On total mean curvatures of foliated half-lightlike submanifolds in semi-Riemannian manifolds

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Abstract

We derive total mean curvature integration formulas of a three co-dimensional foliation \( F^n \) on a screen integrable half-lightlike submanifold, \( M^{n+1} \) in a semi-Riemannian manifold \( M^{n+3} \). We give generalized differential equations relating to mean curvatures of a totally umbilical half-lightlike submanifold admitting a totally umbilical screen distribution, and show that they are generalizations of those given by [K. L. Duggal and B. Sahin, Differential geometry of lightlike submanifolds, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2010].

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1. Introduction

The rapidly growing importance of lightlike submanifolds in semi-Riemannian geometry, particularly Lorentzian geometry, and their applications to mathematical physics–like in general relativity and electromagnetism motivated the study of lightlike geometry in semi-Riemannian manifolds. More precisely, lightlike submanifolds have been shown to represent different black hole horizons (see [4] and [6] for details). Among other motivations for investing in lightlike geometry by many physicists is the idea that the universe we are living in can be viewed as a 4-dimensional hypersurface embedded in \((4 + m)\)-dimensional spacetime manifold, where \(m\) is any arbitrary integer. There are significant differences between lightlike geometry and Riemannian geometry as shown in [4] and [6], and many more references therein. Some of the pioneering work on this topic is due to Duggal-Bejancu [4], Duggal-Sahin [6] and Kupeli [7]. It is upon those books that many other researchers, including but not limited to [3, 5, 8–11], have extended their theories.

Lightlike geometry rests on a number of operators, like shape and algebraic invariants derived from them, such as trace, determinants, and in general the \(r\)-th mean curvature \(S_r\). There is a great deal of work so far on the case \(r = 1\) (see some in [4, 6] and many more) and as far as we know, very little has been done for the case \(r > 1\). This is partly due to the non-linearity of \(S_r\) for \(r > 1\), and hence very complicated to study. A great

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deal of research on higher order mean curvatures $S_r$ in Riemannian geometry has been done with numerous applications, for instance see [2] and [1]. This gap has motivated our introduction of lightlike geometry of $S_r$ for $r > 1$. In this paper we have considered a half-lightlike submanifold admitting an integrable screen distribution, of a semi-Riemannian manifold. On it we have focused on a codimension 3 foliation of its screen distribution and thus derived integral formuals of its total mean curvatures (see Theorems 4.9 and 4.10). Furthermore, we have considered totally umbilical half-lightlike submanifolds, with a totally umbilical screen distribution and generalized Theorem 4.3.7 of [6] (see Theorem 5.2 and its Corollaries). The paper is organized as follows; In Section 2 we summarize the basic notions on lightlike geometry necessary for other sections. In Section 3 we give some basic information on Newton transformations of a foliation $F$ of the screen distribution. Section 4 focuses on integration formuals of $F$ and their consequences. In Section 5 we discus screen umbilical half-lightlike submanifolds and generalizations of some well-known results of [6].

2. Preliminaries

Let $(M^{n+1}, g)$ be a two-co-dimensional submanifold of a semi-Riemannian manifold $(\overline{M}^{n+3}, \overline{g})$, where $g = \overline{g}|_{TM}$. The submanifold $(M^{n+1}, g)$ is called a half-lightlike if the radical distribution $\text{Rad} \, TM = TM \cap TM^\perp$ is a vector subbundle of the tangent bundle $TM$ and the normal bundle $TM^\perp$ of $M$, with rank one. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\text{Rad} \, TM$ in $TM$, and also choose a screen transversal bundle $S(TM^\perp)$, which is semi-Riemannian and complementary to $\text{Rad} \, TM$ in $TM^\perp$. Then,

$$TM = \text{Rad} \, TM \perp S(TM), \quad TM^\perp = \text{Rad} \, TM \perp S(TM^\perp).$$

(2.1)

We will denote by $\Gamma(\Xi)$ the set of smooth sections of the vector bundle $\Xi$. It is well-known from [4] and [6] that for any null section $E$ of $\text{Rad} \, TM$, there exists a unique null section $N$ of the orthogonal complement of $S(TM^\perp)$ in $S(TM)^\perp$ such that $g(E, N) = 1$, it follows that there exists a lightlike transversal vector bundle $\text{ltr}(TM)$ locally spanned by $N$. Let $W \in \Gamma(S(TM^\perp))$ be a unit vector field, then $\overline{g}(N, N) = \overline{g}(N, Z) = \overline{g}(N, W) = 0$, for any $Z \in \Gamma(S(TM))$.

Let $\text{tr}(TM)$ be complementary (but not orthogonal) vector bundle to $TM$ in $\overline{TM}$. Then we have the following decompositions of $\text{tr}(TM)$ and $\overline{TM}$

$$\text{tr}(TM) = \text{ltr}(TM) \perp S(TM^\perp),$$

(2.2)

$$\overline{TM} = S(TM) \perp S(TM^\perp) \perp \{\text{Rad} \, TM \oplus \text{ltr}(TM)\}.$$ (2.3)

It is important to note that the distribution $S(TM)$ is not unique, and is canonically isomorphic to the factor vector bundle $TM/\text{Rad} \, TM$ [4]. Let $P$ be the projection of $TM$ on to $S(TM)$. Then the local Gauss-Weingarten equations of $M$ are the following;

$$\nabla_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)W,$$

(2.4)

$$\nabla_X N = -A_N X + \tau(X)N + \rho(X)W,$$

(2.5)

$$\nabla_X W = -A_W X + \phi(X)N,$$

(2.6)

$$\nabla_X PY = \nabla_X PY + C(X, PY)E,$$

(2.7)

$$\nabla_X E = -A_E^X X - \tau(X)E,$$

(2.8)

for all $E \in \Gamma(\text{Rad} \, TM)$, $N \in \Gamma(\text{ltr}(TM))$ and $W \in \Gamma(S(TM^\perp))$, where $\nabla$ and $\nabla^*$ are induced linear connections on $TM$ and $S(TM)$, respectively, $B$ and $D$ are the local second fundamental forms of $M$, $C$ is the local second fundamental form on $S(TM)$. Furthermore, $\{A_N, A_W\}$ and $A_E^X$ are the shape operators on $TM$ and $S(TM)$ respectively, and $\tau$, $\rho$, $\phi$ and $\delta$ are differential 1-forms on $TM$. Notice that $\nabla^*$ is a metric connection.
on $S(TM)$ while $\nabla$ is generally not a metric connection. In fact, $\nabla$ satisfies the following relation

