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Some qualitative properties of mild solutions of a second-order integro-differential inclusion

Aurelian Cernea[a]

[a] *Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest, Romania, Academy of Romanian Scientists, Splaiul Independenţei 54, 050094 Bucharest, Romania.*

Abstract

We prove the Lipschitz dependence on the initial data of the solution set of a Cauchy problem associated to a second-order integro-differential inclusion by using the contraction principle in the space of selections of the multifunction instead of the space of solutions. A Filippov type existence theorem for this problem is also provided.

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1. Introduction

This paper is concerned with the problem of Lipschitz dependence on the initial data of the solution set for the following second order integro-differential inclusion

$$x''(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, x'(0) = y_0, \tag{1.1}$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a Lipschitz-continuous set-valued map with respect to the second variable, X is a separable Banach space, $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{G(t,s)\}_{t,s\in[0,T]}$, $\Delta = \{(t,s) \in [0,T] \times [0,T]; t \geq s\}$, $K(.,.) : \Delta \to \mathbb{R}$ is

Email address: `acernea@fmi.unibuc.ro` (Aurelian Cernea)

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continuous and $x_0, y_0 \in X$. The general framework of evolution operators $\{A(t)\}_{t\geq 0}$ that define problem (1.1) has been developed by Kozak ([14]) and improved by Henriquez ([11]).

We study the properties of the map that associates to given initial conditions the set of mild solutions of problem (1.1) and the main purpose is to prove that this solution map depends Lipschitz-continuously on the initial conditions. Our approach is based on an idea of Tallos ([13,16]) applying the set-valued contraction principle in the space of selections of the multifunction instead of the space of solutions as usual. This approach allows us to obtain also a Filippov type existence result for mild solutions of problem (1.1). Recall that for a differential inclusion defined by a Lipschitz-continuous set-valued map with nonconvex values, Filippov's theorem consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained.

In several recent papers ([1-3], [7], [11-12]) existence results and qualitative properties of mild solutions have been obtained for the following problem

$$x''(t) \in A(t)x(t) + F(t, x(t)), \quad x(0) = x_0, x'(0) = y_0, \tag{1.2}$$

with $A(.)$ and $F(.,.)$ as above. All the results quoted above are proved by using fixed point techniques.

On one hand, the result in the present paper extends to the integro-differential framework (1.1) the results in [7] obtained for problem (1.2) and, on the other hand, this paper extends to second-order integro-differential inclusions similar results in [5] and [6] obtained for a class of first-order integro-differential inclusions.

The paper is organized as follows: in Section 2 we recall some preliminary results that we use in the sequel and in Section 3 we prove our main results.

2. Preliminaries

Let denote by I the interval $[0, T]$, $T > 0$ and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$. As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|\mathrm{d}t$. With $B(X)$ we denote the Banach space of linear bounded operators on X and with B we denote the closed unit ball in X.

In what follows $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{G(t, s)\}_{t,s\in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t.

Definition 2.1. *([11,14]) A family of bounded linear operators $G(t, s) : X \to X$, $(t, s) \in \Delta := \{(t, s) \in I \times I; s \leq t\}$ is called an evolution operator of the equation*

$$x''(t) = A(t)x(t) \tag{2.1}$$

if

i) For any $x \in X$, the map $(t, s) \to G(t, s)x$ is continuously differentiable and

a) $G(t, t) = 0$, $t \in I$.

b) If $t \in I, x \in X$ then $\frac{\partial}{\partial t}G(t, s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}G(t, s)x|_{t=s} = -x$.

ii) If $(t, s) \in \Delta$, then $\frac{\partial}{\partial s}G(t, s)x \in D(A(t))$, the map $(t, s) \to G(t, s)x$ is of class C^2 and

a) $\frac{\partial^2}{\partial t^2}G(t, s)x \equiv A(t)G(t, s)x$.

b) $\frac{\partial^2}{\partial s^2}G(t, s)x \equiv G(t, s)A(t)x$.

c) $\frac{\partial^2}{\partial s\partial t}G(t, s)x|_{t=s} = 0$.

iii) If $(t, s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2\partial s}G(t, s)x$, $\frac{\partial^3}{\partial s^2\partial t}G(t, s)x$ and

a) $\frac{\partial^3}{\partial t^2\partial s}G(t, s)x \equiv A(t)\frac{\partial}{\partial s}G(t, s)x$ and the map $(t, s) \to A(t)\frac{\partial}{\partial s}G(t, s)x$ is continuous.

b) $\frac{\partial^3}{\partial s^2\partial t}G(t, s)x \equiv \frac{\partial}{\partial t}G(t, s)A(s)x$.

