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Research Article

Multiplicative order convergence in f-algebras

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Abstract

A net (x_α) in an f-algebra E is said to be multiplicative order convergent to $x \in E$ if $|x_\alpha - x|\, u \xrightarrow{o} 0$ for all $u \in E_+$. In this paper, we introduce the notions mo-convergence, mo-Cauchy, mo-complete, mo-continuous, and mo-KB-space. Moreover, we study the basic properties of these notions.

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1. Introductory facts

In spite of the nature of the classical theory of Riesz algebra and f-algebra, as far as we know, the concept of convergence in f-algebras related to multiplication has not been done before. However, there are some close studies under the name unbounded convergence in some kinds of vector lattices; see for example [2–6]. In the light of this information, we define a new concept of the convergence, which is called the mo-convergence, on f-algebras. Our aim is to introduce the concept of the mo-convergence by using the multiplication in f-algebras and examine the relationship between other types of convergence.

First of all, let us remember some notations and terminologies used in this paper. Let E be a real vector space. Then E is called *ordered vector space* if it has an order relation $\leq$ (i.e, $\leq$ is reflexive, antisymmetric, and transitive) that is compatible with the algebraic structure of E that means $y \leq x$ implies $y + z \leq x + z$ for all $z \in E$ and $\lambda y \leq \lambda x$ for each $\lambda \geq 0$. An ordered vector E is said to be *vector lattice* (or, *Riesz space*) if, for each pair of vectors $x, y \in E$, the supremum $x \vee y = \sup\{x, y\}$ and the infimum $x \wedge y = \inf\{x, y\}$ both exist in E. Moreover, $x^+ := x \vee 0$, $x^- := (-x) \vee 0$, and $|x| := x \vee (-x)$ are called the *positive* part, the *negative* part, and the *absolute value* of $x \in E$, respectively. Also, two vectors x, y in a vector lattice are said to be *disjoint* whenever $|x| \wedge |y| = 0$. A vector lattice E is called *order complete* if $0 \leq x_\alpha \uparrow\leq x$ implies the existence of $\sup x_\alpha$ in E. A subset A of a vector lattice is called *solid* whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace is referred to as an *order ideal*. An order closed ideal is referred to as a *band*. A sublattice Y of a vector lattice is majorizing E if, for every $x \in E$, there exists $y \in Y$ with $x \leq y$. A partially ordered set I is called *directed* if, for each $a_1, a_2 \in I$, there is another $a \in I$ such that $a \geq a_1$ and $a \geq a_2$ (or, $a \leq a_1$ and $a \leq a_2$). A function from a directed set I into a set E is called a *net* in E. A net $(x_\alpha)_{\alpha \in A}$ in a vector lattice X is called *order convergent* (or shortly, *o-convergent*) to $x \in X$, if there exists another net

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$(y_\beta)_{\beta\in B}$ satisfying $y_\beta \downarrow 0$, and for any $\beta \in B$ there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \le y_\beta$ for all $\alpha \ge \alpha_\beta$. In this case, we write $x_\alpha \xrightarrow{o} x$; for more details see for example [1,7,8].

A vector lattice E under an associative multiplication is said to be a *Riesz algebra* whenever the multiplication makes E an algebra (with the usual properties), and in addition, it satisfies the following property: $x, y \in E_+$ implies $xy \in E_+$. A Riesz algebra E is called *commutative* if $xy = yx$ for all $x, y \in E$. A Riesz algebra E is called *f-algebra* if E has additionally property that $x \wedge y = 0$ implies $(xz) \wedge y = (zx) \wedge y = 0$ for all $z \in E_+$; see for example [1]. A vector lattice E is called *Archimedean* whenever $\frac{1}{n}x \downarrow 0$ holds in E for each $x \in E_+$. Every Archimedean f-algebra is commutative; see Theorem 140.10 [8]. Assume E is an Archimedean f-algebra with a multiplicative unit vector e. Then, by applying Theorem 142.1(v) [8], in view of $e = ee = e^2 \ge 0$, it can be seen that e is a positive vector. In this article, unless otherwise stated, all f-algebras are semiprime, and all vector lattices are assumed to be real and Archimedean, and so f-algebras are commutative.

Recall that a net (x_α) in a vector lattice E is *unbounded order convergent* (or shortly, *uo-convergent*) to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in E_+$. In this case, we write $x_\alpha \xrightarrow{uo} x$; see for example [6] and [2–4]. Motivated from this definition, we give the following notion.

Definition 1.1. Let E be an f-algebra. A net (x_α) in E is said to be *multiplicative order convergent* to $x \in E$ (shortly, (x_α) *mo*-converges to x) if $|x_\alpha - x|\, u \xrightarrow{o} 0$ for all $u \in E_+$. Abbreviated as $x_\alpha \xrightarrow{mo} x$.

