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STANCU TYPE (p,q)-SZÁSZ-MIRAKYAN-BASKAKOV OPERATORS

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Abstract. In the present paper, we introduce Stancu type generalization of (p,q)-Szasz-Mirakyan-Baskakov operators and investigate their approximation properties such as weighted approximation, rate of convergence and pointwise convergence.

1. Introduction

In the last two decades, there has been intensive research on the approximation of functions by positive linear operators introduced by making use of q-calculus. Many q-generalizations of approximation operators and their approximation behaviors were intensively studied. Nowadays, approximation by linear positive operators in post-quantum calculus, namely the (p,q)-calculus is very active area. Mursaleen et al. introduced the (p,q)-analogues of some well-known operators such as Bernstein operators [15], Bernstein-Stancu operators [16], Bleimann-Butzer-Hahn operators [17], Bernstein-Schurer operators [18]. They investigated the approximation properties of above mentioned operators using the techniques of (p,q)-calculus. Also we can refer the readers to some recent papers: (p,q)-Baskakov-Kantorovich operators [3], bivariate (p,q)-Bernstein Kantorovich operators [4], bivariate (p,q)-Baskakov-Kantorovich operators [12].

Let us recall certain notations and definitions of (p,q)-calculus. Let $0<q<p\leq 1$. For each nonnegative integer k, n, $n\geq k\geq 0$, the (p,q)-numbers $[k]_{p,q}$, (p,q)-factorial $[k]_{p,q}!$ and (p,q)-binomial are defined by

$$[k]_{p,q} := \frac{p^k-q^k}{p-q},$$

$$[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q}\ldots 1 \quad , & k\geq 1, \\ 1, & k=0 \end{cases}$$

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and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}.$$

In the case of $p=1$, the above notations reduce to q-analogues and one can easily see that $[n]_{p,q} = p^{n-1}\,[n]_{q/p}$.

Further, the (p,q)-power basis is defined by

$$(x \oplus a)^n_{p,q} = (x+a)(px+qa)(p^2x+q^2a)\cdots(p^{n-1}x+q^{n-1}a),$$

and

$$(x \ominus a)^n_{p,q} = (x-a)(px-qa)(p^2x-q^2a)\cdots(p^{n-1}x-q^{n-1}a).$$

Also the (p,q)-derivative of a function f, denoted by $D_{p,q}f$, is defined by

$$\left(D_{p,q}f\right)(x) := \frac{f(px)-f(qx)}{(p-q)\,x},\ \ x\neq 0,\ \ \left(D_{p,q}f\right)(0) := f'(0)$$

provided that f is differentiable at 0. The formula for the (p,q)-derivative of a product is

$$D_{p,q}\left(u(x)\,v(x)\right) := D_{p,q}\left(u(x)\right)v(qx) + D_{p,q}\left(v(x)\right)u(px).$$

For more details on (p,q)-calculus, we refer the readers to [9, 10, 20] and the references therein. There are two (p,q)-analogues of the exponential function, see [10],

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}}x^n}{[n]_{p,q}!},$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}x^n}{[n]_{p,q}!}, \tag{1.1}$$

which satisfy the equality $e_{p,q}(x)\,E_{p,q}(-x) = 1$. For $p=1$, $e_{p,q}(x)$ and $E_{p,q}(x)$ reduce to q-exponential functions. Here we note that the interval of convergence of $e_{p,q}(x)$ is $|x| < 1/(p-q)$, $|p|<1$ and $|q|<1$ and the series (1.1) is convergent for all $x\in\mathbb{R}$, $|p|<1$ and $|q|<1$.

In the recent paper [1], Acar introduced (p,q)-Szász-Mirakyan operators as

$$S_{n,p,q}(f;x) := \frac{1}{E\left([n]_{p,q}\,x\right)} \sum_{k=0}^{\infty} f\left(\frac{[k]_{p,q}}{q^{k-2}\,[n]_{p,q}}\right) q^{\frac{k(k-1)}{2}} \frac{[n]^k_{p,q}\,x^k}{[k]_{p,q}!}. \tag{1.2}$$

The operators (1.2) are linear and positive and for $p=1$, the operators (1.2) turn out to be q-Szász-Mirakyan operators defined in [14]. King type modification of (p,q)-Szász-Mirakyan operators was introduced in [2].

