathbb{F}(\mathrm{SP}_q^{2|1})$. The following theorem asserts that the superalgebra $\mathbb{F}(\mathrm{SP}_q^{2|1})$ is a super-Hopf algebra:

Theorem 3.5. *The algebra $\mathbb{F}(\mathrm{SP}_q^{2|1})$ is a $\mathbb{Z}_2$-graded Hopf algebra. The definitions of a coproduct, a counit and a coinverse on the algebra $\mathbb{F}(\mathrm{SP}_q^{2|1})$ are as follows*
(i) *the coproduct $\Delta : \mathbb{F}(\mathrm{SP}_q^{2|1}) \longrightarrow \mathbb{F}(\mathrm{SP}_q^{2|1}) \otimes \mathbb{F}(\mathrm{SP}_q^{2|1})$ is defined by*

$$\Delta(x_+) = x_+ \otimes x_+, \quad \Delta(\theta) = \theta \otimes \mathbf{1} + \mathbf{1} \otimes \theta, \quad \Delta(x_-) \quad = x_+^{-1} \otimes x_- + x_- \otimes x_+^{-1}, \quad (3.3)$$

(ii) *the counit $\epsilon : \mathbb{F}(\mathrm{SP}_q^{2|1}) \longrightarrow \mathbb{C}$ is given by*

$$\epsilon(x_+) = 1, \quad \epsilon(\theta) = 0, \quad \epsilon(x_-) = 0,$$

(iii) the algebra $\mathbb{F}(\mathrm{SP}_q^{2|1})$ admits a $\mathbb{C}$-algebra antihomomorphism $S : \mathbb{F}(\mathrm{SP}_q^{2|1}) \longrightarrow \mathbb{F}(\mathrm{SP}_{q^{-1}}^{2|1})$ defined by

$$S(x_+) = x_+^{-1}, \quad S(\theta) = -\theta, \quad S(x_-) = -x_+x_-x_+.$$

Proof. The axioms (2.3)-(5) are satisfied automatically. It is also not difficult to show that the co-maps preserve the relations (3.2). In fact, for instance,

$$\begin{aligned}\Delta([x_+, x_-]) &= \Delta(x_+x_- - x_-x_+) = \mathbf{1} \otimes [x_+, x_-] + [x_+, x_-] \otimes \mathbf{1}\\ &= q^{-1/2}(q+1)(\mathbf{1} \otimes \theta^2 + \theta^2 \otimes \mathbf{1})\\ \Delta(\theta^2) &= \mathbf{1} \otimes \theta^2 + \theta^2 \otimes \mathbf{1},\end{aligned}$$

and

$$S([x_+,x_-]) = -[x_+,x_-], \quad S(\theta^2) = -\theta^2.$$

Since $S^2(a) = \mathrm{id}(a)$ for all $a \in \mathbb{F}(\mathrm{SP}_q^{2|1})$, the coinverse S is of second order. □

The set $\{x^k\theta^l y^m : k,l,m \in \mathbb{N}_0\}$ form a vector space basis of $\mathbb{F}(\mathrm{SP}_q^{2|1})$. The formula (3.3) gives the action of the coproduct Δ only on the generators. The action of Δ on product on generators can be calculated by taking into account that Δ is a homomorphism.

Corollary 3.6. *For the quantum supersphere r, we have*

$$\Delta(r) = r\otimes \mathbf{1} + \mathbf{1}\otimes r, \quad \epsilon(r) = 0, \quad S(r) = -r.$$

Proof. Using the definition of Δ, as an algebra homomorphism, on the generators of $\mathbb{F}(\mathrm{SP}_q^{2|1})$ in (3.3), it is easy to see that the element $r \in \mathbb{F}(\mathrm{SP}_q^{2|1})$ is a primitive element, that is,

$$\begin{aligned}
\Delta(r) &= q^{1/2}(x_+^{-1}\otimes x_- + x_-\otimes x_+^{-1})(x_+\otimes x_+) + (\theta\otimes\mathbf{1} + \mathbf{1}\otimes\theta)(\theta\otimes\mathbf{1}+\mathbf{1}\otimes\theta)\\
&\quad - q^{-1/2}(x_+\otimes x_+)(x_+^{-1}\otimes x_- + x_-\otimes x_+^{-1})\\
&= q^{1/2}(\mathbf{1}\otimes x_-x_+ + x_-x_+\otimes\mathbf{1}) + \theta^2\otimes\mathbf{1} + \mathbf{1}\otimes\theta^2 - q^{-1/2}(\mathbf{1}\otimes x_+x_- + x_+x_-\otimes\mathbf{1})\\
&= \mathbf{1}\otimes(q^{1/2}x_-x_+ + \theta^2 - q^{-1/2}x_+x_-) + (q^{1/2}x_-x_+ + \theta^2 - q^{-1/2}x_+x_-)\otimes\mathbf{1}.
\end{aligned}$$