It is clear that $x_\alpha \xrightarrow{mo} x$ in an f-algebra E implies $x_\alpha y \xrightarrow{mo} xy$ for all $y \in E$ because of $|xy| = |x|\,|y|$ for all $x, y \in E$. Also, in general, the mo-convergence and uo-convergence are not the same. To see that we consider the following example.

Example 1.2. Let E be a vector lattice and consider $Orth(E) := \{T \in L_b(E) : x \perp y \text{ implies } Tx \perp y\}$ the set of orthomorphisms on E. The space $Orth(E)$ is not only vector lattice but also an f-algebra. The *mo*-convergence and the *uo*-convergence are different in $Orth(E)$.

We shall keep in mind the following useful lemma, obtained from the property of $xy \in E_+$ for every $x, y \in E_+$.

Lemma 1.3. *If $y \le x$ is provided in an f-algebra E then $uy \le ux$ for all $u \in E_+$.*

Recall that multiplication by a positive element in f-algebras is a vector lattice homomorphism, i.e., $u(x \wedge y) = (ux) \wedge (uy)$ and $u(x \vee y) = (ux) \vee (uy)$ for every positive element u; see for example Theorem 142.1(i) [8]. We will denote an l-algebra E as *infinite distributive* l-algebra whenever the following condition holds: if $\inf(A)$ exists for any subset A of E_+ then the infimum of the subset uA exists and $\inf(uA) = u \inf(A)$ for each positive vector $u \in E_+$. For a net $(x_\alpha) \downarrow 0$ in an infinite distributive l-algebra, the net (ux_α) is also decreasing to zero for all positive vector u. Fortunately, every f-algebra has the infinite distributive property.

Remark 1.4. The order convergence implies the *mo*-convergence in f-algebras. The converse holds true in f-algebras with multiplication unit. Indeed, assume a net $(x_\alpha)_{\alpha\in A}$ order converges to x in an f-algebra E. Then there exists another net $(y_\beta)_{\beta\in B}$ satisfying $y_\beta \downarrow 0$, and, for any $\beta \in B$, there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \le y_\beta$. Hence, we have $|x_\alpha - x|u \le y_\beta u$ for all $\alpha \ge \alpha_\beta$ and for each $u \in E_+$. Since $y_\beta \downarrow$, we have $uy_\beta \downarrow$ for each $u \in E_+$ by Lemma 1.3, and $\inf(uy_\beta) = u \inf(y_\beta) = 0$ because of $\inf(y_\beta) = 0$. Therefore, $|x_\alpha - x|\, u \xrightarrow{o} 0$ for each $u \in E_+$. That means $x_\alpha \xrightarrow{mo} x$.

For the converse, assume E is an f-algebra with multiplication unit e and $x_\alpha \xrightarrow{mo} x$ in E. That is, $|x_\alpha - x|\, u \xrightarrow{o} 0$ for all $u \in E_+$. Since $e \in E_+$, in particular, choose $u = e$, and so we have $|x_\alpha - x| = |x_\alpha - x|\, e \xrightarrow{o} 0$, or $x_\alpha \xrightarrow{o} x$ in E.

By considering Example 141.5 [8], we give the following example.

Example 1.5. Let $[a, b]$ be a closed interval in $\mathbb{R}$ and let E be vector lattice of all reel continuous functions on $[a, b]$ such that the graph of functions consists of a finite number of line segments. In view of Theorem 141.1 [8], every positive orthomorphism π in E is trivial orthomorphism, i.e., there is a reel number λ such that $\pi(f) = \lambda f$ for all $f \in E$. Therefore, a net of positive orthomorphism (π_α) is order convergent to π if and only if it is mo-convergent to π whenever the multiplication is the natural multiplicative, i.e., $\pi_1\pi_2(f) = \pi_1(\pi_2 f)$ for all $\pi_1, \pi_2 \in Orth(E)$ and all $f \in E$. Indeed, $Orth(E)$ is Archimedean f-algebra with the identity operator as a unit element; see Theorem 140.4 [8]. So, by applying Remark 1.4, the mo-convergence implies the order convergence of the net (π_α).

Conversely, assume the net of positive orthomorphisms $\pi_\alpha \xrightarrow{o} \pi$ in $Orth(E)$. Then we have $\pi_\alpha(f) \xrightarrow{o} \pi(f)$ for all $f \in E$; see Theorem VIII.2.3 [7]. For fixed $0 \leq \mu \in Orth(E)$, there is a reel number λ_μ such that $\mu(f) = \lambda_\mu f$ for all $f \in E$. Since $|\pi_\alpha(f) - \pi(f)| = |\lambda_{\pi_\alpha} f - \lambda_\pi f| \xrightarrow{o} 0$, we have

$$|(\pi_\alpha)f - (\pi)f|\,\mu = |\mu\lambda_{\pi_\alpha} f - \mu\lambda_\pi f| = |\lambda_\mu\lambda_{\pi_\alpha} f - \lambda_\mu\lambda_\pi f| = |\lambda_\mu|\,|\lambda_{\pi_\alpha} f - \lambda_\pi f| \xrightarrow{o} 0$$

for all $f \in E$. Since μ is arbitrary, we get $\pi_\alpha \xrightarrow{mo} \pi$.