Further, Aral et al. proposed (p,q)-Beta function of second kind for $m,n\in\mathbb{N}$ as

$$B_{p,q}(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1\oplus px)^{m+n}_{p,q}}\,d_{p,q}x$$

and considering the (p,q)-Beta function, Gupta [7] introduced (p,q)-Szász-Mirakyan-Baskakov operators as

$$D_n^{p,q}(f;x) = [n-1]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\, q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \int_0^{\infty} b_{n,k}^{p,q}(t)\, f\left(p^k t\right) d_{p,q}t, \tag{1.3}$$

where

$$s_{n,k}^{p,q}(x) = \frac{1}{E_{p,q}\left([n]_{p,q}\, x\right)} q^{\frac{k(k-1)}{2}} \frac{[n]_{p,q}^k\, x^k}{[k]_{p,q}!},\ \ b_{n,k}^{p,q}(t) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} \frac{t^k}{(1 \oplus pt)_{p,q}^{n+k}}.$$

Similar consideration of the operators given by (1.3) in classical calculus, we refer the readers to [5].

In the present paper, we extend the operators (1.3) for $0 \le \alpha \le \beta$, and every $n \in \mathbb{N}$, $q \in (0,1)$, $p \in (q,1]$, Stancu type modification of the operators (1.3) can be defined by

$$D_{n,\alpha,\beta}^{p,q}(f;x) = [n-1]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\, q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \int_0^{\infty} b_{n,k}^{p,q}(t)\, f\left(\frac{p^k [n]_{p,q}\, t + \alpha}{[n]_{p,q} + \beta}\right) d_{p,q}t, \tag{1.4}$$

for $x \in [0,\infty)$ and for every real valued continuous function f on $[0,\infty)$. In case $\alpha = \beta = 0$, the operators (1.4) reduce to (1.3).

We shall first give some lemmas which will be necessary to prove our main results. We obtain local approximation behaviors of the operators (1.3) in terms of second order modulus of smoothness and classical modulus of continuity. We also present uniform convergence theorems via weighted Korovkin theorem for the functions belong to weighted spaces. In the last section, we prove the pointwise estimates for the functions satisfying the Lipschitz conditions.

2. Auxiliary Results

Lemma 1. *([7]) For $x \in [0,\infty)$, $0 < q < p \le 1$, we have*

(i) $D_n^{p,q}(1;x) = 1$

(ii) $D_n^{p,q}(t;x) = \frac{1}{qp^2[n-2]_{p,q}} + \frac{[n]_{p,q}x}{pq^2[n-2]_{p,q}}$

(iii) $D_n^{p,q}\left(t^2;x\right) = \frac{[2]_{p,q}}{p^5q^3[n-2]_{p,q}[n-3]_{p,q}} + \frac{\left[q\left([2]_{p,q}+p\right)+p^2\right][n]_{p,q}}{p^4q^5[n-2]_{p,q}[n-3]_{p,q}}x + \frac{[n]_{p,q}^2}{pq^6[n-2]_{p,q}[n-3]_{p,q}}x^2.$

Lemma 2. *For $x \in [0,\infty)$, $0 < q < p \le 1$, we have*

$$D_{n,\alpha,\beta}^{p,q}(1;x) = 1, \tag{2.1}$$

$$D_{n,\alpha,\beta}^{p,q}(t;x) = \frac{[n]_{p,q}^2 x}{\left([n]_{p,q}+\beta\right) pq^2 [n-2]_{p,q}} + \frac{[n]_{p,q}+\alpha qp^2 [n-2]_{p,q}}{\left([n]_{p,q}+\beta\right) qp^2 [n-2]_{p,q}}, \tag{2.2}$$

$$\begin{aligned} D_{n,\alpha,\beta}^{p,q}\left(t^2;x\right) &= \frac{[n]_{p,q}^4}{pq^6 [n-2]_{p,q} [n-3]_{p,q} \left([n]_{p,q}+\beta\right)^2} x^2 \\ &+ \left(\frac{\left(q\left([2]_{p,q}+p\right)+p^2\right)[n]_{p,q}^3 + 2\alpha [n]_{p,q}^2 p^3 q^3 [n-3]_{p,q}}{p^4 q^5 [n-2]_{p,q} [n-3]_{p,q} \left([n]_{p,q}+\beta\right)^2}\right) x \\ &+ \frac{[2]_{p,q} [n]_{p,q}^2}{p^5 q^3 [n-2]_{p,q} [n-3]_{p,q} \left([n]_{p,q}+\beta\right)^2} + \frac{2\alpha [n]_{p,q}}{\left([n]_{p,q}+\beta\right)^2 qp^2 [n-2]_{p,q}} \\ &+ \frac{\alpha^2}{\left([n]_{p,q}+\beta\right)^2}. \end{aligned}$$