$$ (\nabla_X g)(Y, Z) = B(X, Y)\lambda(Z) + B(X, Z)\lambda(Y), $$

for all $X, Y, Z \in \Gamma(TM)$, where $\lambda$ is a 1-form on $TM$ given $\lambda(\cdot) = \bar{g}(\cdot, N)$. It is well-known from [4] and [6] that $B$ and $D$ are independent of the choice of $S(TM)$ and they satisfy

$$ B(X, E) = 0, \quad D(X, E) = -\phi(X), \quad \forall X \in \Gamma(TM). $$

The local second fundamental forms $B$, $D$ and $C$ are related to their shape operators by the following equations

$$ g(A^*_E X, Y) = B(X, Y), \quad \bar{g}(A^*_E X, N) = 0, $$
$$ g(A_W X, Y) = \varepsilon D(X, Y) + \phi(X)\lambda(Y), $$
$$ g(A_N X, P_Y) = C(X, P_Y), \quad \bar{g}(A_N X, N) = 0, $$
$$ \bar{g}(A_W X, N) = \varepsilon \rho(X), \quad \text{where} \quad \varepsilon = \bar{g}(W; W), $$

for all $X, Y \in \Gamma(TM)$. From equations (2.11) we deduce that $A^*_E$ is $S(TM)$-valued, self-adjoint and satisfies $A^*_E E = 0$. Let $\mathcal{R}$ denote the curvature tensor of $\mathcal{M}$, then

$$ \bar{g}(\mathcal{R}(X, Y)P_Z, N) = g((\nabla_X A_N) Y, P_Z) - g((\nabla_Y A_N) X, P_Z) + \tau(Y)C(X, P_Z) - \varepsilon \tau(X)C(Y, P_Z)\{\rho(Y)D(X, P_Z) $$
$$ - \rho(X)D(Y, P_Z)\}, \quad \forall X, Y, Z \in \Gamma(TM). $$

A half-lightlike submanifold $(M, g)$ of a semi-Riemannian manifold $\mathcal{M}$ is said to be totally umbilical [6] if on each coordinate neighborhood $U$ there exist smooth functions $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\text{tr}(TM)$ and $S(TM^\perp)$ respect such that

$$ B(X, Y) = \mathcal{H}_1 g(X, Y), \quad D(X, Y) = \mathcal{H}_2 g(X, Y), \quad \forall X, Y \in \Gamma(TM). $$

Furthermore, when $M$ is totally umbilical then the following relations follows by straightforward calculations

$$ A^*_E X = \mathcal{H}_1 PX, \quad P(A_W X) = \varepsilon \mathcal{H}_2 PX, \quad D(X, E) = 0, \quad \rho(E) = 0, $$

for all $X, Y \in \Gamma(TM)$.

Next, we suppose that $M$ is a half-lightlike submanifold of $\mathcal{M}$, with an integrable screen distribution $S(TM)$. Let $M'$ be a leaf of $S(TM)$. Notice that for any screen integrable half-lightlike $M$, the leaf $M'$ of $S(TM)$ is a co-dimension 3 submanifold of $\mathcal{M}$ whose normal bundle is $\{\text{Rad} TM \perp \text{tr}(TM)\} \perp S(TM^\perp)$. Now, using (2.4) and (2.7) we have

$$ \nabla_X Y = \nabla_X^\prime Y + C(X, P_Y)E + B(X, Y)N + D(X, Y)W, $$

for all $X, Y \in \Gamma(TM')$. Since $S(TM)$ is integrable, then its leave is semi-Riemannian and hence we have

$$ \nabla_X Y = \nabla_X^\prime Y + h'(X, Y), \quad \forall X, Y \in \Gamma(TM'), $$

where $h'$ and $\nabla^\prime$ are second fundamental form and the Levi-Civita connection of $M'$ in $\mathcal{M}$. From (2.18) and (2.19) we can see that

$$ h'(X, Y) = C(X, P_Y)E + B(X, Y)N + D(X, Y)W, $$

for all $X, Y \in \Gamma(TM')$. Since $S(TM)$ is integrable, then it is well-known from [6] that $C$ is symmetric on $S(TM)$ and also $A_N$ is self-adjoint on $S(TM)$ (see Theorem 4.1.2 for details). Thus, $h'$ given by (2.20) is symmetric on $TM'$.

Let $L \in \Gamma(\{\text{Rad} TM \perp \text{tr}(TM)\} \perp S(TM^\perp))$, then we can decompose $L$ as

$$ L = aE + bN + cW, $$

(2.21)
for non-vanishing smooth functions on \( \overline{M} \) given by \( a = \mathfrak{g}(L, N), \ b = \mathfrak{g}(L, E) \) and \( c = \varepsilon \mathfrak{g}(L, W) \). Suppose that \( \mathfrak{g}(L, L) > 0 \), then using (2.21) we obtain a unit normal vector \( \hat{W} \) to \( M' \) given by

\[
\hat{W} = \frac{1}{\mathfrak{g}(L, L)}(aE + bN + cW) = \frac{1}{\mathfrak{g}(L, L)}L.
\]

(2.22)

Next we define a \((1,1)\) tensor \( \mathcal{A}_{\hat{W}}\) in terms of the operators \( \mathcal{A}_E^*, \mathcal{A}_N\) and \( \mathcal{A}_W\) by

\[
\mathcal{A}_{\hat{W}}X = \frac{1}{\mathfrak{g}(L, L)}(aA_E^*X + bA_NX + cA_WX),
\]

(2.23)

for all \( X \in \Gamma(TM)\). Notice that \( \mathcal{A}_{\hat{W}}\) is self-adjoint on \( S(TM) \). Applying \( \nabla_X \) to \( \hat{W} \) and using equations (2.23) (2.4) and (2.11)-(2.13), we have

\[
g(\mathcal{A}_{\hat{W}}X, PY) = -g(\nabla_X \hat{W}, PY), \ \forall X, Y \in \Gamma(TM).
\]