2. Main results

We begin the section with the next list of properties of the mo-convergence which follows directly from Lemma 1.3, and the inequalities $|x - y| \leq |x - x_\alpha| + |x_\alpha - y|$ and $||x_\alpha| - |x|| \leq |x_\alpha - x|$.

Lemma 2.1. *Let $x_\alpha \xrightarrow{mo} x$ and $y_\alpha \xrightarrow{mo} y$ in an f-algebra E. Then the following holds:*

(i) *$x_\alpha \xrightarrow{mo} x$ if and only if $(x_\alpha - x) \xrightarrow{mo} 0$;*
(ii) *if $x_\alpha \xrightarrow{mo} x$ then $y_\beta \xrightarrow{mo} x$ for each subnet (y_β) of (x_α);*
(iii) *suppose $x_\alpha \xrightarrow{mo} x$ and $y_\beta \xrightarrow{mo} y$ then $ax_\alpha + by_\beta \xrightarrow{mo} ax + by$ for any $a, b \in \mathbb{R}$;*
(iv) *if $x_\alpha \xrightarrow{mo} x$ and $x_\alpha \xrightarrow{mo} y$ then $x = y$;*
(v) *if $x_\alpha \xrightarrow{mo} x$ then $|x_\alpha| \xrightarrow{mo} |x|$.*

Recall that an order complete vector lattice E^δ is said to be an order completion of the vector lattice E whenever E is Riesz isomorphic to a majorizing order dense vector lattice subspace of E^δ. Every Archimedean Riesz space has a unique order completion; see Theorem 2.24 [1].

Proposition 2.2. *Let (x_α) be a net in an f-algebra E. Then $x_\alpha \xrightarrow{mo} 0$ in E if and only if $x_\alpha \xrightarrow{mo} 0$ in the order completion E^δ of E.*

Proof. Assume $x_\alpha \xrightarrow{mo} 0$ in E. Then $|x_\alpha|\,u \xrightarrow{o} 0$ in E for all $u \in E_+$, and so $|x_\alpha|\,u \xrightarrow{o} 0$ in E^δ for all $u \in E_+$; see Corollary 2.9 [6]. Now, let us fix $v \in E^\delta_+$. Then there exists $x_v \in E_+$ such that $v \leq x_v$ because E majorizes E^δ. Then we have $|x_\alpha|\,v \leq |x_\alpha|\,x_v$. From $|x_\alpha|\,x_v \xrightarrow{o} 0$ in E^δ it follows that $|x_\alpha|\,v \xrightarrow{o} 0$ in E^δ, that is, $x_\alpha \xrightarrow{mo} 0$ in the order completion E^δ because $v \in E^\delta_+$ is arbitrary.

Conversely, assume $x_\alpha \xrightarrow{mo} 0$ in E^δ. Then, for all $u \in E^\delta_+$, we have $|x_\alpha|\,u \xrightarrow{o} 0$ in E^δ. In particular, for all $x \in E_+$, $|x_\alpha|\,x \xrightarrow{o} 0$ in E^δ. By Corollary 2.9 [6], we get $|x_\alpha|\,x \xrightarrow{o} 0$ in E for all $x \in E_+$. Hence $x_\alpha \xrightarrow{mo}$ in E. □

The multiplication in f-algebra is mo-continuous in the following sense.

Theorem 2.3. *Let E be an f-algebra, and $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ be two nets in E. If $x_\alpha \xrightarrow{mo} x$ and $y_\beta \xrightarrow{mo} y$ for some $x, y \in E$ and each positive element of E can be written as a multiplication of two positive elements then $x_\alpha y_\beta \xrightarrow{mo} xy$.*

Proof. Assume $x_\alpha \xrightarrow{\text{mo}} x$ and $y_\beta \xrightarrow{\text{mo}} y$. Then $|x_\alpha - x|\,u \xrightarrow{\text{o}} 0$ and $|y_\beta - y|\,u \xrightarrow{\text{o}} 0$ for every $u \in E_+$. Let us fix $u \in E_+$. So, there exist another two nets $(z_\gamma)_{\gamma\in\Gamma} \downarrow 0$ and $(z_\xi)_{\xi\in\Xi} \downarrow 0$ in E such that, for all $(\gamma,\xi) \in \Gamma \times \Xi$ there are $\alpha_\gamma \in A$ and $\beta_\xi \in B$ with $|x_\alpha - x|\,u \le z_\gamma$ and $|y_\beta - y|\,u \le z_\xi$ for all $\alpha \ge \alpha_\gamma$ and $\beta \ge \beta_\xi$.