Proof. Using Lemma 1, we have $D_{n,\alpha,\beta}^{p,q}(1;x) = 1$. Also we get

$$\begin{aligned} D_{n,\alpha,\beta}^{p,q}(t;x) &= [n-1]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\, q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \int_0^{\infty} b_{n,k}^{p,q}(t) \left(\frac{p^k [n]_{p,q} t+\alpha}{[n]_{p,q}+\beta}\right) d_{p,q}t \\ &= \frac{[n]_{p,q}}{[n]_{p,q}+\beta} D_n^{p,q}(t;x) + \frac{\alpha}{[n]_{p,q}+\beta} D_n^{p,q}(1;x) \\ &= \frac{[n]_{p,q}}{[n]_{p,q}+\beta}\left(\frac{1}{qp^2[n-2]_{p,q}} + \frac{[n]_{p,q} x}{pq^2 [n-2]_{p,q}}\right) + \frac{\alpha}{[n]_{p,q}+\beta} \\ &= \frac{[n]_{p,q}^2 x}{\left([n]_{p,q}+\beta\right) pq^2 [n-2]_{p,q}} + \frac{[n]_{p,q}+\alpha qp^2 [n-2]_{p,q}}{\left([n]_{p,q}+\beta\right) qp^2 [n-2]_{p,q}} \end{aligned}$$

and

$$D_{n,\alpha,\beta}^{p,q}\left(t^2;x\right) = [n-1]_{p,q} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\, q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \int_0^{\infty} b_{n,k}^{p,q}(t) \left(\frac{p^k [n]_{p,q} t+\alpha}{[n]_{p,q}+\beta}\right)^2 d_{p,q}t \tag{2.3}$$

$$\begin{aligned} &= \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}\right)^2 D_n^{p,q}\left(t^2;x\right) + \frac{2\alpha [n]_{p,q}}{\left([n]_{p,q}+\beta\right)^2} D_n^{p,q}(t;x) + \frac{\alpha^2}{\left([n]_{p,q}+\beta\right)^2} D_n^{p,q}(1;x) \\ &= \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}\right)^2 \left(\frac{q^3 [2]_{p,q} + pq\left(q\left([2]_{p,q}+p\right)+p^2\right)[n]_{p,q} x + p^4 [n]_{p,q}^2 x^2}{p^5 q^6 [n-2]_{p,q} [n-3]_{p,q}}\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{2\alpha [n]_{p,q}}{\left([n]_{p,q}+\beta\right)^2}\left(\frac{q+p[n]_{p,q}x}{q^2p^2[n-2]_{p,q}}\right)+\frac{\alpha^2}{\left([n]_{p,q}+\beta\right)^2} \\
&= \frac{[n]_{p,q}^4}{pq^6[n-2]_{p,q}[n-3]_{p,q}\left([n]_{p,q}+\beta\right)^2}x^2 \\
&+\left(\frac{\left(q\left([2]_{p,q}+p\right)+p^2\right)[n]_{p,q}^3+2\alpha [n]_{p,q}^2 p^3q^3[n-3]_{p,q}}{p^4q^5[n-2]_{p,q}[n-3]_{p,q}\left([n]_{p,q}+\beta\right)^2}\right)x \\
&+\frac{[2]_{p,q}[n]_{p,q}^2}{p^5q^3[n-2]_{p,q}[n-3]_{p,q}\left([n]_{p,q}+\beta\right)^2}+\frac{2\alpha [n]_{p,q}}{\left([n]_{p,q}+\beta\right)^2 qp^2[n-2]_{p,q}} \\
&+\frac{\alpha^2}{\left([n]_{p,q}+\beta\right)^2}.
\end{aligned}
$$

□

Corollary 1. *Using Lemma 2 we set*

$$
\begin{aligned}
&D_{n,\alpha,\beta}^{p,q}\left((t-x)^2;x\right) \\
&= x^2\left(\frac{[n]_{p,q}^4}{pq^6[n-2]_{p,q}[n-3]_{p,q}\left([n]_{p,q}+\beta\right)^2}-\frac{2[n]_{p,q}^2x}{\left([n]_{p,q}+\beta\right)pq^2[n-2]_{p,q}}+1\right) \\
&\quad +x\left(\frac{\left[q\left([2]_{p,q}+p\right)+p^2\right][n]_{p,q}^3}{p^4q^5[n-2]_{p,q}[n-3]_{p,q}\left([n]_{p,q}+\beta\right)^2}\right. \\
&\quad \left.+\frac{2\alpha [n]_{p,q}^2}{\left([n]_{p,q}+\beta\right)^2 pq^2[n-2]_{p,q}}-\frac{2[n]_{p,q}+2\alpha qp^2[n-2]_{p,q}}{\left([n]_{p,q}+\beta\right)qp^2[n-2]_{p,q}}\right) \\
&\quad +\frac{[2]_{p,q}[n]_{p,q}^2}{p^5q^3[n-2]_{p,q}[n-3]_{p,q}\left([n]_{p,q}+\beta\right)^2}+\frac{2\alpha [n]_{p,q}}{\left([n]_{p,q}+\beta\right)^2 qp^2[n-2]_{p,q}} \\
&\quad +\frac{\alpha^2}{\left([n]_{p,q}+\beta\right)^2} \\
&=\ :\delta_n(p,q,x)
\end{aligned}
$$

and

$$
\beta_n(p,q,x):=\left(\frac{[n]_{p,q}^2}{\left([n]_{p,q}+\beta\right)pq^2[n-2]_{p,q}}-1\right)x+\frac{[n]_{p,q}+\alpha qp^2[n-2]_{p,q}}{\left([n]_{p,q}+\beta\right)qp^2[n-2]_{p,q}}.
$$