(2.24)

Let \( \nabla^*\perp \) be the connection on the normal bundle \( \{\text{Rad } TM \oplus \text{ltr}(TM)\} \cap S(TM^\perp) \). Then from (2.24) we have

\[
\nabla_X \hat{W} = -\mathcal{A}_{\hat{W}}X + \nabla^*\perp_X \hat{W}, \ \forall X \in \Gamma(TM),
\]

(2.25)

where

\[
\nabla^*\perp_X \hat{W} = -\frac{1}{\mathfrak{g}(L, L)}X(\mathfrak{g}(L, L))\hat{W} + \frac{1}{\mathfrak{g}(L, L)}\left\{\mathfrak{g}(X(a) - a\tau(X))E + \{X(b) + b\tau(X) + c\phi(X)\}N + \{X(c) + aD(X, E) + b\rho(X)\}W\right\}.
\]

**Example 2.1.** Let \( \overline{M} = (\mathbb{R}^5, \mathfrak{g}) \) be a semi-Riemannian manifold, where \( \mathfrak{g} \) is of signature \((-++,+++)\) with respect to canonical basis \((\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5)\), where \((x_1, \cdots, x_5)\) are the usual coordinates on \( \overline{M} \). Let \( M \) be a submanifold of \( \overline{M} \) and given parametrically by the following equations

\[
x_1 = \varphi_1, \ x_2 = \sin \varphi_2 \sin \varphi_3, \ x_3 = \varphi_1, \ x_4 = \cos \varphi_2 \sin \varphi_3, \ x_5 = \cos \varphi_3, \ \text{where} \ \varphi_2 \in [0, 2\pi]\ \text{and} \ \varphi_3 \in (0, \pi/2).
\]

Then we have \( TM = \text{span}\{E, Z_1, Z_2\} \) and \( \text{ltr}(TM) = \text{span}\{N\} \), where

\[
E = \partial x_1 + \partial x_3, \ Z_1 = \cos \varphi_3 \partial x_2 - \sin \varphi_2 \sin \varphi_3 \partial x_5, \ Z_2 = \cos \varphi_3 \partial x_4 - \cos \varphi_2 \sin \varphi_3 \partial x_5 \ \text{and} \ N = \frac{1}{2}(-\partial x_1 + \partial x_3).
\]

Also, by straightforward calculations, we have

\[
W = \sin \varphi_2 \sin \varphi_3 \partial x_2 + \cos \varphi_2 \sin \varphi_3 \partial x_4 + \cos \varphi_3 \partial x_5.
\]

Thus, \( S(TM^\perp) = \text{span}\{W\} \) and hence \( M \) is a half-lightlike submanifold of \( \overline{M} \). Furthermore we have \([Z_1, Z_2] = \cos \varphi_2 \sin \varphi_3 \partial x_2 - \sin \varphi_2 \sin \varphi_3 \partial x_4, \ \text{which leads to} \ [Z_1, Z_2] = \cos \varphi_2 \tan \varphi_3 Z_1 - \sin \varphi_2 \tan \varphi_3 Z_2 \in \Gamma(S(TM)) \). Thus, \( M \) is a screen integrable half-lightlike submanifold of \( \overline{M} \). Finally, it is easy to see that \( \mathcal{A}_W\) is self-adjoint operator on \( S(TM) \).

In the next sections we shall consider screen integrable half-lightlike submanifolds of semi-Riemannian manifold \( \overline{M} \) and derive special integral formulas for a foliation of \( S(TM) \), whose normal vector is \( \hat{W} \) and with shape operator \( \mathcal{A}_{\hat{W}}\).
3. Newton transformations of $A_{\tilde{W}}$

Let $(\mathcal{M}^{m+3}, \tilde{g})$ be a semi-Riemannian manifold and let $(M^{n+1}, g)$ be a screen integrable half-lightlike submanifold of $\mathcal{M}$. Then $S(TM)$ admits a foliation and let $\mathcal{F}$ be a such foliation. Then, the leaves of $\mathcal{F}$ are co-dimension three submanifolds of $\mathcal{M}$, whose normal bundle is $S(TM)^{\perp}$. Let $\tilde{W}$ be unit normal vector to $\mathcal{F}$ such that the orientation of $\mathcal{M}$ coincides with that given by $\mathcal{F}$ and $\tilde{W}$. The Levi-Civita connection $\nabla$ on the tangent bundle of $\mathcal{M}$ induces a metric connection $\nabla'$ on $\mathcal{F}$. Furthermore, $h'$ and $A_{\tilde{W}}\hat{\ }$ are the second fundamental form and shape operator of $\mathcal{F}$. Notice that $A_{\tilde{W}}\hat{\ }$ is self-adjoint on $T\mathcal{F}$ and at each point $p \in \mathcal{F}$ has $n$ real eigenvalues (or principal curvatures) $\kappa_1(p), \cdots, \kappa_n(p).$

Attached to the shape operator $A_{\tilde{W}}\hat{\ }$ are $n$ algebraic invariants

\[ S_r = \sigma_r(\kappa_1, \cdots, \kappa_n), \quad 1 \leq r \leq n, \]

where $\sigma_r : M^n \to \mathbb{R}$ are symmetric functions given by

\[ \sigma_r(\kappa_1, \cdots, \kappa_n) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \kappa_{i_1} \cdots \kappa_{i_r}. \]  

(3.1)

Then, the characteristic polynomial of $A_{\tilde{W}}\hat{\ }$ is given by

\[ \det(A_{\tilde{W}} - tI) = \sum_{\alpha=0}^{n} (-1)^{\alpha} S_{\alpha} t^{n-\alpha}, \]

where $I$ is the identity in $\Gamma(T\mathcal{F})$. The normalized $r$-th mean curvature $H_r$ of $M'$ is defined by

\[ H_r = \left( \binom{n}{r} \right)^{-1} S_r \quad \text{and} \quad H_0 = 1. \]  

(a constant function 1).

In particular, when $r = 1$ then $H_1 = \frac{1}{n} \text{tr}(A_{\tilde{W}}\hat{\ })$ which is the mean curvature of a $\mathcal{F}$. On the other hand, $H_2$ relates directly with the (intrinsic) scalar curvature of $\mathcal{F}$. Moreover, the functions $S_r$ ($H_r$ respectively) are smooth on the whole $M$ and, for any point $p \in \mathcal{F}$, $S_r$ coincides with the $r$-th mean curvature at $p$. In this paper, we shall use $S_r$ instead of $H_r$.