Since $\epsilon(\mathbf{1}) = 1$ and

$$m(\mathrm{id}\otimes\epsilon)\Delta(r) = r\epsilon(\mathbf{1}) + \epsilon(r)\mathbf{1} = r = m(\epsilon\otimes\mathrm{id})\Delta(r),$$

we obtain $\epsilon(r) = 0$. Finally, using the fact that S is an anti-homomorphism we get

$$\begin{aligned}
S(r) &= q^{1/2}x_+^{-1}(-x_+x_-x_+) - (-\theta)(-\theta) - q^{-1/2}(-x_+x_-x_+)x_+^{-1}\\
&= -(q^{1/2}x_-x_+ + \theta^2 - q^{-1/2}x_+x_-),
\end{aligned}$$

as desired. □

3.4. Coactions on the quantum symplectic superspace

Let $a,b,c,d,e,\gamma,\alpha,\delta,\beta$ be elements of an algebra $\mathbb{A}$. Assuming that the generators of $\mathbb{O}(\mathrm{OSP}_q(2|1))$ super-commute with the elements of $\mathbb{O}(\mathrm{SP}_q^{2|1})$, define the components of the vectors $X' = (x',\theta',y')^t$ and $X'' = (x'',\theta'',y'')^t$ using the following matrix equalities

$$X' = T\,X \quad \text{and} \quad (X'')^t = X^t\,T \tag{3.4}$$

where $X = (x,\theta,y)^t \in \mathrm{SP}_q^{2|1}$ and $T \in \mathrm{OSP}_q(2|1)$. If we assume that $q \neq 1$ then we have the following theorem proving straightforward computations.

Theorem 3.7. *If the transformations in* (3.4) *preserve the relations* (3.1)*, then the entries of T satisfy the relations* (2.1) *and then generate the algebra* $\mathbb{O}(\mathrm{OSP}_q(2|1))$ *together with q-orthosymplectic condition.*

A left quantum space (or left comodule algebra) for a Hopf algebra H is an algebra $\mathbb{X}$ together with an algebra homomorphism (left coaction) $\delta_L : \mathbb{X} \longrightarrow H\otimes\mathbb{X}$ such that

$$(\mathrm{id}\otimes\delta_L)\circ\delta_L = (\Delta\otimes\mathrm{id})\circ\delta_L \quad \text{and} \quad (\epsilon\otimes\mathrm{id})\circ\delta_L = \mathrm{id}.$$

Right comodule algebra can be defined in a similar way.

Theorem 3.8. (i) *The algebra* $\mathbb{O}(\mathrm{SP}_q^{2|1})$ *is a left and right comodule algebra of the Hopf algebra* $\mathbb{O}(\mathrm{OSP}_q(2|1))$ *with left coaction δ_L and right coaction δ_R such that*

$$\delta_L(X_i) = \sum_{k=1}^{3} t_{ik}\otimes X_k, \quad \delta_R(X_i) = \sum_{k=1}^{3} X_k\otimes t_{ki}. \tag{3.5}$$

(ii) *The quantum supersphere r belongs to the center of* $\mathbb{O}(\mathrm{SP}_q^{2|1})$ *and satisfies* $\delta_L(r) = \mathbf{1}\otimes r$ *and* $\delta_R(r) = r\otimes \mathbf{1}$.

Proof. (i) These assertions are obtained from the relations in (2.1) and (2.2) together with (3.1).

(ii) That r is a central element of $\mathbb{O}(\mathrm{SP}_q^{2|1})$ is shown by using the relations in (3.1). To show that $\delta_L(r) = \mathbf{1}\otimes r$ and $\delta_R(r) = r\otimes \mathbf{1}$ we use the definitions of δ_L and δ_R in (3.5) and the relations (2.1) and (2.2) with $D_q = \mathbf{1}$. □

4. An h-deformation of the superspace $\mathrm{SP}^{2|1}$

In this section, we introduce an h-deformation of the superspace $\mathrm{SP}^{2|1}$ from the q-deformation via a contraction following the method of [1]. Consider the q-deformed algebra of functions on the quantum superspace $\mathrm{SP}_q^{2|1}$ generated by $x_\pm$ and θ with the relations (3.2).

We introduce new coordinates $X_\pm$ and Θ by

$$\mathbf{x} = \begin{pmatrix} x_+ \\ \theta \\ x_- \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{h}{q-1} & 0 & 1 \end{pmatrix} \begin{pmatrix} X_+ \\ \Theta \\ X_- \end{pmatrix} = g\,\mathbf{X}.$$

When the relations (3.2) are used, taking the limit $q\to 1$ we obtain the following exchange relations, which define the h-superspace $\mathrm{SP}_h^{2|1}$:

Definition 4.1. Let $\mathbb{O}(\mathrm{SP}_h^{2|1})$ be the algebra with the generators $X_\pm$ and Θ satisfying the relations

$$X_+\Theta = \Theta X_+, \quad X_-\Theta = \Theta X_- - 2h\Theta X_+, \quad X_+X_- = X_-X_+ + 2\Theta^2 \tag{4.1}$$

where the coordinates $X_\pm$ are even and the coordinate Θ is odd. We call $\mathbb{O}(\mathrm{SP}_h^{2|1})$ the algebra of functions on the $\mathbb{Z}_2$-graded quantum space $\mathrm{SP}_h^{2|1}$.

h-deformed supersphere on the symplectic h-superspace is given by

$$r_h = X_-X_+ + \Theta^2 + h\,X_+^2 - X_+X_- = h\,X_+^2 - \Theta^2.$$

It is easily seen that the quantum supersphere r_h belongs to the center of the superalgebra $\mathbb{O}(\mathrm{SP}_h^{2|1})$.

