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Research Article

Two problems in the theory of disjointness preserving operators

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Abstract

In this short note, our aim is to solve two problems in the theory of disjointness preserving operators. Firstly, we obtain the converse direction of Hart's Theorem which was given in [D.R. Hart, Some properties of disjointness preserving operators, Mathematics Proceedings, 1985]. As a result, we get an affirmative solution of an open problem given by Y.A. Abramovich and A.K. Kitover in [Inverses of disjointness preserving operators, Mem. Amer. Math. Soc., 2000].

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1. Introduction

All vector spaces are considered over the reals only. Let G and H be Riesz spaces. An operator $S : G \to H$ is called *disjointness preserving* if $Sx \perp Sy$ for all $x, y \in G$ satisfying $x \perp y$ (i.e., $|x| \wedge |y| = 0$). A positive operator $S : G \to H$ is disjointness preserving if and only if S is a Riesz homomorphism. An operator $\pi : G \to G$ on a Riesz space is said to be *band preserving* whenever $\pi(B) \subseteq B$ holds for each band B of G. π is a band preserving operator if and only if $\pi(x) \perp y$ whenever $x \perp y$ in G. A band preserving and order bounded operator π is called *orthomorphism* of G and the set of all orthomorphisms of G is denoted by $Orth(G)$. Every orthomorphism is disjointness preserving and order continuous. The order ideal generated by the identity operator I in $Orth(G)$ is called the *ideal centre* of G and is denoted by $Z(G)$. Both of $Orth(G)$ and $Z(G)$ under composition are f-algebras, having the identity operator I as a unit.

We refer to [2, 4, 6, 8] for definitions and notations which are not explained here. All Riesz spaces in this paper are Archimedean.

The following theorem was proved by Hart in [3].

Theorem 1.1 ([3, Theorem 2.1]). *Let G, H be Riesz spaces and $S : G \to H$ be a surjective, order bounded, disjointness preserving operator. Then there exists a uniquely defined disjointness preserving operator $\Psi : Z(G) \to Z(H)$ satisfying $\Psi(\pi)S = S\pi$ for every $\pi \in Z(G)$.*

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The following two problems will be our main concern in this work. The first problem deals with the converse of Hart's Theorem. The second problem is an open problem given by Abramovich and Kitover in [1].

Problem 1.2. *Let G, H be Riesz spaces and $S : G \to H$ be an operator. If there exists a disjointness preserving operator $\Psi : Z(G) \to Z(H)$ such that $\Psi(\pi)S = S\pi$ for every $\pi \in Z(G)$, then S is disjointness preserving operator.*

Problem 1.3 ([1, Problem 8.2])**.** *Assume that both vector lattices G and H have a cofinal family of projection bands. Let $S : G \to H$ be a bijective operator such that the mapping $\pi \to S\pi S^{-1}$ (or resp. $\pi \to S^{-1}\pi S$) is an isomorphism from $Z(G)$ onto $Z(H)$ (or resp. from $Z(H)$ onto $Z(G)$). Does this imply that S is disjointness preserving?*

If the operator $\pi \to \Psi(\pi) = S\pi S^{-1}$ is an isomorphism from $Z(G)$ onto $Z(H)$, then Ψ is a disjointness preserving map as Ψ is an algebra homomorphism from the f-algebra $Z(G)$ to the f-algebra $Z(H)$ and Ψ provides $\Psi(\pi)S = S\pi$ for every $\pi \in Z(G)$. So, if we solve the first above mentioned problem, we get the second problem as a result. In general, the answer to the first problem is negative.

Example 1.4. *Let G be the Riesz space of piecewise affine, continuous functions on* $[0, 1]$. *Then, $Z(G)$ consists only of multiples of identity (i.e., $Z(G) = \{\lambda I : \lambda \in \mathbb{R}\}$)* [5]. *Let $H = G$ and take S to be an arbitrary operator that does not preserve disjointness. However, there exists a disjointness preserving operator $\Psi : Z(G) \to Z(H)$ such that $\Psi(\pi)S = S\pi$ for every $\pi \in Z(G)$.*

2. Main results

Now, we recall some definitions.

Definition 2.1.
- (i) A Riesz space G *has rich center* if for each $u, v \in G$ satisfying $|u| \leq |v|$ there exists a central operator $\pi \in Z(G)$ such that $|\pi| \leq I$ and $\pi v = u$.
- (ii) A Riesz space G is said to *have the principal projection property* if every principal band in G is a projection band.
- (iii) A Riesz space G has *a cofinal family of projection bands* if for each non-zero band B there is a non-zero projection band $D \subseteq B$.

Every σ-Dedekind complete Riesz space satisfies the definitions (i), (ii) and (iii). Any $C(K)$ space of continuous functions on a zero-dimensional compact space K is a typical example of a Riesz space with a cofinal family of projection bands. If G satisfies the principal projection property, then the affirmative solution of the Problem 1.2 was given in [7].

Proposition 2.2 ([7, Proposition 3.7])**.** *Let G, H be vector lattices, G having the principal projection property. Let $S : G \to H$ be a linear operator. If there exists a disjointness preserving map Ψ from $Z(G)$ to $Z(H)$ such that $\Psi(\pi)S = S\pi$ for every $\pi \in Z(G)$, then S is a disjointness preserving operator.*

If G has rich center, then the affirmative solution of the Problem 1.2 is given in the next proposition.