Next, we introduce the Newton transformations with respect to the operator $A_{\tilde{W}}\hat{\ }$. The Newton transformations $T_r : \Gamma(T\mathcal{F}) \to \Gamma(T\mathcal{F})$ of a foliation $\mathcal{F}$ of a screen integrable half-lightlike submanifold $M$ of an $(n + 3)$-dimensional semi-Riemannian manifold $\mathcal{M}$ with respect to $A_{\tilde{W}}\hat{\ }$ are given by the inductive formula

\[ T_0 = I, \quad T_r = (-1)^r S_r I + A_{\tilde{W}}\hat{\ } \circ T_{r-1}, \quad 1 \leq r \leq n. \]  

(3.2)

By Cayley-Hamilton theorem, we have $T_n = 0$. Moreover, $T_r$ are also self-adjoint and commutes with $A_{\tilde{W}}\hat{\ }$. Furthermore, the following algebraic properties of $T_r$ are well-known (see [2], [1] and references therein for details).

\[ \text{tr}(T_r) = (-1)^r (n - r) S_r, \]  

(3.3)

\[ \text{tr}(A_{\tilde{W}}\hat{\ } \circ T_r) = (-1)^r (r + 1) S_{r+1}, \]  

(3.4)

\[ \text{tr}(A_{\tilde{W}}^2 \circ T_r) = (-1)^r (r + 2) S_{r+2}, \]  

(3.5)

\[ \text{tr}(T_r \circ \nabla_X A_{\tilde{W}}\hat{\ }) = (-1)^r X(S_{r+1}) = (-1)^r g(\nabla' S_{r+1}, X), \]  

(3.6)

for all $X \in \Gamma(T\mathcal{M})$. We will also need the following divergence formula for the operators $T_r$

\[ \text{div}^{\nabla'}(T_r) = \text{tr}(\nabla T_r) = \sum_{\beta=1}^{n} (\nabla'_{Z_\beta} T_r) Z_\beta, \]  

(3.7)

where $\{Z_1, \cdots, Z_n\}$ is a local orthonormal frame field of $T\mathcal{F}$. 

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4. Integration formulas for $\mathcal{F}$

This section is devoted to derivation of integral formulas of foliation $\mathcal{F}$ of $S(TM)$ with a unit normal vector $\hat{W}$ given by (2.22). By the fact that $\nabla$ is a metric connection then $\mathcal{g}(\nabla_{\hat{W}} \hat{W}, \hat{W}) = 0$. This implies that the vector field $\nabla_{\hat{W}} \hat{W}$ is always tangent to $\mathcal{F}$. Our main goal will be to compute the divergence of the vectors $T_r \nabla_{\hat{W}} \hat{W}$ and $T_r \nabla_{\hat{W}} \hat{W} + (-1)^r S_{r+1} \hat{W}$. The following technical lemmas are fundamentally important to this paper. Let $\{E, Z_i, N, W\}$, for $i = 1, \ldots, n$ be a quasi-orthonormal field of frame of $TM$, such that $S(TM) = \text{span}\{Z_i\}$ and $\epsilon_i = \mathcal{g}(Z_i, Z_i)$.

**Lemma 4.1.** Let $M$ be a screen integrable half-lightlike submanifold of $\mathcal{M}^{n+3}$ and let $M'$ be a foliation of $S(TM)$. Let $A_{\hat{W}}$ be its shape operator, where $\hat{W}$ is a unit normal vector to $\mathcal{F}$. Then

$$\mathcal{g}(\nabla_X A_{\hat{W}} Y, Z) = \mathcal{g}(Y, (\nabla_X A_{\hat{W}}) Z),$$

$$\mathcal{g}(\nabla_Y T_r, Z) = \mathcal{g}(Y, (\nabla_X T_r) Z),$$

for all $X, Y, Z \in \Gamma(TM)$.

**Proof.** By simple calculations we have

$$\mathcal{g}(\nabla_X A_{\hat{W}} Y, Z) = \mathcal{g}(\nabla_X (A_{\hat{W}} Y), Z) - \mathcal{g}(\nabla_X Y, A_{\hat{W}} Z). \quad (4.1)$$

Using the fact that $\nabla'$ is a metric connection and the symmetry of $A_{\hat{W}}$, (4.1) gives

$$\mathcal{g}(\nabla_X A_{\hat{W}} Y, Z) = \mathcal{g}(Y, \nabla'_X (A_{\hat{W}} Z)) - \mathcal{g}(Y, A_{\hat{W}} (\nabla'_X Z)). \quad (4.2)$$

Then, from (4.2) we deduce the first relation of the lemma. A proof of the second relation follows in the same way, which completes the proof. \qed

**Lemma 4.2.** Let $M$ be a screen integrable half-lightlike submanifold of $\mathcal{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Let $A_{\hat{W}}$ be its shape operator, where $\hat{W}$ is a unit normal vector to $\mathcal{F}$. Denote by $R$ the curvature tensor of $\mathcal{M}$. Then

$$\text{div}^\mathcal{F}(T_0) = 0,$$

$$\text{div}^\mathcal{F}(T_r) = A_{\hat{W}} \text{div}^\mathcal{F}(T_{r-1}) + \sum_{i=1}^{n} \epsilon_i \mathcal{g}(\hat{R}(\hat{W}, T_{r-1} Z_i), Z_i),$$

where $(\hat{R}(\hat{W}, X) Z)'$ denotes the tangential component of $\hat{R}(\hat{W}, X) Z$ for $X, Z \in \Gamma(TM)$. Equivalently, for any $Y \in \Gamma(TM)$ then

$$\mathcal{g}(\text{div}^\mathcal{F}(T_r), Y) = \sum_{j=1}^{r} \sum_{i=1}^{n} \epsilon_i \mathcal{g}(\hat{R}(T_{r-1} Z_i, \hat{W})(-A_{\hat{W}})^{j-1} Y, Z_i). \quad (4.3)$$