By $C_B[0,\infty)$, we denote the space of real-valued uniformly continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\|f\| = \sup_{0\leq x<\infty} |f(x)|.$$

Further let us consider the following $\mathcal{K}$-functional:

$$\mathcal{K}_2(f,\delta) = \inf_{g\in W^2} \{\|f-g\| + \delta\|g''\|\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [6, p. 177, Theorem 2.4] there exists an absolute constant $C > 0$ such that

$$\mathcal{K}_2(f,\delta) \leq C\omega_2\left(f,\sqrt{\delta}\right), \tag{2.4}$$

where

$$\omega_2(f,\delta) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

Let $B_m[0,\infty)$ be the set of all functions satisfying the condition $|f(x)| \leq M_f(1+x^m)$, $x \in [0,\infty)$, $m > 0$ with some constant depending on f. $C_m[0,\infty) = B_m[0,\infty) \cap C[0,\infty)$ endowed with the norm

$$\|f\|_m = \sup_{x\geq 0} \frac{|f(x)|}{1+x^m}$$

and

$$C_m^*[0,\infty) = \left\{f \in C_m[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^m} < \infty\right\}.$$

Lemma 3. *Let $f \in C_B[0,\infty)$. Then for all $g \in C_B^2[0,\infty)$, we have*

$$\left|\hat{D}_{n,\alpha,\beta}^{p,q}(g;x) - g(x)\right| \leq \|g''\|\left(\delta_n(p,q,x) + \beta_n^2(p,q,x)\right), \tag{2.5}$$

where

$$\hat{D}_{n,\alpha,\beta}^{p,q}(g;x) = D_{n,\alpha,\beta}^{p,q}(g;x) + g(x) - g\left(D_{n,\alpha,\beta}^{p,q}(t;x)\right). \tag{2.6}$$

Proof. By the definition of $\hat{D}_{n,\alpha,\beta}^{p,q}$ and Lemma 2, it is obvious that

$$\hat{D}_{n,\alpha,\beta}^{p,q}(t-x;x) = 0. \tag{2.7}$$

Since $g \in C_B^2[0,\infty)$, using the Taylor's expansion for $x \in [0,\infty)$ we have

$$g(t) = g(x) + g(x)(t-x) + \int_x^t (t-u)\,g''(u)\,du.$$

Applying the operators $\hat{D}_{n,\alpha,\beta}^{p,q}$ to both sides of above equality and considering the fact (2.7) we obtain

$$\begin{aligned} &\hat{D}_{n,\alpha,\beta}^{p,q}\left(g;x\right)-g\left(x\right) \\ &=\hat{D}_{n,\alpha,\beta}^{p,q}\left(\int\limits_{x}^{t}\left(t-u\right)g''\left(u\right)du;x\right)=D_{n,\alpha,\beta}^{p,q}\left(\int\limits_{x}^{t}\left(t-u\right)g''\left(u\right)du;x\right) \qquad (2.8) \\ &-\int\limits_{x}^{D_{n,\alpha,\beta}^{p,q}(t;x)}\left(\frac{[n]_{p,q}^{2}x}{\left([n]_{p,q}+\beta\right)pq^{2}[n-2]_{p,q}}+\frac{[n]_{p,q}+\alpha qp^{2}[n-2]_{p,q}}{\left([n]_{p,q}+\beta\right)qp^{2}[n-2]_{p,q}}-u\right)g''\left(u\right)du. \end{aligned}$$