Next, we show the *mo*-convergence of $(x_\alpha y_\beta)$ to xy. By considering the equality $|xy| = |x|\,|y|$ and Lemma 1.3, we have

$$\begin{aligned} |x_\alpha y_\beta - xy|\,u &= |x_\alpha y_\beta - x_\alpha y + x_\alpha y - xy|\,u \\ &\le |x_\alpha|\,|y_\beta - y|\,u + |x_\alpha - x|\,|y|\,u \\ &\le |x_\alpha - x|\,|y_\beta - y|\,u + |x|\,|y_\beta - y|\,u + |x_\alpha - x|\,|y|\,u. \end{aligned}$$

The second and the third terms in the last inequality both order converge to zero as $\beta \to \infty$ and $\alpha \to \infty$ respectively because of $|x|\,u, |y|\,u \in E_+$, $x_\alpha \xrightarrow{\text{mo}} x$ and $y_\beta \xrightarrow{\text{mo}} y$.

Now, let us show the convergence of the first term of last inequality. There are two positive elements $u_1, u_2 \in E_+$ such that $u = u_1 u_2$ because the positive element of E can be written as a multiplication of two positive elements. So, we get $|x_\alpha - x|\,|y_\beta - y|\,u = (|x_\alpha - x|\,u_1)(|y_\beta - y|\,u_2)$. Since $(z_\gamma)_{\gamma\in\Gamma} \downarrow 0$ and $(z_\xi)_{\xi\in\Xi} \downarrow 0$, the multiplication $(z_\gamma z_\xi) \downarrow 0$. Indeed, we firstly show that the multiplication is decreasing. For indexes $(\gamma_1,\xi_1)(\gamma_2,\xi_2) \in \Gamma \times \Xi$, we have $z_{\gamma_2} \le z_{\gamma_1}$ and $z_{\xi_2} \le z_{\xi_1}$ because both of them are decreasing. Since the nets are positive, it follows from $z_{\xi_2} \le z_{\xi_1}$ that $z_{\gamma_2} z_{\xi_2} \le z_{\gamma_2} z_{\xi_1} \le z_{\gamma_1} z_{\xi_1}$. As a result $(z_\gamma z_\xi)_{(\gamma,\xi)\in\Gamma\times\Xi} \downarrow$. Now, we show that the infimum of multiplication is zero. For a fixed index γ_0, we have $z_\gamma z_\xi \le z_{\gamma_0} z_\xi$ for $\gamma \ge \gamma_0$ because (z_γ) is decreasing. Thus, we get $\inf(z_\gamma z_\xi) = 0$ because of $\inf(z_{\gamma_0} z_\xi) = z_{\gamma_0} \inf(z_\xi) = 0$. Therefore, we see $(|x_\alpha - x|\,u_1)(|y_\beta - y|\,u_2) \xrightarrow{\text{o}} 0$. Hence, we get $x_\alpha y_\beta \xrightarrow{\text{mo}} xy$. □

The lattice operations in an f-algebra are *mo*-continuous in the following sense.

Proposition 2.4. *Let $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ be two nets in an f-algebra E. If $x_\alpha \xrightarrow{\text{mo}} x$ and $y_\beta \xrightarrow{\text{mo}} y$ then $(x_\alpha \vee y_\beta)_{(\alpha,\beta)\in A\times B} \xrightarrow{\text{mo}} x \vee y$. In particular, $x_\alpha \xrightarrow{\text{mo}} x$ implies $x_\alpha^+ \xrightarrow{\text{mo}} x^+$.*

Proof. Assume $x_\alpha \xrightarrow{\text{mo}} x$ and $y_\beta \xrightarrow{\text{mo}} y$. Then there exist two nets $(z_\gamma)_{\gamma\in\Gamma}$ and $(w_\lambda)_{\lambda\in\Lambda}$ in E satisfying $z_\gamma \downarrow 0$ and $w_\lambda \downarrow 0$, and for all $(\gamma,\lambda) \in \Gamma \times \Lambda$ there are $\alpha_\gamma \in A$ and $\beta_\lambda \in B$ such that $|x_\alpha - x|\,u \le z_\gamma$ and $|y_\beta - y|\,u \le w_\lambda$ for all $\alpha \ge \alpha_\gamma$ and $\beta \ge \beta_\lambda$ and for every $u \in E_+$. It follows from the inequality $|a \vee b - a \vee c| \le |b - c|$ in vector lattices that

$$\begin{aligned} |x_\alpha \vee y_\beta - x \vee y|\,u &\le |x_\alpha \vee y_\beta - x_\alpha \vee y|\,u + |x_\alpha \vee y - x \vee y|\,u \\ &\le |y_\beta - y|\,u + |x_\alpha - x|\,u \le w_\lambda + z_\gamma \end{aligned}$$

for all $\alpha \ge \alpha_\gamma$ and $\beta \ge \beta_\lambda$ and for every $u \in E_+$. Since $(w_\lambda + z_\gamma) \downarrow 0$, $|x_\alpha \vee y_\beta - x \vee y|\,u$ order converges to 0 for all $u \in E_+$. That is, $(x_\alpha \vee y_\beta)_{(\alpha,\beta)\in A\times B} \xrightarrow{\text{mo}} x \vee y$. □

Lemma 2.5. *Let (x_α) be a net in an f-algebra E. Then*

(i) $0 \le x_\alpha \xrightarrow{\text{mo}} x$ *implies* $x \in E_+$.