**Proof.** The first equation of the lemma is obvious since $T_0 = I$. We turn to the second relation. By direct calculations using the recurrence relation (3.2) we derive

$$\text{div}^\mathcal{F}(T_r) = (-1)^r \text{div}^\mathcal{F}(S_r I) + \text{div}^\mathcal{F}(A_{\hat{W}} \circ T_{r-1})$$

$$= (-1)^r \nabla S_r + A_{\hat{W}} \text{div}^\mathcal{F}(T_{r-1}) + \sum_{i=1}^{n} \epsilon_i (\nabla Z_i A_{\hat{W}}) T_{r-1} Z_i. \quad (4.4)$$

Using Codazzi equation

$$\mathcal{g}(\hat{R}(X, Y) Z, \hat{W}) = \mathcal{g}((\nabla'_X A_{\hat{W}}) Y, Z) - \mathcal{g}((\nabla'_Y A_{\hat{W}}) X, Z),$$

for any $X, Y, Z \in \Gamma(TM)$ and Lemma 4.1, we have

$$\mathcal{g}((\nabla Z_i A_{\hat{W}}) Y, T_{r-1} Z_i) = \mathcal{g}((\nabla'_Y A_{\hat{W}}) Z_i, T_{r-1} Z_i) + \mathcal{g}(\hat{R}(Y, Z_i) T_{r-1} Z_i, \hat{W})$$

$$= \mathcal{g}(T_{r-1} (\nabla'_Y A_{\hat{W}}) Z_i, Z_i) + \mathcal{g}(\hat{R}(\hat{W}, T_{r-1} Z_i) Z_i, Y), \quad (4.5)$$
for any $Y \in \Gamma(T\mathcal{F})$. Then applying (4.4) and (4.5) we get
\[ g(\text{div}^r(T_r), Y) = (-1)^r g(\nabla^r S_r, Y) + \text{tr}(T_{r-1}(\nabla^r_1 A^r_0)) \]
\[ + g(\text{div}^r(T_{r-1}), Y) + g(Y, \sum_{i=1}^n \epsilon_i R(\tilde{W}, T_{r-1} Z_i) Z_i). \]  
(4.6)

Then, applying (4.6) and (3.6) we get the second equation of the lemma. Finally, (4.3) follows immediately by an induction argument.

Notice that when the ambient manifold is a space form of constant sectional curvature, then $(\mathcal{R}(\tilde{W}, X)Y)' = 0$, for each $X, Y \in \Gamma(T\mathcal{F})$. Hence, from Lemma (4.2) we have $\text{div}^r(T_r) = 0$.

**Lemma 4.3.** Let $M$ be a screen integrable half-lightlike submanifold of $\mathcal{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Let $A^r_0$ be its shape operator, where $\tilde{W}$ is a unit normal vector to $\mathcal{F}$. Let $\{Z_i\}$ be a local field such $(\nabla_X Z_i)p = 0$, for $i = 1, \cdots, n$ and any vector field $X \in \Gamma(T\mathcal{M})$. Then at $p \in \mathcal{F}$ we have
\[ g(\nabla_{\tilde{W}} Z_i \tilde{W}, Z_j) = g(A^r_0 Z_i Z_j) - g(\mathcal{R}(Z_i, \tilde{W}) Z_j, \tilde{W}) \]
\[ - g((\nabla^r_1 A^r_0) Z_i Z_j) + g(\tilde{W}, \tilde{W}) g(Z_j, \nabla_{\tilde{W}} Z_i). \]

**Proof.** Applying $\nabla_{\tilde{W}} Z_i$ to $g(\nabla_{\tilde{W}} \tilde{W}, Z_j)$ and $g(\tilde{W}, \nabla_{\tilde{W}} Z_j)$ in turn and then using the two resulting equations, we have

\[ -g(\nabla_{\tilde{W}} \tilde{W}, Z_i Z_j) = g(\nabla_{\tilde{W}} \tilde{W}, Z_i Z_j) + g(\nabla_{\tilde{W}} \tilde{W}, \nabla_{\tilde{W}} Z_j) \]
\[ + g(\tilde{W}, \nabla_{\tilde{W}} Z_i Z_j). \]

Furthermore, by direct calculations using $(\nabla_X Z_i)p = 0$ we have
\[ g((\nabla^r_1 A^r_0) Z_i Z_j) = g(\nabla_{\tilde{W}} \tilde{W}, Z_i Z_j) + g(\tilde{W}, \nabla_{\tilde{W}} Z_i Z_j), \]
and hence
\[ g(A^r_0 Z_i Z_j) - g(\mathcal{R}(Z_i, \tilde{W}) Z_j, \tilde{W}) - g((\nabla^r_1 A^r_0) Z_i Z_j) \]
\[ = g(A^r_0 Z_i Z_j) - g(\mathcal{R}(Z_i, \tilde{W}) Z_j, \tilde{W}) \]
\[ - g((\nabla^r_1 A^r_0) Z_i Z_j) - g(\tilde{W}, \nabla_{\tilde{W}} Z_i Z_j) \]
\[ = g(A^r_0 Z_i Z_j) - g(\nabla_{\tilde{W}} Z_i Z_j, \tilde{W}) \]
\[ - g(\nabla_{\tilde{W}} Z_i Z_j, \tilde{W}) + g(\nabla_{\tilde{W}} Z_i Z_j, \tilde{W}). \]

(4.8)

Now, applying (4.7), the condition at $p$ and the following relations
\[ \nabla_{\tilde{W}} \tilde{W} = \sum_{k=1}^n \epsilon_k g(\nabla_{\tilde{W}} Z_k Z_k, \tilde{W}) \]
\[ \nabla_{\tilde{W}} Z_j = g(\nabla_{\tilde{W}} Z_j, \tilde{W}), \]
and $g(A^r_0 Z_i Z_j) = - \sum_{k=1}^n \epsilon_k g(\nabla_{\tilde{W}} Z_k Z_k, g(\nabla_{\tilde{W}} Z_i Z_j, \tilde{W})$ to the last line of (4.8) and the fact that $S(TM)$ is integrable we get
\[ g(A^r_0 Z_i Z_j) - g(\mathcal{R}(Z_i, \tilde{W}) Z_j, \tilde{W}) - g((\nabla^r_1 A^r_0) Z_i Z_j) \]
\[ = g(\nabla_{\tilde{W}} Z_i Z_j, \tilde{W}) - g(\nabla_{\tilde{W}} Z_i Z_j, g(Z_j, \nabla_{\tilde{W}} \tilde{W}), \]
from which the lemma follows by rearrangement.