As an example for equation (2.1) one may consider the problem (e.g., [11])

$$\frac{\partial^2 z}{\partial t^2}(t,\tau)=\frac{\partial^2 z}{\partial \tau^2}(t,\tau)+a(t)\frac{\partial z}{\partial t}(t,\tau), \quad t\in[0,T],\tau\in[0,2\pi],$$

$$z(t,0)=z(t,\pi)=0, \quad \frac{\partial z}{\partial \tau}(t,0)=\frac{\partial z}{\partial \tau}(t,2\pi),\ t\in[0,T],$$

where $a(.):I\to\mathbb{R}$ is a continuous function. This problem is modeled in the space $X=L^2(\mathbb{R},\mathbb{C})$ of 2π-periodic 2-integrable functions from $\mathbb{R}$ to $\mathbb{C}$, $A_1z=\frac{d^2z(\tau)}{d\tau^2}$ with domain $H^2(\mathbb{R},\mathbb{C})$ the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbb{R},\mathbb{C})$. It is well known that$A_1$ is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2$, $n\in\mathbb{Z}$ with associated eigenvectors $z_n(\tau)=\frac{1}{\sqrt{2\pi}}e^{in\tau}$, $n\in\mathbb{N}$. The set z_n, $n\in\mathbb{N}$ is an orthonormal basis of X. In particular, $A_1z=\sum_{n\in\mathbb{Z}}-n^2<z,z_n>z_n$, $z\in D(A_1)$. The cosine function is given by $C(t)z=\sum_{n\in\mathbb{Z}}\cos(nt)<z,z_n>z_n$ with the associated sine function $S(t)z=t<z,z_0>z_0+\sum_{n\in\mathbb{Z}^*}\frac{\sin(nt)}{n}<z,z_n>z_n$.

For $t\in I$ define the operator $A_2(t)z=a(t)\frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t))=H^1(\mathbb{R},\mathbb{C})$. Set $A(t)=A_1+A_2(t)$. It has been proved in [11] that this family generates an evolution operator as in Definition 2.1.

Definition 2.2. *A continuous mapping $x(.)\in C(I,X)$ is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function $f(.)\in L^1(I,X)$ such that*

$$f(t)\in F(t,x(t)) \quad a.e.\,(I), \tag{2.2}$$

$$x(t)=-\frac{\partial}{\partial s}G(t,0)x_0+G(t,0)y_0+\int_0^t G(t,s)\int_0^s K(s,\tau)f(\tau)d\tau,\ t\in I. \tag{2.3}$$

We shall call $(x(.),f(.))$ a *trajectory-selection pair* of (1.1) if $f(.)$ verifies (2.2) and $x(.)$ is defined by (2.3). We shall use the following notations for the solution sets of (1.1).

$$\mathcal{S}(x_0,y_0)=\{(x(.),f(.));\quad (x(.),f(.)) \text{ is a trajectory-selection pair of (1.1)}\}, \tag{2.4}$$

$$\mathcal{S}_1(x_0,y_0)=\{x(.);\quad x(.) \text{ is a mild solution of (1.1)}\}. \tag{2.5}$$

In what follows we assume the following hypothesis.

Hypothesis. i) There exists an evolution operator $\{G(t,s)\}_{t,s\in I}$ associated to the family $\{A(t)\}_{t\geq 0}$.

ii) There exist $M,M_0\geq 0$ such that $|G(t,s)|_{B(X)}\leq M$, $|\frac{\partial}{\partial s}G(t,s)|\leq M_0$, for all $(t,s)\in\Delta$.

iii) $F(.,.):I\times X\to\mathcal{P}(X)$ has nonempty closed values and for every $x\in X$, $F(.,x)$ is measurable.

iv) There exists $L(.)\in L^1(I,\mathbb{R}_+)$ such that for almost all $t\in I$, $F(t,.)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t,x),F(t,y))\leq L(t)|x-y| \quad \forall\ x,y\in X,$$

here $d_H(A,B)$ is the Hausdorff distance between $A,B\subset X$

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\},\quad d^*(A,B)=\sup\{d(a,B);\ a\in A\},$$

$d(a,B)=\inf\{d(a,b);\ b\in B\}$.

v) $d(0,F(t,0))\leq L(t)\quad a.e.\ (I)$

Let $m(t)=\int_0^t L(u)du$ and for given $\alpha\in\mathbb{R}$ we consider on $L^1(I,X)$ the following norm

$$|f|_1=\int_0^T e^{-\alpha m(t)}|f(t)|dt,\quad f\in L^1(I,X),$$

which is equivalent with the usual norm on $L^1(I,X)$.