(ii) *if (x_α) is monotone and $x_\alpha \xrightarrow{\text{mo}} x$ then implies $x_\alpha \xrightarrow{\text{o}} x$.*

Proof. (i) Assume $0 \le x_\alpha \xrightarrow{\text{mo}} x$. Then we have $x_\alpha = x_\alpha^+ \xrightarrow{\text{mo}} x^+ = x$ by Proposition 2.4. Hence, we get $x \in E_+$.

(ii) We show that $x_\alpha \uparrow$ and $x_\alpha \xrightarrow{\text{mo}} x$ implies $x_\alpha \uparrow x$. Fix an index α. Then we have $x_\beta - x_\alpha \in X_+$ for $\beta \ge \alpha$. By (i), $x_\beta - x_\alpha \xrightarrow{\text{mo}} x - x_\alpha \in X_+$. Therefore, $x \ge x_\alpha$ for any α. Since α is arbitrary, then x is an upper bound of (x_α). Assume y is another upper bound of (x_α), i.e., $y \ge x_\alpha$ for all α. So, $y - x_\alpha \xrightarrow{\text{mo}} y - x \in X_+$, or $y \ge x$, and so $x_\alpha \uparrow x$. □

The following simple observation is useful in its own right.

Proposition 2.6. *Decreasing disjoint sequence in an f-algebra mo-converges to zero.*

Proof. Suppose (x_n) is a disjoint decreasing sequence in an f-algebra E. So, $|x_n|\,u$ is also a disjoint sequence in E for all $u \in E_+$; see Theorem 142.1(iii) [8]. Fix $u \in E_+$, by Corollary 3.6 [6], we have $|x_n|\,u \xrightarrow{\text{uo}} 0$ in E. So, $|x_n|\,u \wedge w \xrightarrow{\text{o}} 0$ in E for all $w \in E_+$. Thus, in particular, for fixed n_0, taking w as $|x_{n_0}|\,u$, Then, for all $n \geq n_0$, we get

$$|x_n|\,u = |x_n|\,u \wedge |x_{n_0}|\,u = |x_n|\,u \wedge w \xrightarrow{\text{o}} 0.$$

because of $|x_n|\,u \leq |x_{n_0}|\,u$. Therefore, $x_n \xrightarrow{\text{mo}} 0$ in E. □

For the next two facts, observe the following fact. Let E be a vector lattice, I be an order ideal of E and (x_α) be a net in I. If $x_\alpha \xrightarrow{\text{o}} x$ in I then $x_\alpha \xrightarrow{\text{o}} x$ in E. Conversely, if (x_α) is order bounded in I and $x_\alpha \xrightarrow{\text{o}} x$ in E then $x_\alpha \xrightarrow{\text{o}} x$ in I.

Proposition 2.7. *Let E be an f-algebra, B be a projection band of E and P_B be the corresponding band projection. If $x_\alpha \xrightarrow{\text{mo}} x$ in E then $P_B(x_\alpha) \xrightarrow{\text{mo}} P_B(x)$ in both E and B.*

Proof. It is known that P_B is a lattice homomorphism and $0 \leq P_B \leq I$. It follows from $|P_B(x_\alpha) - P_B(x)| = P_B|x_\alpha - x| \leq |x_\alpha - x|$ that $|P_B(x_\alpha) - P_B(x)|u \leq |x_\alpha - x|u$ for all $u \in E_+$. Then it follows easily that $P_B(x_\alpha) \xrightarrow{\text{mo}} P_B(x)$ in both X and B. □

Theorem 2.8. *Let E be an f-algebra and I be an order ideal and sub-f-algebra of E. For an order bounded net (x_α) in I, $x_\alpha \xrightarrow{\text{mo}} 0$ in I if and only if $x_\alpha \xrightarrow{\text{mo}} 0$ in E.*

Proof. Suppose $x_\alpha \xrightarrow{\text{mo}} 0$ in E. Then for any $u \in I_+$, we have $|x_\alpha|u \xrightarrow{\text{o}} 0$ in E. So, the preceding remark implies $|x_\alpha|u \xrightarrow{\text{o}} 0$ in I because $|x_\alpha|u$ is order bounded in I. Therefore, we get $x_\alpha \xrightarrow{\text{mo}} 0$ in I.