Notice that, using parallel transport, we can always construct a frame field from the above lemma.
Proposition 4.4. Let $M$ be a screen integrable half-lightlike submanifold of an indefinite almost contact manifold $\overline{M}$ and let $\mathcal{F}$ be a foliation of $S(TM)$. Then
\[
\text{div}^\langle (T_r \nabla_{\overline{W}} \overline{W}) = g(\text{div}^\langle (T_r), \nabla_{\overline{W}} \overline{W}) + (-1)^{r+1} \overline{W} (S_{r+1}) \\
+ (-1)^{r+1} (-S_1 S_{r+1} + (r + 2)S_{r+2}) - \sum_{i=1}^{n} \epsilon_i g(\mathcal{R}(Z_i, \overline{W})) T_r Z_i, \overline{W} \\
+ g(\nabla_{\overline{W}} \overline{W}, T_r \nabla_{\overline{W}} \overline{W}),
\]
where $\{Z_i\}$ is a field of frame tangent to the leaves of $\mathcal{F}$.

**Proof.** From (3.7), we deduce that
\[
\text{div}^\langle (T_r, Z) = g(\text{div}^\langle (T_r), Z) + \sum_{i=1}^{n} \epsilon_i g(\nabla Z_i, T_r Z_i),
\] (4.9)
for all $Z \in \Gamma(T\mathcal{F})$. Then using (4.9), Lemmas 4.2 and 4.3, we obtain the desired result. Hence the proof.

From Proposition 4.4 we have

Theorem 4.5. Let $M$ be a screen integrable half-lightlike submanifold of an indefinite almost contact manifold $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Then
\[
\text{div}^\langle (T_r \nabla_{\overline{W}} \overline{W}) = g(\text{div}^\langle (T_r), \nabla_{\overline{W}} \overline{W}) + (-1)^{r+1} \overline{W} (S_{r+1}) \\
+ (-1)^{r+1} (-S_1 S_{r+1} + (r + 2)S_{r+2}) \\
- \sum_{i=1}^{n} \epsilon_i g(\mathcal{R}(Z_i, \overline{W})) T_r Z_i, \overline{W}.
\]

**Proof.** A proof follows easily from Proposition 4.4 by recognizing the fact that
\[
\text{div}^\langle (T_r \nabla_{\overline{W}} \overline{W}) = \text{div}^\langle (T_r, \nabla_{\overline{W}} \overline{W}) - g(\nabla_{\overline{W}} \overline{W}, T_r \nabla_{\overline{W}} \overline{W}),
\]
which completes the proof.

**Theorem 4.6.** Let $M$ be a screen integrable half-lightlike submanifold of $\overline{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Then,
\[
\text{div}^\langle (T_r \nabla_{\overline{W}} \overline{W} + (-1)^{r} \overline{W}) = g(\text{div}^\langle (T_r), \nabla_{\overline{W}} \overline{W}) \\
+ (-1)^{r+1} (r + 2)S_{r+2} - \sum_{i=1}^{n} \epsilon_i g(\mathcal{R}(Z_i, \overline{W})) T_r Z_i, \overline{W}.
\]

**Proof.** By straightforward calculations we have
\[
S_1 = \text{tr}(A_{\overline{W}}) \\
= - \sum_{i=1}^{n} \epsilon_i g(\nabla Z_i, \overline{W}, Z_i) \\
= - \sum_{i=1}^{n+1} \epsilon_i g(\nabla Z_i, \overline{W}, Z_i) \\
= - \text{div}^\langle (\overline{W}),
\]
where $Z_{n+1} = \overline{W}$. From this equation we deduce
\[
\text{div}^\langle (S_{r+1} \overline{W}) = -S_1 S_{r+1} + \overline{W} (S_{r+1}).
\] (4.10)
Then from (4.10) and Theorem 4.5 we get our assertion, hence the proof.
Next, we let $dV$ denote the volume form $\mathcal{M}$. Then from Theorem 4.6 we have the following

**Corollary 4.7.** Let $M$ be a screen integrable half-lightlike submanifold of a compact semi-Riemannian manifold $\mathcal{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$. Then

$$\int_{\mathcal{M}} g(\text{div}^\nu (T_r), \nabla\hat{W})dV = \int_{\mathcal{M}} ((-1)^r(r + 2)S_{r+2}$$

$$+ \sum_{i=1}^{n} \epsilon_i g(S_i, T_r) + \hat{W})dV.$$

Setting $r = 0$ in the above corollary we get

**Corollary 4.8.** Let $M$ be a screen integrable half-lightlike submanifold of a compact semi-Riemannian manifold $\mathcal{M}$ and let $\mathcal{F}$ be a co-dimension three foliation of $S(TM)$ with mean curvatures $S_r$. Then for $r = 0$ we have

$$\int_{\mathcal{M}} 2S_2dV = \int_{\mathcal{M}} \text{Ric}(\hat{W}, \hat{W})dV,$$

where $\text{Ric}(\hat{W}, \hat{W}) = \sum_{i=1}^{n} \epsilon_i g(S_i, \hat{W})\hat{W}, Z_i)$.

Notice that the equation in Corollary 4.8 is the lightlike analogue of (3.5) in [2] for co-dimension one foliations on Riemannian manifolds.

Next, we will discuss some consequences of the integral formulas developed so far.

A semi-Riemannian manifold $\mathcal{M}$ of constant sectional curvature $c$ is called a semi-Riemannian space form [4, 6] and is denoted by $\mathcal{M}(c)$. Then, the curvature tensor $\mathcal{R}$ of $\mathcal{M}(c)$ is given by

$$\mathcal{R}(\nabla\nu, \nu)\nu = c(\mathcal{g}(\nu, \nu)\nu - \mathcal{g}(\nu, \nu)\nu), \quad \forall \nabla\nu, \nu, \nu \in \Gamma(TM). \quad (4.11)$$

**Theorem 4.9.** Let $M$ be a screen integrable half-lightlike submanifold of a compact semi-Riemannian space form $\mathcal{M}(c)$ of constant sectional curvature $c$. Let $\mathcal{F}$ be a co-dimension three foliation of its screen distribution $S(TM)$. If $V$ is the total volume of $\mathcal{M}$, then

$$\int_{\mathcal{M}} S_r dV = \begin{cases} 0, & r = 2k + 1, \\ c^2 \left( \frac{n}{r+2} \right) V, & r = 2k, \end{cases} \quad (4.12)$$

for positive integers $k$.