Consider the following norm on $C(I,X)\times L^1(I,X)$

$$|(x,f)|_{C\times L}=|x|_C+|f|_1 \quad \forall\,(x,f)\in C(I,X)\times L^1(I,X).$$

Finally we recall some basic results concerning set valued contractions that we shall use in the sequel.

Let (Z,d) be a metric space and consider a set valued map N on Z with nonempty closed values in Z. N is said to be a γ-contraction if there exists $0<\gamma<1$ such that:

$$d_H(N(x),N(y))\le \gamma d(x,y) \quad \forall x,y\in Z$$

If Z is complete, then every set valued contraction has a fixed point, i.e. a point $z\in Z$ such that $z\in N(z)$ ([8]).

We denote by $Fix(N)$ the set of all fixed point of the multifunction N. Obviously, $Fix(N)$ is closed.

Proposition 2.3. *([15]) Let Z be a complete metric space and suppose that N_1,N_2 are λ-contractions with closed values in Z. Then*

$$d_H(Fix(N_1),Fix(N_2))\le \frac{1}{1-\gamma}\sup_{z\in Z} d_H(N_1(z),N_2(z)).$$

Finally, we note that condition (2.3) can be rewritten as

$$x(t)=-\frac{\partial}{\partial s}G(t,0)x_0+G(t,0)y_0+\int_0^t U(t,s)f(s)ds \quad \forall t\in I, \tag{2.6}$$

where $U(t,s)=\int_s^t G(t,\tau)K(\tau,s)d\tau$.

Denote $K_0:=\sup_{(t,s)\in\Delta}|K(t,s)|$ and remark that $|U(t,s)|\le MK_0(t-s)\le MK_0T$.

3. The main results

We show first that the set of all trajectory-selection pairs of (1.1) depends Lipschitz-continuously on the initial condition.

Theorem 3.1. *Let Hypothesis be satisfied and let $\alpha>MK_0T$.*

Then the map $(x_0,y_0)\to\mathcal{S}(x_0,y_0)$ is Lipschitz-continuous on $X\times X$ with nonempty closed values in $C(I,X)\times L^1(I,X)$.

Proof. Let us consider $x_0,y_0\in X, f(.)\in L^1(I,X)$ and define the following set valued maps

$$B_{x_0,y_0,f}(t)=F(t,-\frac{\partial}{\partial s}G(t,0)x_0+G(t,0)y_0+\int_0^t U(t,s)f(s)ds), \quad t\in I, \tag{3.1}$$

$$N_{x_0,y_0}(f)=\{\phi(.)\in L^1(I,X);\quad \phi(t)\in B_{x_0,y_0,f}(t)\quad a.e.\ (I)\}. \tag{3.2}$$

At the begining we prove that $N_{x_0,y_0}(f)$ is nonempty and closed for every $f\in L^1(I,X)$. The fact that that the set valued map $B_{x_0,y_0,f}(.)$ is measurable is well known. For example, the map $t\to -\frac{\partial}{\partial s}G(t,0)x_0+G(t,0)y_0+\int_0^t U(t,s)f(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [4]. Since the values of F are closed and X is separable with the measurable selection theorem (Theorem III.6 in [4]) we infer that $B_{x_0,y_0,f}(.)$ admits a measurable selection ϕ. According to Hypothesis one has

$$|\phi(t)|\le d(0,F(t,0))+d_H(F(t,0),F(t,x(t)))\le L(t)(1+|x(t)|)$$

$$\le L(t)(1+M_0|x_0|+M|y_0|+\int_0^t MK_0T|f(s)|ds).$$

Thus integrating by parts we obtain

$$\int_0^T e^{-\alpha m(t)}|\phi(t)|dt \le \int_0^T e^{-\alpha m(t)} L(t)(1 + M_0|x_0| + M|y_0|+$$

$$\int_0^t MK_0T|f(s)|ds)dt \le \frac{1 + M_0|x_0|}{\alpha} + \frac{M|y_0|}{\alpha} + \frac{MK_0T|f|_1}{\alpha}.$$

Hence, if $\phi(.)$ is a measurable selection of $B_{x_0,y_0,f}(.)$, then $\phi(.) \in L^1(I, X)$ and thus $N_{x_0,y_0}(f) \ne \emptyset$.