Conversely, assume that (x_α) *mo*-converges to zero in I. For any $u \in I_+$, we have $|x_\alpha|u \xrightarrow{\text{o}} 0$ in I, and so in E. Then, by applying Theorem 142.1(iv) [8], we have $x_\alpha w = 0$ for all $w \in I^d = \{x \in E : x \perp y \ for\ all\ y \in I\}$ and for each α because (x_α) in I. For any $u \in I_+$ and each $0 \leq w \in I^d$, it follows that

$$|x_\alpha|(u + w) = |x_\alpha|u + |x_\alpha|w = |x_\alpha|u \xrightarrow{\text{o}} 0$$

in E. So that, for each $z \in (I \oplus I^d)_+$, we get $|x_\alpha|z \xrightarrow{\text{o}} 0$ in E. It is known that $I \oplus I^d$ is order dense in E; see Theorem 1.36 [1]. Fix $v \in E_+$. Then there exists some $u \in (I \oplus I^d)$ such that $v \leq u$. Thus, we have $|x_\alpha|v \leq |x_\alpha|u \xrightarrow{\text{o}} 0$ in E. Therefore, $|x_\alpha|v \xrightarrow{\text{o}} 0$, and so $x_\alpha \xrightarrow{\text{mo}} 0$ in E. □

The following proposition extends Theorem 3.8 [2] to the general setting.

Theorem 2.9. *Let E be an f-algebra with a unit e and $(x_n) \downarrow$ be a sequence in E. Then $x_n \xrightarrow{\text{mo}} 0$ if and only if $|x_n|\,(u \wedge e) \xrightarrow{\text{o}} 0$ for all $u \in E_+$.*

Proof. For the forward implication, assume $x_n \xrightarrow{\text{mo}} 0$. Hence, $|x|\,u \xrightarrow{\text{o}} 0$ for all $u \in E_+$, and so $|x_n|\,(u \wedge e) \leq |x_n|\,u \xrightarrow{\text{o}} 0$ because of $e \in E_+$. Therefore, $|x_n|\,(u \wedge e) \xrightarrow{\text{o}} 0$.

For the reverse implication, fix $u \in E_+$. By applying Theorem 2.57 [1] and Theorem 142.1(i) [8], note that

$$|x_n|\,u \leq |x_n|\,(u - u \wedge ne) + |x_n|\,(u \wedge ne) \leq \frac{1}{n}u^2\,|x_n| + n\,|x_n|\,(u \wedge e)$$

Since $(x_n) \downarrow$ and E is Archimedean, we have $\frac{1}{n}u^2\,|x_n| \downarrow 0$. Furthermore, it follows from $|x_n|\,(u \wedge e) \xrightarrow{\text{o}} 0$ for each $u \in E_+$ that there exists another sequence $(y_m)_{m \in B}$ satisfying $y_m \downarrow 0$, and for any $m \in B$, there exists n_m such that $|x_n|\,(u \wedge e) \leq \frac{1}{n}y_m$, or $n\,|x_n|\,(u \wedge e) \leq y_m$ for all $n \geq n_m$. Hence, we get $n\,|x_n|\,(u \wedge e) \xrightarrow{\text{o}} 0$. Therefore, we have $|x_n|\,u \xrightarrow{\text{o}} 0$, and so $x_n \xrightarrow{\text{mo}} 0$. □

The *mo*-convergence passes obviously to any sub-f-algebra Y of E, i.e., for any net (y_α) in Y, $y_\alpha \xrightarrow{\text{mo}} 0$ in E implies $y_\alpha \xrightarrow{\text{mo}} 0$ in Y. For the converse, we give the following theorem.

Theorem 2.10. *Let Y be a sub-f-algebra of an f-algebra E and (y_α) be a net in Y. If $y_\alpha \xrightarrow{mo} 0$ in Y then it mo-converges to zero in E for each of the following cases;*

(i) *Y is majorizing in E;*
(ii) *Y is a projection band in E;*
(iii) *if, for each $u \in E$, there are element $x, y \in Y$ such that $|u - y| \leq |x|$.*

Proof. Assume (y_α) is a net in Y and $y_\alpha \xrightarrow{mo} 0$ in Y. Let us fix $u \in E_+$.

(i) Since Y is majorizing in E, there exists $v \in Y_+$ such that $u \leq v$. It follows from

$$0 \leq |y_\alpha|u \leq |y_\alpha|v \xrightarrow{o} 0,$$

that $|y_\alpha|u \xrightarrow{o} 0$ in E. That is, $y_\alpha \xrightarrow{mo} 0$ in E.