**Proof.** By setting $X = Z_i, Y = \hat{W}$ and $Z = T_rZ_i$ in (4.11) we deduce that

$$\mathcal{R}(Z_i, \hat{W})T_r = -c\mathcal{g}(Z_i, T_r)\hat{W}.$$  

Then substituting this equation in Corollary 4.7 we obtain

$$\int_{\mathcal{M}} g(\text{div}^\nu (T_r), \nabla\hat{W})dV = \int_{\mathcal{M}} ((-1)^r(r + 2)S_{r+2} - c\nu(T_r))dV.$$  

Since $\mathcal{M}$ is of constant sectional curvature $c$, then Lemma 4.2 implies that $T_r = 0$ for any $r$ and hence the above equation simplifies to

$$(r + 2) \int_{\mathcal{M}} S_{r+2}dV = c(n - r) \int_{\mathcal{M}} S_r dV. \quad (4.13)$$  

Since $S_1 = \text{div}^\nu(\hat{W})$ and that $\mathcal{M}$ is compact, then $\int_{\mathcal{M}} S_1 dV = 0$. Using this fact together with (4.13), mathematical induction gives $\int_{\mathcal{M}} S_r dV = 0$ for all $r = 2k + 1$ (i.e., $r$ odd).
For \( r \) even we will consider \( r = 2m \) and \( n = 2l \) (i.e., both \( M \) and \( \overline{M} \) are odd dimensional). With these conditions, (4.13) reduces to
\[
\int_{\overline{M}} S_{2m+2}dV = c\frac{l-m}{1+m} \int_{\overline{M}} S_{2m}dV.
\]
(4.14)

Now setting \( m = 0, 1, \cdots \) and \( S_0 = 1 \) in (4.14) we obtain
\[
\int_{\overline{M}} S_2dV = c l V, \quad \int_{\overline{M}} S_4dV = c^2 \frac{(l-1)l}{2} V,
\]
and more generally
\[
\int_{\overline{M}} S_{2k}dV = c^k \frac{(l-k+1)(l-k+2)(l-k+3)\cdots l}{k!} V.
\]
(4.15)

Hence, our assertion follows from (4.15), which completes the proof. \( \square \)

Next, when \( \overline{M} \) is Einstein i.e., \( \overline{\text{Ric}} = \mu \overline{g} \) we have the following.

**Theorem 4.10.** Let \( M \) be a screen integrable half-lightlike submanifold of an Einstein compact semi-Riemannian manifold \( \overline{M} \). Let \( \mathcal{F} \) be a co-dimension three foliation of its screen distribution \( S(TM) \) with totally umbilical leaves. If \( V \) is the total volume of \( \overline{M} \), then
\[
\int_{\overline{M}} S_r dV = \begin{cases} 0, & r = 2k + 1, \\ \left(\frac{n}{2}\right)^\frac{n}{2} \left(\frac{n}{2}\right)^V, & r = 2k, \end{cases}
\]
for positive integers \( k \).

**Proof.** Suppose that \( A_N \overline{W} = \frac{1}{n} S_r \mathcal{L} \mathcal{L} \). Then by direct calculations using the formula for \( T_r \) we derive \( T_r = (-1)^{r+1} \frac{(n-r)}{n} S_r \mathcal{L} \mathcal{L} \). Then, under the assumptions of the theorem we obtain \( \overline{\text{Ric}}(\overline{W}, \nabla \overline{W}) = 0 \) and \( \overline{\text{Ric}}(\overline{W}, \overline{W}) = \mu \) and hence, Corollary 4.7 reduces to
\[
n(r+2) \int_{\overline{M}} S_{r+2}dV = \lambda(n-r) \int_{\overline{M}} S_r dV.
\]
(4.17)

Notice that (4.17) is similar to (4.13) and hence following similar steps as in the previous theorem we get \( \int_{\overline{M}} S_r dV = 0 \) for \( r \) odd and for \( r \) even we get
\[
\int_{\overline{M}} S_{2k}dV = \left(\frac{n}{2}\right)^k \frac{(l-k+1)(l-k+2)(l-k+3)\cdots l}{k!} V,
\]
which complete the proof. \( \square \)

5. Screen umbilical half-lightlike submanifolds

In this section we consider totally umbilical half-lightlike submanifolds of semi-Riemannian manifold, with a totally umbilical screen distribution and thus, give a generalized version of Theorem 4.3.7 of [6] and its Corollaries, via Newton transformations of the operator \( A_N \).

A screen distribution \( S(TM) \) of a half-lightlike submanifold \( M \) of a semi-Riemannian manifold \( \overline{M} \) is said to be totally umbilical [6] if on any coordinate neighborhood \( \mathcal{U} \) there exist a function \( K \) such that
\[
C(X, PY) = Kg(X, PY), \quad \forall X, Y \in \Gamma(TM).
\]
(5.1)

In case \( K = 0 \), we say that \( S(TM) \) is totally geodesic. Furthermore, if \( S(TM) \) is totally umbilical then by straightforward calculations we have
\[
A_N X = PX, \quad C(E, PX) = 0, \quad \forall X \in \Gamma(TM).
\]
(5.2)
Let \(\{E, Z_i\}\), for \(i = 1, \cdots, n\), be a quasi-orthonormal frame field of \(TM\) which diagonalizes \(A_N\). Let \(l_0, l_1, \cdots, l_n\) be the respective eigenvalues (or principal curvatures). Then as before, the \(r\)-th mean curvature \(S_r\) is given by

\[
S_r = \sigma_r(l_0, \cdots, l_n) \quad \text{and} \quad S_0 = 1.
\]