The set $N_{x_0,y_0}(f)$ is closed. Indeed, if $\phi_n \in N_{x_0,y_0}(f)$ and $|\phi_n - \phi|_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in N_{x_0,y_0}(f)$.

We prove now that $N_{x_0,y_0}(.)$ is a contraction on $L^1(I, X)$.

Let $f, g \in L^1(I, X)$ be given, $\phi \in N_{x_0,y_0}(f)$ and let $\varepsilon > 0$. Consider the following set valued map

$$G(t) = B_{x_0,y_0,g}(t) \cap \{x \in X;\ |\phi(t) - x| \le L(t)|\int_0^t U(t, s)(f(s) - g(s))ds| + \varepsilon\}.$$

Since

$$d(\phi(t), B_{x_0,y_0,g}(t)) \le d(F(t, -\tfrac{\partial}{\partial s}G(t, 0)x_0 + G(t, 0)y_0 + \textstyle\int_0^t U(t, s)f(s)ds),$$
$$F(t, -\tfrac{\partial}{\partial s}G(t, 0)x_0 + G(t, 0)y_0 + \textstyle\int_0^t U(t, s)g(s)ds)) \le L(t)|\int_0^t U(t, s)(f(s)-$$
$$g(s))ds|$$

we find that $G(.)$ has nonempty closed values. Moreover, according to Proposition III.4 in [4], $G(.)$ is measurable. Let $\psi(.)$ be a measurable selection of $G(.)$. It follows that $\psi \in N_{x_0,y_0}(g)$ and

$$|\phi - \psi|_1 = \int_0^T e^{-\alpha m(t)}|\phi(t) - \psi(t)|dt \le \int_0^T e^{-\alpha m(t)} L(t)(\int_0^t MK_0T|f(s)-$$

$$g(s)|ds)dt + \int_0^T \varepsilon e^{-\alpha m(t)}dt \le \frac{MK_0T}{\alpha}|f - g|_1 + \varepsilon \int_0^T e^{-\alpha m(t)}dt.$$

ε is arbitrary, hence

$$d(\phi, N_{x_0,y_0}(g)) \le \frac{MK_0T}{\alpha}|f - g|_1.$$

If we replace f by g we obtain

$$d(N_{x_0,y_0}(f), N_{x_0,y_0}(g)) \le \frac{MK_0T}{\alpha}|f - g|_1,$$

thus $N_{x_0,y_0}(.)$ is a contraction on $L^1(I, X)$.

Therefore, $N_{x_0,y_0}(.)$ admits a fixed point $f(.) \in L^1(I, X)$. We define $x(t) = -\frac{\partial}{\partial s}G(t, 0)x_0 + G(t, 0)y_0 + \int_0^t U(t, s)f(s)ds$.

We show that $\mathcal{S}(x_0, y_0) \subset C(I, X) \times L^1(I, X)$ is a closed subset. Let $(x_n, f_n) \in \mathcal{S}(x_0, y_0)$, $|(x_n, f_n) - (x, f)|_{C\times L} \to 0$. In particular,
$f_n \in Fix(N_{x_0,y_0})$, which is a closed set, and thus $f(.) \in Fix(N_{x_0,y_0})$. We put $y(t) = -\frac{\partial}{\partial s}G(t, 0)x_0 + G(t, 0)y_0 + \int_0^t U(t, s)f(s)ds$ and we prove that $y(.) = x(.)$. One may write

$$|y - x_n|_C = \sup_{t\in I} |y(t) - x_n(t)| \le \sup_{t\in I} MK_0T \textstyle\int_0^t |f_n(u) - f(u)|du \le$$
$$MK_0Te^{\alpha m(T)}|f_n - f|_1$$

and finally we get that $y(.) = x(.)$.