(ii) Since Y is a projection band in E, we have $Y = Y^{\perp\perp}$ and $E = Y \oplus Y^{\perp}$. Hence $u = u_1 + u_2$ with $u_1 \in Y_+$ and $u_2 \in Y_+^{\perp}$. Thus, we have $y_\alpha \wedge u_2 = 0$ because (y_α) in Y and $u_2 \in Y^{\perp}$. Hence, by applying Theorem 142.1(iii) [8], we see $y_\alpha u = 0$ for all index α. It follows from

$$|y_\alpha|\, u = |y_\alpha|\,(u_1 + u_2) = |y_\alpha|\, u_1 \xrightarrow{o} 0$$

that $|y_\alpha|\, u \xrightarrow{o} 0$ in E. Therefore, $y_\alpha \xrightarrow{mo} 0$ in E.

(iii) For the given $u \in E_+$, there exists elements $x, y \in Y$ with $|u - y| \leq |x|$. Then

$$|y_\alpha|\, u \leq |y_\alpha|\, |u - y| + |y_\alpha|\, |y| \leq |y_\alpha|\, |x| + |y_\alpha|\, |y|\, .$$

By *mo*-convergence of (y_α) in Y, we have $|y_\alpha|\, |x| \xrightarrow{o} 0$ and $|y_\alpha|\, |y| \xrightarrow{o} 0$, and so $|y_\alpha|\, u \xrightarrow{o} 0$. That means $y_\alpha \xrightarrow{mo} 0$ in E because u is arbitrary in E_+. □

We continue with some basic notions in f-algebra, which are motivated by their analogies from vector lattice theory.

Definition 2.11. Let $(x_\alpha)_{\alpha \in A}$ be a net in f-algebra E. Then

(i) (x_α) is said to be *mo-Cauchy* if the net $(x_\alpha - x_{\alpha'})_{(\alpha,\alpha') \in A \times A}$ *mo*-converges to 0,
(ii) E is called *mo-complete* if every *mo*-Cauchy net in E is *mo*-convergent,
(iii) E is called *mo-continuous* if $x_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{mo} 0$,
(iv) E is called a *mo-KB-space* if every order bounded increasing net in E_+ is *mo*-convergent.

Remark 2.12. An f-algebra E is *mo*-continuous if and only if $x_\alpha \downarrow 0$ in E implies $x_\alpha \xrightarrow{mo} 0$. Indeed, the implication is obvious. For the converse, consider a net $x_\alpha \xrightarrow{o} 0$. Then there exists a net $z_\beta \downarrow 0$ in X such that, for any β there exists α_β so that $|x_\alpha| \leq z_\beta$ for all $\alpha \geq \alpha_\beta$. Hence, by *mo*-continuity of E, we have $z_\beta \xrightarrow{mo} 0$, and so $x_\alpha \xrightarrow{mo} 0$.

Proposition 2.13. *Let (x_α) be a net in an f-algebra E. If $x_\alpha \xrightarrow{mo} x$ and (x_α) is an o-Cauchy net then $x_\alpha \xrightarrow{o} x$. Moreover, if $x_\alpha \xrightarrow{mo} x$ and (x_α) is uo-Cauchy then $x_\alpha \xrightarrow{uo} x$.*

Proof. Assume $x_\alpha \xrightarrow{mo} x$ and (x_α) is an order Cauchy net in E. Then $x_\alpha - x_\beta \xrightarrow{o} 0$ as $\alpha, \beta \to \infty$. Thus, there exists another net $z_\gamma \downarrow 0$ in E such that, for every γ, there exists α_γ satisfying

$$|x_\alpha - x_\beta| \leq z_\gamma$$

for all $\alpha, \beta \geq \alpha_\gamma$. By taking f-limit over β the above inequality and applying Proposition 2.4, i.e., $|x_\alpha - x_\beta| \xrightarrow{mo} |x_\alpha - x|$, we get $|x_\alpha - x| \leq z_\gamma$ for all $\alpha \geq \alpha_\gamma$. That means $x_\alpha \xrightarrow{o} x$. The similar argument can be applied for the *uo*-convergence case, and so the proof is omitted. □

In the case of *mo*-complete in f-algebras, we have conditions for *mo*-continuity.

Theorem 2.14. *For an mo-complete f-algebra E, the following statements are equivalent:*

(i) *E is mo-continuous;*

(ii) *if* $0 \leq x_\alpha \uparrow\leq x$ *holds in* E *then* x_α *is a mo-Cauchy net;*
(iii) $x_\alpha \downarrow 0$ *implies* $x_\alpha \xrightarrow{\text{mo}} 0$ *in* E.

Proof. $(i) \Rightarrow (ii)$ Consider the increasing and bounded net $0 \leq x_\alpha \uparrow\leq x$ in E. Then there exists a net (y_β) in E such that $(y_\beta - x_\alpha)_{\alpha,\beta} \downarrow 0$; see Lemma 12.8 [1]. Thus, by applying Remark 2.12, we have $(y_\beta - x_\alpha)_{\alpha,\beta} \xrightarrow{\text{mo}} 0$, and so the net (x_α) is *mo*-Cauchy because of $|x_\alpha - x_{\alpha'}|_{\alpha,\alpha' \in A} \leq |x_\alpha - y_\beta| + |y_\beta - x_{\alpha'}|$.