Also we get

$$\left|\int\limits_{x}^{t}\left(t-u\right)g''\left(u\right)du\right|\leq\left|\int\limits_{x}^{t}\left|t-u\right|\left|g''\left(u\right)\right|du\right|\leq\left\|g''\right\|\left|\int\limits_{x}^{t}\left|t-u\right|du\right|\leq\left\|g''\right\|\left(t-x\right)^{2} \tag{2.9}$$

and

$$\begin{aligned} &\left|\int\limits_{x}^{D_{n,\alpha,\beta}^{p,q}(t;x)}\left(\frac{[n]_{p,q}^{2}x}{\left([n]_{p,q}+\beta\right)pq^{2}[n-2]_{p,q}}+\frac{[n]_{p,q}+\alpha qp^{2}[n-2]_{p,q}}{\left([n]_{p,q}+\beta\right)qp^{2}[n-2]_{p,q}}-u\right)g''\left(u\right)du\right| \\ &\leq\left(\frac{[n]_{p,q}^{2}x}{\left([n]_{p,q}+\beta\right)pq^{2}[n-2]_{p,q}}+\frac{[n]_{p,q}+\alpha qp^{2}[n-2]_{p,q}}{\left([n]_{p,q}+\beta\right)qp^{2}[n-2]_{p,q}}-x\right)^{2}\left\|g''\right\|:=\beta_{n}^{2}\left(p,q,x\right)\left\|g''\right\|. \end{aligned} \tag{2.10}$$

Using the inequalities (2.9) and (2.10) in (2.8) we immediately have

$$\left|\hat{D}_{n,\alpha,\beta}^{p,q}\left(g;x\right)-g\left(x\right)\right|\leq\left\|g''\right\|\left(\delta_{n}\left(p,q,x\right)+\beta_{n}^{2}\left(p,q,x\right)\right).$$

□

Lemma 4. *For $f\in C_{B}\left[0,\infty\right)$, one has*

$$\left\|D_{n,\alpha,\beta}^{p,q}f\right\|\leq\left\|f\right\|.$$

Proof. In view of (1.4) and Lemma 2, the proof easily follows. □

3. Local Approximation

Theorem 1. *Let $f\in C_{B}\left[0,\infty\right)$. Then for every $x\in\left[0,\infty\right)$, there exists a constant $L>0$ such that*

$$\left|D_{n,\alpha,\beta}^{p,q}\left(f;x\right)-f\left(x\right)\right|\leq L\omega_{2}\left(f;\sqrt{\delta_{n}\left(p,q,x\right)+\beta_{n}^{2}\left(p,q,x\right)}\right)+\omega\left(f;\beta_{n}\left(p,q,x\right)\right).$$

Proof. By (2.6), for every $g \in C_B^2[0,\infty)$ one can obtain

$$\begin{aligned}\left|D_{n,\alpha,\beta}^{p,q}(f;x)-f(x)\right| &\le \left|\hat{D}_{n,\alpha,\beta}^{p,q}(f;x)-f(x)\right| \\ &+\left|f(x)-f\left(\frac{[n]_{p,q}^2 x}{\left([n]_{p,q}+\beta\right)pq^2[n-2]_{p,q}}+\frac{[n]_{p,q}+\alpha qp^2[n-2]_{p,q}}{\left([n]_{p,q}+\beta\right)qp^2[n-2]_{p,q}}\right)\right| \\ &\le \left|\hat{D}_{n,\alpha,\beta}^{p,q}(f-g;x)-(f-g)(x)\right| \\ &+\left|f(x)-f\left(\frac{[n]_{p,q}^2 x}{\left([n]_{p,q}+\beta\right)pq^2[n-2]_{p,q}}+\frac{[n]_{p,q}+\alpha qp^2[n-2]_{p,q}}{\left([n]_{p,q}+\beta\right)qp^2[n-2]_{p,q}}\right)\right| \\ &+\left|\hat{D}_{n,\alpha,\beta}^{p,q}(g;x)-g(x)\right|.\end{aligned}$$

Taking into account Lemma 4 and Lemma 3 we get

$$\begin{aligned}\left|D_{n,\alpha,\beta}^{p,q}(f;x)-f(x)\right| &\le 4\|f-g\| \\ &+\left|f(x)-f\left(\frac{[n]_{p,q}^2 x}{\left([n]_{p,q}+\beta\right)pq^2[n-2]_{p,q}}+\frac{[n]_{p,q}+\alpha qp^2[n-2]_{p,q}}{\left([n]_{p,q}+\beta\right)qp^2[n-2]_{p,q}}\right)\right| \\ &+\|g''\|\left(\delta_n(p,q,x)+\beta_n^2(p,q,x)\right)\end{aligned}$$

and taking infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and using (2.4), we deduce

$$\begin{aligned}\left|D_{n,\alpha,\beta}^{p,q}(f;x)-f(x)\right| &\le 4K_2\left(f;\delta_n(p,q,x)+\beta_n^2(p,q,x)\right)+\omega\left(f;\beta_n(p,q,x)\right) \\ &\le 4\omega_2\left(f;\sqrt{\delta_n(p,q,x)+\beta_n^2(p,q,x)}\right)+\omega\left(f;\beta_n(p,q,x)\right) \\ &= L\omega_2\left(f;\sqrt{\delta_n(p,q,x)+\beta_n^2(p,q,x)}\right)+\omega\left(f;\beta_n(p,q,x)\right),\end{aligned}$$