At the next step of the proof we obtain the following inequality

$$d_H(N_{x_1,y_1}(f), N_{x_2,y_2}(f)) \le \frac{1}{\alpha}(M_0|x_1 - x_2| + M|y_1 - y_2|) \tag{3.3}$$

$\forall f \in L^1(I,X), x_1, x_2, y_1, y_2 \in X$. Define

$$G_1(t) = B_{x_1,x_2,f}(t) \cap \{z \in X;\ |\phi(t) - z| \leq L(t)(|\tfrac{\partial}{\partial s}G(t,0)||x_1 - x_2| + |G(t,0)||y_1 - y_2| + \varepsilon\},$$

$t \in I$, where $\phi(.)$ is a measurable selection of $B_{x_1,y_1,f}(.)$ and $\varepsilon > 0$.

Repeating the arguments used for the set valued map $G(.)$, we obtain that $G_1(.)$ is measurable with nonempty closed values. Let $\psi(.)$ be a measurable selection of $G_1(.)$. It follows that $\psi(.) \in N_{x_2,y_2}(f)$ and

$$|\phi - \psi|_1 = \int_0^T e^{-\alpha m(t)}|\phi(t) - \psi(t)|dt \leq$$

$$\int_0^T e^{-\alpha m(t)}L(t)(|\frac{\partial}{\partial s}G(t,0)||x_1 - x_2| + |G(t,0)||y_1 - y_2|)dt +$$

$$\varepsilon \int_0^T e^{-\alpha m(t)}dt \leq \frac{M_0}{\alpha}|x_1 - x_2| + \frac{M}{\alpha}|y_1 - y_2| + \varepsilon \int_0^T e^{-\alpha m(t)}dt.$$

Since ε was arbitrary, we deduce that

$$d(\phi, N_{x_2,y_2}(f)) \leq \frac{1}{\alpha}(M_0|x_1 - x_2| + M|y_1 - y_2|).$$

If we replace (x_1, y_1) by (x_2, y_2) we obtain (3.3).

From (3.3) and Proposition 2.3 we obtain

$$d_H(Fix(N_{x_1,y_1}), Fix(N_{x_2,y_2})) \leq \frac{1}{\alpha - MK_0T}(M_0|x_1 - x_2| + M|y_1 - y_2|).$$

Let $x_1, x_2, y_1, y_2 \in X$ and $(x(.), f(.)) \in \mathcal{S}(x_1, y_1)$. In particular, $f(.) \in Fix(N_{x_1,y_1})$ and thus, for every $\varepsilon > 0$ there exists $g(.) \in Fix(N_{x_2,y_2})$ such that

$$|f - g|_1 \leq \frac{1}{\alpha - MK_0T}(M_0|x_1 - x_2| + M|y_1 - y_2|) + \varepsilon. \tag{3.4}$$

Put $z(t) = -\frac{\partial}{\partial s}G(t,0)x_0 + G(t,0)y_0 + \int_0^t U(t,s)g(s)ds$. One has

$$|x - z|_C = \sup_{t \in I}|x(t) - z(t)| \leq M_0|x_1 - x_2| + M|y_1 - y_2| +$$

$$\sup_{t \in I}\int_0^t MK_0T|f(s) - g(s)|ds \leq M_0|x_1 - x_2| + M|y_1 - y_2| + MK_0Te^{\alpha m(t)}|f - g|_1$$

$$\leq (1 + \frac{MK_0Te^{\alpha m(t)}}{\alpha - MK_0T})(M_0|x_1 - x_2| + M|y_1 - y_2|) + \frac{MK_0Te^{\alpha m(t)}}{\alpha - MK_0T}\varepsilon.$$

It remains to denote $k = \max\{M_0 + \frac{MK_0TM_0e^{\alpha m(T)}}{\alpha - MK_0T}, M + \frac{M^2K_0Te^{\alpha m(T)}}{\alpha - MK_0T}\}$ to get first that

$$d((x,f), \mathcal{S}(x_2, y_2)) \leq k[|x_1 - x_2| + |y_1 - y_2|].$$

By interchanging (x_1, y_1) and (x_2, y_2) we obtain

$$d_H(\mathcal{S}(x_1, y_1), \mathcal{S}(x_2, y_2)) \leq k[|x_1 - x_2| + |y_1 - y_2|]$$

and the proof is complete. □

An easy consequence of Theorem 3.1 is

Corollary 3.2. *Let Hypothesis be satisfied and let $\alpha > MK_0T$. Then the map $(x_0, y_0) \to \mathcal{S}_1(x_0, y_0)$ is Lipschitz continuous on $X \times X$ with nonempty values in $C(I, X)$.*