The characteristic polynomial of \(A_N\) is given by

\[
\det(A_N - tI) = \sum_{\alpha=0}^{n} (-1)^\alpha S_\alpha t^{n-\alpha},
\]

where \(I\) is the identity in \(\Gamma(TM)\). The normalized \(r\)-th mean curvature \(H_r\) of \(M\) is defined by

\[
\left(\begin{array}{c}
\sum_{\alpha=0}^{n} (-1)^\alpha S_\alpha t^{n-\alpha} \end{array}\right)\left(\begin{array}{c}
H_r = S_r \\
0
\end{array}\right) \quad \text{and} \quad H_0 = 1.
\]

The Newton transformations \(T_r : \Gamma(TM) \to \Gamma(TM)\) of \(A_N\) are given by the inductive formula

\[
T_0 = I, \quad T_r = (-1)^r S_r I + A_N \circ T_{r-1}, \quad 1 \leq r \leq n.
\]

By Cayley-Hamilton theorem, we have \(T_{n+1} = 0\). Also, \(T_r\) satisfies the following properties.

\[
\begin{align*}
\text{tr}(T_r) &= (-1)^r (n + 1 - r) S_r, \\
\text{tr}(A_N \circ T_r) &= (-1)^r (r + 1) S_{r+1}, \\
\text{tr}(A_N^2 \circ T_r) &= (-1)^r S_{r+1} + (r + 2) S_{r+2}, \\
\text{tr}(T_r \circ \nabla_X A_N) &= (-1)^r X(S_{r+1}),
\end{align*}
\]

for all \(X \in \Gamma(TM)\).

**Proposition 5.1.** Let \((M, g)\) be a totally umbilical half-lightlike submanifold of a semi-Riemannian manifold \(\overline{M}\) of constant sectional curvature \(c\). Then

\[
g(\nabla_X (T_r), X) = (-1)^{r-1} \lambda(X) E(S_r) - \tau(X) \text{tr}(A_N \circ T_{r-1}) \\
- c\lambda(X) \text{tr}(T_{r-1}) + g(\nabla_X (T_{r-1}), A_N X) + g((\nabla_X A_N) T_{r-1} E, X) \\
+ \sum_{i=1}^{n} \epsilon_i \{ -\lambda(X) B(Z_i, A_N(T_{r-1} Z_i)) \\
+ \varepsilon \tau(Z_i) C(X, T_{r-1} Z_i) \{ \rho(X) D(Z_i, T_{r-1} Z_i) - \rho(Z_i) D(X, T_{r-1} Z_i) \} \},
\]

for any \(X \in \Gamma(TM)\).

**Proof.** From the recurrence relation (5.3), we derive

\[
g(\nabla_X (T_r), X) = (-1)^r PX(S_r) + g((\nabla_X A_N) T_{r-1} E, X) \\
+ g(\nabla_X (T_{r-1}), A_N X) + \sum_{i=1}^{n} \epsilon_i g((\nabla_X A_N) T_{r-1} Z_i, X),
\]

for any \(X \in \Gamma(TM)\). But

\[
g((\nabla_X A_N) T_{r-1} Z_i, X) = g(T_{r-1} Z_i, (\nabla_X A_N) X) + g((\nabla_X A_N) T_{r-1} Z_i, X) \\
- g((\nabla_X A_N) T_{r-1} Z_i, X) + g(A_N(\nabla_X Z_i), T_{r-1} Z_i) \\
- g(A_N(\nabla_X T_{r-1} Z_i), X),
\]

for all \(X \in \Gamma(TM)\). \(\square\)

Then applying (2.9) to (5.9) while considering the fact that \(A_N\) is screen-valued, we get

\[
g((\nabla_X A_N) T_{r-1} Z_i, X) = g(T_{r-1} Z_i, (\nabla_X A_N) X) - \lambda(X) B(Z_i, A_N(T_{r-1} Z_i)).
\]
2.15 K.L. Duggal and B. Sahin,

5.10 4.11 Under the hypothesis of Theorem 5.3

5.4 K. Andrzejewski and Pawel G. Walczak,

5.2 5.2 5.11 Replacing

5.8 Corollary 5.4.

K. Andrzejewski, W. Kozlowski and K. Niedzialomski,

and then using (5.5) we have the following.

5.5 Furthermore, using (2.16) we get the desired result.

Theorem 5.2. Let \((M, g)\) be a half-lightlike submanifold of a semi-Riemannian manifold \(\mathcal{M}(c)\) of constant curvature \(c\), with a proper totally umbilical screen distribution \(S(TM)\). If \(M\) is also totally umbilical, then the \(r\)-th mean curvature \(S_r\), for \(r = 0, 1, \ldots, n\), with respect to \(A_N\) are solution of the following differential equation

\[
E(S_{r+1}) - \tau(E)(r+1)S_{r+1} - c(-1)^r(n+1-r)S_1 = \mathcal{H}_1(r+1)S_{r+1}.
\]

Proof. Replacing \(X\) with \(E\) in the Proposition 5.1 and then using (2.16) and (5.2) we obtain, for all \(r = 0, 1, \ldots, n\),

\[
E(S_{r+1}) - (-1)^r\tau(E)\text{tr}(A_N \circ T_r) - c(-1)^r\text{tr}(T_r) = (-1)^r\mathcal{H}_1\text{tr}(A_N \circ T_r),
\]

from which the result follows by applying (5.4) and (5.5). \(\Box\)

Corollary 5.3. Under the hypothesis of Theorem 5.2, the induced connection \(\nabla\) on \(M\) is a metric connection, if and only if, the \(r\)-th mean curvature \(S_r\) with respect to \(A_N\) are solution of the following equation

\[
E(S_{r+1}) - \tau(E)(r+1)S_{r+1} - c(-1)^r(n+1-r)S_r = 0.
\]

Also the following holds.

Corollary 5.4. Under the hypothesis of Theorem 5.2, \(\mathcal{M}(c)\) is a semi-Euclidean space, if and only if, the \(r\)-th mean curvature \(S_r\) with respect to \(A_N\) are solution of the following equation

\[
E(S_{r+1}) - \tau(E)(r+1)S_{r+1} = \mathcal{H}_1(r+1)S_{r+1}.
\]

Notice that Theorem 5.2 and Corollary 5.3 are generalizations of Theorem 4.3.7 and Corollary 4.3.8, respectively, given in [6].

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References


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