where $L=4M>0$. □

Theorem 2. *Let* $f \in C_2[0,\infty)$, $p_n, q_n \in (0,1)$ *such that* $0<q_n<p_n\le 1$ *and* $\omega_{a+1}(f,\delta)$ *be its modulus of continuity on the finite interval* $[0,a+1]\subset[0,\infty)$, *where* $a>0$. *Then, for every* $n>2$,

$$\left|D_{n,\alpha,\beta}^{p,q}(f;x)-f(x)\right| \le 4M_f\left(1+a^2\right)\delta_n(p_n,q_n,x)+2\omega_{a+1}\left(f,\sqrt{\delta_n(p_n,q_n,x)}\right).$$

Proof. By [11], $\omega_{a+1}(\cdot,\delta)$ has the property

$$|f(t)-f(x)| \le 4M_f\left(1+a^2\right)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta),\ \delta>0.$$

Applying Cauchy-Schwarz inequality and choosing $\delta=\sqrt{\delta_n\left(p_n,q_n,x\right)}$, we have

$$\begin{aligned}\left|D_{n,\alpha,\beta}^{p,q}\left(f;x\right)-f\left(x\right)\right| &\leq 4M_f\left(1+a^2\right)D_{n,\alpha,\beta}^{p,q}\left(\left(t-x\right)^2;x\right)\\ &\quad+\omega_{a+1}\left(f,\delta\right)\left(1+\frac{1}{\delta}\left(D_{n,\alpha,\beta}^{p,q}\left(\left(t-x\right)^2;x\right)\right)^{1/2}\right)\\ &= 4M_f\left(1+a^2\right)\delta_n\left(p_n,q_n,x\right)+2\omega_{a+1}\left(f,\sqrt{\delta_n\left(p_n,q_n,x\right)}\right),\end{aligned}$$

which completes the proof. □

4. Weighted Approximation

First, let us recall the definitions of weighted spaces and corresponding modulus of continuity. Let $C\left[0,\infty\right)$ be the set of all continuous functions f defined on $\left[0,\infty\right)$ and $B_2\left[0,\infty\right)$ the set of all functions f defined on $\left[0,\infty\right)$ satisfying the condition $\left|f\left(x\right)\right|\leq M\left(1+x^2\right)$ with some positive constant M which may depend only on f. $C_2\left[0,\infty\right)$ denotes the subspace of all continuous functions in $B_2\left[0,\infty\right)$. By $C_2^*\left[0,\infty\right)$, we denote the subspace of all functions $f\in C_2\left[0,\infty\right)$ for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. $B_2\left[0,\infty\right)$ is a linear normed space with the norm $\left\|f\right\|_2=\sup_{x\geq0}\frac{|f(x)|}{1+x^2}$.

Theorem 3. *Let $q=q_n\in(0,1)$, $p=p_n\in(q,1]$ such that $q_n\to1$, $p_n\to1$ as $n\to\infty$. Then for each function $f\in C_2^*\left[0,\infty\right)$ we get*

$$\lim_{n\to\infty}\left\|D_{n,\alpha,\beta}^{p_n,q_n}f-f\right\|_2=0.$$

Proof. According to weighted Korovkin theorem proved in [8], it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\left\|D_{n,\alpha,\beta}^{p_n,q_n}e_i-e_i\right\|_2=0,\ i=0,1,2. \tag{4.1}$$

By (2.1), (4.1) holds for $i=0$. By (2.2) and (2.3) we have

$$\begin{aligned}\left\|D_{n,\alpha,\beta}^{p_n,q_n}e_1-e_1\right\|_2 &= \sup_{x\geq0}\frac{\beta_n\left(p_n,q_n,x\right)}{1+x^2}\\ &\leq \left(\frac{[n]_{p_n,q_n}^2}{\left([n]_{p_n,q_n}+\beta\right)p_nq_n^2\left[n-2\right]_{p_n,q_n}}-1\right)\sup_{x\geq0}\frac{x}{1+x^2}\\ &\quad+\frac{[n]_{p_n,q_n}+\alpha q_np_n^2\left[n-2\right]_{p_n,q_n}}{\left([n]_{p,q}+\beta\right)qp^2\left[n-2\right]_{p,q}}\\ &\leq \left(\frac{[n]_{p_n,q_n}^2}{\left([n]_{p_n,q_n}+\beta\right)p_nq_n^2\left[n-2\right]_{p_n,q_n}}-1\right)\end{aligned}$$