If the assumptions of Theorem 3.1 are satisfied the solution set $\mathcal{S}_1(x_0, y_0)$ is not closed in $C(I, X)$. In the following result one see that if X is reflexive and the set-valued map $F(.,.)$ is convex valued and integrably bounded then $\mathcal{S}_1(x_0, y_0) \subset C(I, X)$ is closed.

Proposition 3.3. *Assume that X is reflexive, $\alpha > MK_0T$ and let $F(.,.) : I \times X \to \mathcal{P}(X)$ be a convex valued set-valued map that satisfies Hypothesis. Assume that there exists $k(.) \in L^1(I, X)$ such that for almost all $t \in I$ and for all $x \in X$, $F(t, x) \subset k(t)B$.*

Then for every $x_0, y_0 \in X$, the set $\mathcal{S}_1(x_0, y_0) \subset C(I, X)$ is closed.

Proof. Take $x_n(.) \in \mathcal{S}_1(x_0, y_0)$ such that $|x_n - x|_C \to 0$. There exists $f_n(.) \in L^1(I, X)$ such that $(x_n(.), f_n(.))$ is a trajectory-selection pair of (1.1) $\forall n \in \mathbb{N}$. We put $h_n(t) = e^{-\alpha m(t)} f_n(t),\ t \in I$.

Since $F(.,.)$ is integrably bounded, we deduce that $f_n(.)$ is bounded in $L^1(I, X)$ and $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall E \subset I,\ \mu(E) < \delta,\ \ |\int_E h_n(s)ds| < \varepsilon$ uniformly with respect to n. X is reflexive and so by the Dunford-Pettis criterion ([9]), taking a subsequence and keeping the same notations, we may assume that $h_n(.)$ converges weakly in $L^1(I, X)$ to some $h(.) \in L^1(I, X)$.

We recall that for convex subsets of a Banach space the strong closure coincides with the weak closure. Since $h_n(.)$ converges weakly in $L^1(I, X)$ to $h(.) \in L^1(I, X)$ then for all $p \geq 0$, $h(.)$ belongs to the weak closure of the convex hull $co\{h_n(.)\}_{n \geq p}$ of the subset $\{h_n(.)\}_{n \geq p}$. It coincides with the strong closure of $co\{h_n(.)\}_{n \geq p}$. So, there exist $\lambda_i^n > 0, i = n, \ldots k(n)$ such that

$$\sum_{i=1}^{k(n)} \lambda_i^n = 1, \quad g_n(.) = \sum_{i=n}^{k(n)} \lambda_i^n h_i(.) \in co\{h_n(.)\}_{n \geq h}$$

and such that $g_n(.)$ converges strongly to $f(.)$ in $L^1(I, X)$. Define $r_n(.) = \sum_{i=n}^{k(n)} \lambda_i^n f_i(.)$ Therefore, there exists a subsequence $g_{n_j}(.)$ that converges to $h(.)$ almost everywhere. In particular, $r_{n_j}(.)$ converges almost everywhere to $r(.) = e^{\alpha m(.)} h(.) \in L^1(I, X)$. With Lebesgue's dominated convergence theorem, for every $t \in I$ we obtain

$$\lim_{j \to \infty} \int_0^t U(t, s) r_{n_j}(s) ds = \int_0^t U(t, s) r(s) ds$$

Define

$$y(t) = -\frac{\partial}{\partial s} G(t, 0) x_0 + G(t, 0) y_0 + \int_0^t U(t, s) r(s) ds, \quad t \in I$$

and note that

$$|x(t) - y(t)| \leq |x(.) - x_{n_j}(.)|_C + |\int_0^t U(t, s) r_{n_j}(s) ds - \int_0^t U(t, s) r(s) ds|,$$

which yields $x(t) = y(t)\ \forall t \in I$.