$(ii) \Rightarrow (iii)$ Suppose that $x_\alpha \downarrow 0$ in E, and fix arbitrary α_0. Then we have $x_\alpha \leq x_{\alpha_0}$ for all $\alpha \geq \alpha_0$. Thus we can get $0 \leq (x_{\alpha_0} - x_\alpha)_{\alpha \geq \alpha_0} \uparrow\leq x_{\alpha_0}$. So, it follows from (ii) that the net $(x_{\alpha_0} - x_\alpha)_{\alpha \geq \alpha_0}$ is *mo*-Cauchy, i.e., $(x_{\alpha'} - x_\alpha) \xrightarrow{\text{mo}} 0$ as $\alpha_0 \leq \alpha, \alpha' \to \infty$. Then there exists $x \in E$ satisfying $x_\alpha \xrightarrow{\text{mo}} x$ as $\alpha_0 \leq \alpha \to \infty$ because E is *mo*-complete. Since $x_\alpha \downarrow$ and $x_\alpha \xrightarrow{\text{mo}} 0$, it follows from Lemma 2.5 that $x_\alpha \downarrow 0$, and so we have $x = 0$. Therefore, we get $x_\alpha \xrightarrow{\text{mo}} 0$.

$(iii) \Rightarrow (i)$ It is just the implication of Remark 2.12. □

Corollary 2.15. *Let* E *be an mo-continuous and mo-complete* f*-algebra. Then* E *is order complete.*

Proof. Suppose $0 \leq x_\alpha \uparrow\leq u$ in E. We show the existence of supremum of (x_α). By considering Theorem 2.14 (ii), we see that (x_α) is an *mo*-Cauchy net. Hence, there is $x \in E$ such that $x_\alpha \xrightarrow{\text{mo}} x$ because E is *mo*-complete. It follows from Lemma 2.5 that $x_\alpha \uparrow x$ because of $x_\alpha \uparrow$ and $x_\alpha \xrightarrow{\text{mo}} x$. Therefore, E is order complete. □

Proposition 2.16. *Every mo-KB-space is mo-continuous.*

Proof. Assume $x_\alpha \downarrow 0$ in E. From Theorem 2.14, it is enough to show $x_\alpha \xrightarrow{\text{mo}} 0$. Let us fix an index α_0, and define another net $y_\alpha := x_{\alpha_0} - x_\alpha$ for $\alpha \geq \alpha_0$. Then it is clear that $0 \leq y_\alpha \uparrow\leq x_{\alpha_0}$, i.e., (y_α) is increasing and order bounded net in E. Since E is a *mo*-KB-space, there exists $y \in E$ such that $y_\alpha \xrightarrow{\text{mo}} y$. Thus, by Lemma 2.5, we have $y_\alpha \xrightarrow{\text{o}} y$. Hence, $y = \sup\limits_{\alpha \geq \alpha_0} y_\alpha = \sup\limits_{\alpha \geq \alpha_0} (x_{\alpha_0} - x_\alpha) = x_{\alpha_0}$ because of $x_\alpha \downarrow 0$. Therefore, we get $y_\alpha = x_{\alpha_0} - x_\alpha \xrightarrow{\text{mo}} x_{\alpha_0}$ or $x_\alpha \xrightarrow{\text{mo}} 0$ because of $y_\alpha \xrightarrow{\text{mo}} y$. □

Proposition 2.17. *Every mo-KB-space is order complete.*

Proof. Suppose $0 \leq x_\alpha \uparrow\leq z$ is an order bounded and increasing net in an *mo*-KB-space E for some $z \in E_+$. Then $x_\alpha \xrightarrow{\text{mo}} x$ for some $x \in E$ because E is *mo*-KB-space. By Lemma 2.5, we have $x_\alpha \uparrow x$ because of $x_\alpha \uparrow$ and $x_\alpha \xrightarrow{\text{mo}} x$. So, E is order complete. □

Proposition 2.18. *Let* Y *be an sub-*f*-algebra and order closed sublattice of an mo-KB-space* E*. Then* Y *is also a mo-KB-space.*

Proof. Let (y_α) be a net in Y such that $0 \leq y_\alpha \uparrow\leq y$ for some $y \in Y_+$. Since E is a *mo*-KB-space, there exists $x \in E_+$ such that $y_\alpha \xrightarrow{\text{mo}} x$. By Lemma 2.5, we have $y_\alpha \uparrow x$, and so $x \in Y$ because Y is order closed. Thus Y is a *mo*-KB-space. □

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