$$+\frac{[n]_{p_n,q_n}+\alpha q_n p_n^2\,[n-2]_{p_n,q_n}}{\left([n]_{p_n,q_n}+\beta\right) q_n p_n^2\,[n-2]_{p_n,q_n}}$$

and by similar consideration we have

$$\begin{aligned}\left\|D^{p_n,q_n}_{n,\alpha,\beta}e_2-e_2\right\|_2 \leq\ & \frac{[n]^4_{p_n,q_n}}{p_n q_n^6\,[n-2]_{p_n,q_n}\,[n-3]_{p_n,q_n}\left([n]_{p_n,q_n}+\beta\right)^2}-1\\ &+\frac{\left[q_n\left([2]_{p_n,q_n}+p_n\right)+p_n^2\right][n]^3_{p_n,q_n}}{p_n^4 q_n^5\,[n-2]_{p_n,q_n}\,[n-3]_{p_n,q_n}\left([n]_{p_n,q_n}+\beta\right)^2}\\ &+\frac{2\alpha\,[n]^2_{p,q}}{\left([n]_{p_n,q_n}+\beta\right)^2 p_n q_n^2\,[n-2]_{p_n,q_n}}\\ &+\frac{[2]_{p_n,q_n}\,[n]^2_{p_n,q_n}}{p_n^5 q_n^3\,[n-2]_{p_n,q_n}\,[n-3]_{p_n,q_n}\left([n]_{p_n,q_n}+\beta\right)^2}\\ &+\frac{2\alpha\,[n]_{p,q}}{\left([n]_{p_n,q_n}+\beta\right)^2 q_n p_n^2\,[n-2]_{p_n,q_n}}\\ &+\frac{\alpha^2}{\left([n]_{p_n,q_n}+\beta\right)^2}.\end{aligned}$$

Last two inequality mean that (4.1) holds for $i=1,2$. Hence, the proof is completed. □

Theorem 4. *Let $p=p_n$ and $q=q_n$ satisfies $0<q_n<p_n\leq 1$ and for n sufficiently large $p_n\to 1$, $q_n\to 1$ and $q_n^n\to 1$ and $p_n^n\to 1$. For each $f\in C^*_{x^2}[0,\ \infty)$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\ \infty)}\frac{\left|D^{p_n,q_n}_{n,\alpha,\beta}(f,x)-f(x)\right|}{(1+x^2)^{1+\alpha}}=0.$$

Proof. For any fixed $x_0>0$,

$$\begin{aligned}&\sup_{x\in[0,\ \infty)}\frac{\left|D^{p_n,q_n}_{n,\alpha,\beta}(f,x)-f(x)\right|}{(1+x^2)^{1+\alpha}}\\ &\leq \sup_{x\leq x_0}\frac{\left|D^{p_n,q_n}_{n,\alpha,\beta}(f,x)-f(x)\right|}{(1+x^2)^{1+\alpha}}+\sup_{x\geq x_0}\frac{\left|D^{p_n,q_n}_{n,\alpha,\beta}(f,x)-f(x)\right|}{(1+x^2)^{1+\alpha}}\\ &\leq \left\|D^{p_n,q_n}_{n,\alpha,\beta}(f)-f\right\|_{C[0,\ x_0]}+\|f\|_{x^2}\sup_{x\geq x_0}\frac{\left|D^{p_n,q_n}_{n,\alpha,\beta}\left(1+t^2,x\right)\right|}{(1+x^2)^{1+\alpha}}\end{aligned}$$

$$+\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}.$$

Since $|f(x)| \leq M(1+x^2)$, we have $\sup\limits_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}} \leq \frac{\|f\|_2}{(1+x_0^2)^{1+\alpha}}$ Let $\varepsilon > 0$ be arbitrary. We can choose x_0 to be so large that

$$\frac{\|f\|_2}{(1+x_0^2)^{1+\alpha}} < \frac{\varepsilon}{3} \tag{4.2}$$

On the other hand, in view of Lemma 2 we get

$$\|f\|_2 \lim_{n\to\infty}\frac{\left|D_{n,\alpha,\beta}^{p_n,q_n}(1+t^2,x)\right|}{(1+x^2)^{1+\alpha}} = \frac{(1+x^2)}{(1+x^2)^{1+\alpha}}\|f\|_2 \leq \frac{\|f\|_2}{(1+x^2)^{\alpha}} \leq \frac{\|f\|_2}{(1+x_0^2)^{\alpha}} < \frac{\varepsilon}{3}. \tag{4.3}$$

Also, the first term of the above inequality tends to zero by well known Korovkin's theorem, that is,

$$\left\|D_{n,\alpha,\beta}^{p_n,q_n}(f)-f\right\|_{C[0,\ x_0]} < \frac{\varepsilon}{3}. \tag{4.4}$$