Moreover, for almost every $t \in I$

$$r_{n_j}(t) \in \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} F(t, x_i(t)) \subset F(t, x(t)) + L(t) \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} |x(t) - x_i(t)| B.$$

Since $\lim_{i \to \infty} |x(t) - x_i(t)| = 0$, we deduce that $f(t) \in F(t, x(t))\ a.e.(I)$ and the proof is complete. □

Following similar ideas as in the proof of Theorem 3.1 we obtain an existence result for problem (1.1).

Theorem 3.4. *Let Hypothesis be satisfied and let* $\alpha > MK_0T$ *and let* $y(.)$ *be a mild solution of the problem*

$$x'' = A(t)x + \int_0^t K(t,s)g(s)ds \quad x(0) = x_1, \quad x'(0) = y_1,$$

where $g(.) \in L^1(I, X)$ *and there exists* $p(.) \in L^1(I, \mathbb{R})$ *such that*

$$d(g(t), F(t, y(t))) \leq p(t), \quad a.e.\ (I).$$

Then for every $\varepsilon > 0$ *there exists* $x(.)$ *a mild solution of (1.1) satisfying for all* $t \in I$

$$\begin{aligned} &|x(t) - y(t)| \leq (1 + \tfrac{MK_0T}{\alpha - MK_0T} e^{\alpha m(t)})(M_0|x_0 - y_0| + M|x_1 - y_1|)+ \\ &\tfrac{\alpha MK_0Te^{\alpha m(t)}}{\alpha - MK_0T} \int_0^T e^{-\alpha m(s)}p(s)ds + \varepsilon. \end{aligned} \tag{3.5}$$

Proof. We keep the same notations as in the proof of Theorem 3.1.

Consider the following set-valued maps

$$H(t,x) = F(t,x) + p(t)B, \quad (t,x) \in I \times X,$$

$$\tilde{B}_{x_1,y_1,f}(t) = H(t, -\frac{\partial}{\partial s}G(t,0)x_0 + G(t,0)y_0 + \int_0^t U(t,s)f(s)ds),\ t \in I,$$

$$\tilde{N}_{x_1,y_1}(f) = \{\phi(.) \in L^1(I,X); \quad \phi(t) \in \tilde{B}_{x_1,y_1,f}(t) \quad a.e.\ (I)\},\ f \in L^1(I,X).$$

Obviously, $H(.,.)$ satisfies Hypothesis.

As in the proof of Theorem 3.1 we deduce that $\tilde{N}_{x_1,y_1}(.)$ is a $\frac{MK_0T}{\alpha}$-contraction on $L^1(I,X)$ with closed nonempty values.

Next we show the following estimate

$$d_H(N_{x_0,y_0}(f), \tilde{N}_{x_1,y_1}(f)) \leq \frac{M_0}{\alpha}|x_0 - x_1| + \frac{M}{\alpha}|y_0 - y_1| + \int_0^T e^{-\alpha m(t)}p(t)dt \tag{3.6}$$

$\forall\ f(.) \in L^1(I,X)$.

Take $\phi \in N_{x_0,y_0}(f)$, $\delta > 0$ and, for $t \in I$, define

$$\begin{aligned} &G_1(t) = \tilde{B}_{x_1,y_1,f}(t) \ \cap \ \{z \in X;\ |\phi(t) - z| \leq \\ &L(t)(|\tfrac{\partial}{\partial s}G(t,0)||x_1 - x_0| + |G(t,0)||y_1 - y_0|) + p(t) + \delta\} \end{aligned}$$

With the same arguments used for the set-valued map $G(.)$ in the proof of Theorem 3.1, we deduce that $G_1(.)$ is measurable with nonempty closed values. Let $\psi(.)$ be a measurable selection of $G_1(.)$. It follows that $\psi(.) \in \tilde{N}_{y_0,y_1}(f)$ and one has

$$\begin{aligned} &|\phi - \psi|_1 = \int_0^T e^{-\alpha m(t)}|\phi(t) - \psi(t)|dt \leq \\ &\int_0^T e^{-\alpha m(t)}[L(t)(|\tfrac{\partial}{\partial s}G(t,0)||x_1 - x_2| + |G(t,0)||y_1 - y_2|) + p(t) + \delta]dt \leq \\ &\tfrac{M_0}{\alpha}|x_0 - x_1| + \tfrac{M}{\alpha}|y_0 - y_1| + \int_0^T e^{-\alpha m(t)}p(t)dt + \delta\int_0^T e^{-\alpha m(t)}p(t)dt. \end{aligned}$$