Proposition 2.3. *Let G, H be vector lattices and G has rich center. Let $S : G \to H$ be a linear operator. If there exists a disjointness preserving map Ψ from $Z(G)$ to $Z(H)$ such that $\Psi(\pi)S = S\pi$ for every $\pi \in Z(G)$, then S is a disjointness preserving operator. In addition, if S is a bijective operator then S^{-1} is a disjointness preserving operator.*

Proof. Let $u \perp v$ in G. We may assume without loss of generality that $u \geq 0$ and $v \geq 0$. Since the center $Z(G)$ is rich there exist π_1, $\pi_2 \in Z(G)_+$ such that $\pi_1(u + v) = u$ and $\pi_2(u + v) = v$. If we take $\gamma_1 = \pi_1 - \pi_1 \wedge \pi_2$ and $\gamma_2 = \pi_2 - \pi_1 \wedge \pi_2$, then $\gamma_1 \wedge \gamma_2 = 0$. Since

$$\pi_1 \wedge \pi_2(u + v) \leq \pi_1(u + v) = u, \pi_1 \wedge \pi_2(u + v) \leq \pi_2(u + v) = v$$

we have $\pi_1 \wedge \pi_2(u+v) = 0$. This yields $\gamma_1(u+v) = u$ and $\gamma_2(u+v) = v$. Now we obtain

$$\begin{aligned}
0 &\leq |Su| \wedge |Sv| \\
&= |S\gamma_1(u+v)| \wedge |S\gamma_2(u+v)| \\
&= |\Psi(\gamma_1)S(u+v)| \wedge |\Psi(\gamma_2)S(u+v)| \\
&= |\Psi(\gamma_1)|\,|S(u+v)| \wedge |\Psi(\gamma_2)|\,|S(u+v)| \\
&= (|\Psi(\gamma_1)| \wedge |\Psi(\gamma_2)|)\,|S(u+v)| \\
&= 0
\end{aligned}$$

which yields $Su \perp Sv$ and S is disjointness preserving. If S is a bijective operator, the proof is completed by Proposition 3.8 in [7]. □

Theorem 2.4. *Let $S : G \to H$ be a bijective operator between vector lattices G and H.*

(1) *If G has a cofinal family of projection bands and there exists a disjointness preserving map Ψ from $Z(G)$ to $Z(H)$ such that $\Psi(\pi)S = S\pi$ for every $\pi \in Z(G)$, then S is a disjointness preserving operator.*
(2) *If H has a cofinal family of projection bands and there exists a disjointness preserving map Ψ from $Z(H)$ to $Z(G)$ such that $S\Psi(\pi) = \pi S$ for every $\pi \in Z(H)$, then S^{-1} is a disjointness preserving operator.*
(3) *If G and H have a cofinal family of projection bands and there exists a surjective disjointness preserving map Ψ from $Z(G)$ to $Z(H)$ such that $\Psi(\pi)S = S\pi$ for every $\pi \in Z(G)$, then S and S^{-1} are disjointness preserving operators.*

Proof. (1) Let us first show that $S^{-1}(B_h) \subseteq B_{S^{-1}h}$ for each $h \in H$ (where B_h is the band generated by h). Assume, contrary to what we claim, that there exists $t \in B_h$ such that $S^{-1}t \notin B_{S^{-1}h}$. Thus, we can find $u \in B^d_{S^{-1}h}$ such that $|S^{-1}t| \wedge |u| \neq 0$. Let $z = |S^{-1}t| \wedge |u|$. From the hypothesis there exists a non-zero projection band $B \subseteq B_z$ and the projection $P : G \to B$ yields $PS^{-1}t \neq 0$ and $PS^{-1}h = 0$. Using the hypothesis, we have $S^{-1}\Psi(P)h = PS^{-1}h = 0$ where Ψ is the operator from $Z(G)$ to $Z(H)$ defined by $\Psi(\pi) = S\pi S^{-1}$ for each $\pi \in Z(G)$. Since S^{-1} is one to one we have $\Psi(P)h = 0$. As $\Psi(P) \in Z(H)$, it can be easly shown that $\Psi(P) = 0$ on B_h. Thus, we obtain $\Psi(P)t = 0$. This implies that $PS^{-1}t = S^{-1}\Psi(P)t = 0$, a contradiction. Now we will show that S is a disjointness preserving operator. Take arbitrary $u, v \in G$ with $u \perp v$. There exist $h, t \in H$ such that $S^{-1}h = u$ and $S^{-1}t = v$. By the first part of the proof, we have

$$S^{-1}(B_h) = S^{-1}(B_{Su}) \subseteq B_{S^{-1}h} = B_u \text{ and } S^{-1}(B_t) = S^{-1}(B_{Sv}) \subseteq B_{S^{-1}t} = B_v$$

which yields $B_{Su} \subseteq S(B_u)$ and $B_{Sv} \subseteq S(B_v)$. Since S is one to one we obtain

$$B_{Su} \cap B_{Sv} \subseteq S(B_u) \cap S(B_v) = S(B_u \cap B_v) = \{0\},$$

which means $Su \perp Sv$. (2) The proof is same as (1). (3) It is clear that Ψ is one to one operator from $Z(G)$ to $Z(H)$. Hence, Ψ^{-1} is the operator from $Z(H)$ to $Z(G)$ and $\Psi^{-1}(\pi) = S^{-1}\pi S$ for each $\pi \in Z(H)$. From (1) and (2), S and S^{-1} are disjointness preserving operators. □

Affirmative solution of the Problem 1.3 with the weaker hypothesis is given in the next corollary.

Corollary 2.5. *Let $S : G \to H$ be a bijective operator between vector lattices G and H. If G has a cofinal family of projection bands and for each $\pi \in Z(G)$ the operator $S\pi S^{-1}$ is an element of $Z(H)$, then S is a disjointness preserving operator.*

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