Therefore, combining (4.2)-(4.4) we get the desired result. □

5. Pointwise Estimates

Theorem 5. *Let $0<\alpha\leq 1$ and E be any subset of the interval $[0,\infty)$. Then, if $f\in C_B[0,\infty)$ is locally $Lip(\alpha)$, i.e., the condition*

$$|f(y)-f(x)| \leq L|y-x|^{\alpha},\ y\in E \text{ and } x\in[0,\infty) \tag{5.1}$$

holds, then, for each $x\in[0,\infty)$, we have

$$\left|D_{n,\alpha,\beta}^{p,q}(f;x)-f(x)\right| \leq L\left\{\delta_n^{\frac{\alpha}{2}}(x)+2(d(x,E))^{\alpha}\right\},$$

where L is a constant depending on α and f; and $d(x,E)$ is the distance between x and E defined by

$$d(x,E)=\inf\{|t-x| : t\in E\}.$$

Proof. Let $\bar{E}$ denote the closure of E in $[0,\infty)$. Then, there exists a point $x_0\in\bar{E}$ such that $|x-x_0| = d(x,E)$. Using the triangle inequality

$$|f(t)-f(x)| \leq |f(t)-f(x_0)|+|f(x)-f(x_0)|$$

we immediately have by (5.1) that

$$\begin{aligned}
\left|D_{n,\alpha,\beta}^{p,q}(f;x)-f(x)\right| &\leq D_{n,\alpha,\beta}^{p,q}(|f(t)-f(x_0)|;x)+D_{n,\alpha,\beta}^{p,q}(|f(x)-f(x_0)|;x)\\
&\leq L\left\{D_{n,\alpha,\beta}^{p,q}(|t-x_0|^{\alpha};x)+|x-x_0|^{\alpha}\right\}\\
&\leq L\left\{D_{n,\alpha,\beta}^{p,q}(|t-x|^{\alpha}+|x-x_0|^{\alpha};x)+|x-x_0|^{\alpha}\right\}\\
&= L\left\{D_{n,\alpha,\beta}^{p,q}(|t-x|^{\alpha};x)+2|x-x_0|^{\alpha}\right\}.
\end{aligned}$$

Using Hölder inequality with $p=2/\alpha$, $q=2/(2-\alpha)$, we obtain

$$\begin{aligned}\left|S^{*}_{n,p,q}(f;x)-f(x)\right| &\leq L\left\{\left[D^{p,q}_{n,\alpha,\beta}\left(|t-x|^{\alpha p};x\right)\right]^{\frac{1}{p}}+2\left(d\left(x,E\right)\right)^{\alpha}\right\}\\ &= L\left\{\left[D^{p,q}_{n,\alpha,\beta}\left(|t-x|^{2};x\right)\right]^{\frac{\alpha}{2}}+2\left(d\left(x,E\right)\right)^{\alpha}\right\}\\ &\leq L\left\{\delta_n^{\frac{\alpha}{2}}(x)+2\left(d\left(x,E\right)\right)^{\alpha}\right\}\end{aligned}$$

□

Next we obtain the local direct estimate of the operators $D^{p,q}_{n,\alpha,\beta}$, using the Lipcshitz type maximal function of order α introduced by Lenze [13] as

$$\tilde{\omega}_a(f,x)=\sup_{t\neq x,t\in[0,\infty)}\frac{|f(t)-f(x)|}{|t-x|^{\alpha}},\ x\in[0,\infty)\ \text{and}\ \alpha\in(0,1]. \tag{5.2}$$

Theorem 6. *Let $f\in C_B[0,\infty)$ and $0<\alpha\leq 1$. Then, for all $x\in[0,\infty)$ we have*

$$\left|D^{p,q}_{n,\alpha,\beta}(f;x)-f(x)\right|\leq\tilde{\omega}_a(f,x)\,\delta_n^{\frac{\alpha}{2}}(x).$$

Proof. From the Eq. (5.2), we have

$$\left|D^{p,q}_{n,\alpha,\beta}(f;x)-f(x)\right|\leq\tilde{\omega}_a(f,x)\,D^{p,q}_{n,\alpha,\beta}\left(|t-x|^{\alpha};x\right).$$

Applying the Hölder inequality with $p=2/\alpha$, $q=2/(2-\alpha)$, we get

$$\begin{aligned}\left|D^{p,q}_{n,\alpha,\beta}(f;x)-f(x)\right| &\leq \tilde{\omega}_a(f,x)\left[D^{p,q}_{n,\alpha,\beta}\left(|t-x|^{2};x\right)\right]^{\frac{\alpha}{2}}\\ &\leq \tilde{\omega}_a(f,x)\,\delta_n^{\frac{\alpha}{2}}(x).\end{aligned}$$

□

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