Since $\delta > 0$ was arbitrary, as above, we obtain (3.6). Applying Proposition 2.3 we find

$$\begin{aligned} &d_H(Fix(N_{x_0,y_0}), Fix(\tilde{N}_{x_1,y_1})) \leq \tfrac{M_0}{\alpha - MK_0T}|x_0 - y_0| \\ &+\tfrac{M}{\alpha - MK_0T}|x_1 - y_1| + \tfrac{\alpha}{\alpha - MK_0T}\int_0^T e^{-\alpha m(t)}p(t)dt. \end{aligned}$$

Since $g(.) \in Fix(\tilde{N}_{x_1,y_1})$ we find that there exists $f(.) \in Fix(N_{x_0,y_0})$ such that for any $\varepsilon > 0$

$$\begin{aligned} &|g - f|_1 \leq \tfrac{M_0}{\alpha - MK_0T}|x_0 - x_1| + \tfrac{M}{\alpha - MK_0T}|y_0 - y_1|+ \\ &\tfrac{\alpha}{\alpha - MK_0T}\int_0^T e^{-\alpha m(t)}p(t)dt + \tfrac{\varepsilon}{MK_0Te^{\alpha m(T)}}. \end{aligned} \tag{3.7}$$

It remains to define $x(t) = -\frac{\partial}{\partial s}G(t,0)x_0 + G(t,0)y_0 + \int_0^t U(t,s)f(s)ds$, $t \in I$. One has

$$|x(t) - y(t)| \leq M_0|x_0 - x_1| + M|y_0 - y_1| + MK_0Te^{\alpha m(t)}|f - g|_1.$$

From the last inequality and (3.7) we obtain (3.5). □

References

[1] A. Baliki, M. Benchohra, J.R. Graef, Global existence and stability of second order functional evolution equations with infinite delay, Electronic J. Qual. Theory Differ. Equations 2016, no. 23, (2016) 1–10.

[2] A. Baliki, M. Benchohra, J.J. Nieto, Qualitative analysis of second-order functional evolution equations, Dynamic Syst. Appl. 24 (2015) 559–572.

[3] M. Benchohra, I. Medjadj, Global existence results for second order neutral functional differential equations with state-dependent delay, Comment. Math. Univ. Carolin. 57 (2016), 169–183.

[4] C. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions, Springer, Berlin, 1977.

[5] A. Cernea, Lipschitz-continuity of the solution map of some nonconvex evolution inclusions, Anal. Univ. Bucureşti, Mat. 57 (2008) 189–198.

[6] A. Cernea, On the existence of mild solutions of a nonconvex evolution inclusion, Math. Commun. 13 (2008) 107–114.

[7] A. Cernea, Some remarks on the solutions of a second-order evolution inclusion, Dynamic Syst. Appl. 27 (2018) 319–330.

[8] H. Covitz, S. B. Nadler jr., Multivalued contraction mapping in generalized metric spaces, Israel J. Math. 8 (1970) 5–11.

[9] N. Dunford, B.J. Pettis, Linear operators and summable functions, Trans. Amer. Math. Soc. 47 (1940) 323–392.

[10] A.F. Filippov, Classical solutions of differential equations with multivalued right-hand side, SIAM J. Control Optim. 5 (1967) 609–621.

[11] H.R. Henriquez, Existence of solutions of nonautonomous second order functional differential equations with infinite delay, Nonlinear Anal. 74 (2011), 3333–3352.

[12] H.R. Henriquez, V. Poblete, J.C. Pozo, Mild solutions of non-autonomous second order problems with nonlocal initial conditions, J. Math. Anal. Appl. 412 (2014) 1064–1083.

[13] Z. Kannai, P. Tallos, Stability of solution sets of differential inclusions, Acta Sci. Math. (Szeged) 61 (1995) 197–207.

[14] M. Kozak, A fundamental solution of a second-order differential equation in a Banach space, Univ. Iagel. Acta. Math. 32 (1995) 275–289.

[15] T. C. Lim, On fixed point stability for set valued contractive mappings with applications to generalized differential equations, J. Math. Anal. Appl. 110 (1985) 436–441.

[16] P. Talllos, A Filippov-Gronwall type inequality in infinite dimensional space, Pure Math. Appl. 5 (1994) 355–362.