The definition of dual q-deformed symplectic superspace is given as follows [2].

Definition 4.2. Let $\mathbb{K}\{\varphi_+, z, \varphi_-\}$ be a free associative algebra generated by z, φ_+, φ_- and I_q be a two-sided ideal generated by $z\varphi_\pm - q^{\pm 1}\varphi_\pm z$, $\varphi_-\varphi_+ + q^{-2}\varphi_+\varphi_- + q^{-2}Qz^2$ and $\varphi_\pm^2$. The quantum superspace $SP_q^{1|2}$ with the function algebra

$$\mathbb{O}(SP_q^{1|2}) = \mathbb{K}\{\varphi_+, z, \varphi_-\}/I_q$$

is called $\mathbb{Z}_2$-graded quantum symplectic space (or quantum symplectic superspace) where $Q = q^{1/2} - q^{3/2}$ and $q\neq 0$.

In case of exterior h-superspace, we use the transformation

$$\hat{\mathbf{x}} = g\hat{\mathbf{X}}$$

with the components φ_+, z and φ_- of $\hat{\mathbf{x}}$. The definition is given below.

Definition 4.3. Let $\Lambda(\mathrm{SP}_h^{2|1})$ be the algebra with the generators $\Phi_\pm$ and Z satisfying the relations

$$\Phi_+Z = Z\Phi_+, \quad Z\Phi_- = \Phi_-Z - 2h\Phi_+Z, \quad \Phi_-\Phi_+ = -\Phi_+\Phi_-,$$
$$\Phi_+^2 = 0, \quad \Phi_-^2 = h(2\Phi_-\Phi_+ - Z^2)$$

where the coordinate Z is even and the coordinates $\Phi_\pm$ are odd. We call $\Lambda(\mathrm{SP}_h^{2|1})$ the quantum exterior algebra of the $\mathbb{Z}_2$-graded quantum space $\mathrm{SP}_h^{2|1}$.

5. A Lie superalgebra derived from $\mathbb{F}(\mathrm{SP}_q^{2|1})$

It is known that an element of a Lie group can be represented by exponential of an element of its Lie algebra. By virtue of this fact, one can define the generators of the algebra $\mathbb{F}(\mathrm{SP}_q^{2|1})$ as

$$x_+ := e^u, \quad \theta := q^{-1/2}\,\xi, \quad x_- := e^{-u}v. \tag{5.1}$$

Then, the following lemma can be proved by direct calculations using the relations

$$x_\pm^k\theta = q^{\pm k}\,\theta x_\pm^k, \quad [x_+^k, x_-] = q^{-1/2}\,\frac{q^{2k}-1}{q-1}\,\theta^2\,x_+^{k-1}, \quad \forall k \geq 1$$

whose the proof follows from induction on k.

Lemma 5.1. *The generators u, ξ, v have the following commutation relations (Lie (anti-)brackets)*

$$[u,\xi] = \hbar\,\xi, \quad [\xi, v] = 0, \quad [u,v] = \frac{2\hbar}{1-e^{-\hbar}}\,\xi^2, \tag{5.2}$$

where $q = e^\hbar$ and $\hbar \in \mathbb{R}$.

We denote the algebra for which the generators obey the relations (5.2) by $\mathbb{L}_\hbar := \mathbb{L}(\mathrm{SP}_q^{2|1})$. The $\mathbb{Z}_2$-graded Hopf algebra structure of $\mathbb{L}_\hbar$ can be read off from Theorem 3.5:

Theorem 5.2. *The algebra $\mathbb{L}_\hbar$ is a $\mathbb{Z}_2$-graded Hopf algebra. The definitions of a coproduct, a counit and a coinverse on the algebra $\mathbb{L}_\hbar$ are as follows:*

$$\Delta(u_i) = u_i \otimes \mathbf{1} + \mathbf{1} \otimes u_i, \quad \epsilon(u_i) = 0, \quad S(u_i) = -u_i$$

for $u_i \in \{u, \xi, v\}$.

The following proposition can be easily proved by using the Proposition 3.4 together with (5.1).

Proposition 5.3. *The algebra $\mathbb{L}_\hbar$ supplied with the $\mathbb{Z}_2$-graded involution determined by*

$$u^\star = \tfrac{1}{2}\,\hbar + \ln(e^{-u}v), \quad \xi^\star = \mathbf{i}\,\xi, \quad v^\star = v$$

becomes a super Lie $\star$-algebra.

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