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HARMONIC CURVATURE OF A STRIP IN E^3

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Abstract. In this work we give a new definition of a helix strip. We study the harmonic curvatures functions of a strip by using harmonic curvature functions and give some characterizations of the strip's harmonic curvature functions and total curvature functions of a strip.

1. Introduction

In 3-dimensional Euclidean Space, a regular curve is described by its curvatures k_1 and k_2 and also a strip is descibed by its curvatures k_n, k_g and t_r. The relations between the curvatures of a strip and the curvatures of the curve can be seen in many differential books and papers. We know that a regular curve is called a general helix if its first and second curvatures k_1 and k_2 are not constant, but $\frac{k_1}{k_2}$ is constant ([2], [7]) and harmonic curvature functions of a curve was given by H. Hilmi Hacısalihoğlu. The ratio $\frac{k_1}{k_2}$ is called first Harmonic curvature of the curve and is denoted by H_1 or H in E^3 ([2]). Thus the ratios $\frac{t_r}{k_n}$ and $\frac{t_r}{k_g}$ the harmonic curvature functions of the strip is denoted $\overline{H}$ and $\overline{H}'$ in E^3.

2. Preliminaries

2.1. The Theory of the Curves.

Definition 2.1. If $\alpha : I \subset \mathbb{R} \to E^n$ is a smooth transformation, then α is called a curve (from the class of C^∞).Here I is an open interval of $\mathbb{R}$ ([8]).

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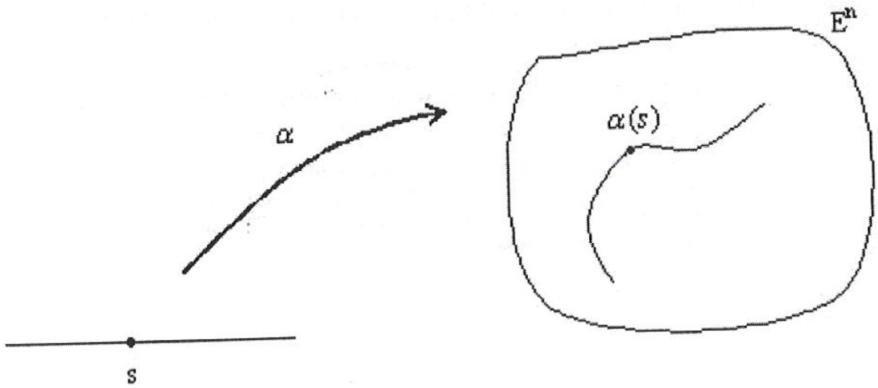

Figure 1 The curve in E^n

Definition 2.2. Let the curve $\alpha \subset E^n$ be a regular curve coordinate neigbourhood and $\{V_1(s), V_2(s), ..., V_r(s)\}$be the Frenet frame at the point $\alpha(s)$ that correspond for every $s \in I$. Accordingly,

$$\begin{aligned} k_i : \quad & I \to R \\ & s \to k_i(s) = \langle V_i'(s), V_{i+1}(s) \rangle . \end{aligned}$$

We know that the function k_i is called $i-th$ curvature function of the curve and the real number $k_i(s)$ is called $i-th$ curvature of the curve for each $s \in I$ ([2]). The relation between the derivatives of the Frenet vectors among α and the curvatures is given with a theorem as follows:

Definition 2.3. Let $M \subset E^n$ be the curve which given by the neigbouring with (I, α). Let $s \in I$ be arc parameter. If $k_i(s)$ and $\{V_1(s), V_2(s), ..., V_r(s)\}$ is the $i-th$ curvature and the Frenet r-frame at the point $\alpha(s)$, then

$$\begin{aligned} \textbf{i. } V_1'(s) &= k_1(s)V_2(s) \\ \textbf{ii. } V_i'(s) &= -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s), \ldots 1 \langle i \langle r, \\ \textbf{iii. } V_r'(s) &= -k_{r-1}(s)V_{r-1}(s) \end{aligned}$$

([2]).

The equations that about the covariant derivatives of the Frenet r-frame $\{V_1(s), V_2(s), ..., V_r(s)\}$ the Frenet vectors $V_i(s)$ along the curve can be written as

$$\begin{bmatrix} V_1'(s) \\ V_2'(s) \\ V_3'(s) \\ \vdots \\ V_{r-2}'(s) \\ V_{r-1}'(s) \\ V_r'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & k_{r-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -k_{r-2} & 0 & k_{r-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_{r-1} & 0 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ \vdots \\ V_{r-2}(s) \\ V_{r-1}(s) \\ V_r(s) \end{bmatrix}$$

These formulas are called Frenet Formulas ([2]).

In special case if we take $n = 3$ above the last matrix equations, we obtain following matrix the equation

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \text{ or } \begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}.$$

2.2. **The Strip Theory.**

Definition 2.4. Let M be a surface in $E^3$and α is a curve in $M \subset E^3$. We define a surface element of M is the part of a tangent plane at the neighbour of the point. The locus of these surface element along the curve α is called a strip or curve-surface pair and is showed by (α, M).

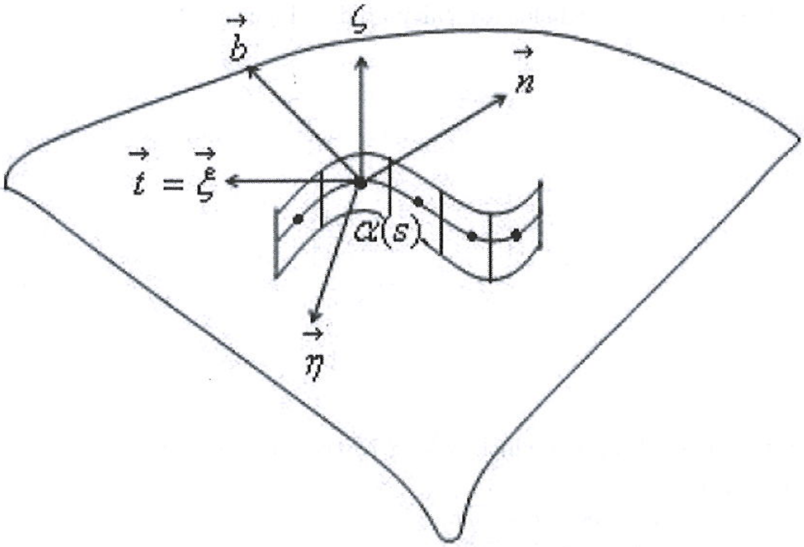

Figure 2 A Strip in E^3 (Hacısaliholu1982)

2.3. Vector Fields of a Strip in E^3.

Definition 2.5. We know the Frenet vectors fields of a curve α in $M \subset E^3$ are $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$. $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$is called Frenet Frame or Frenet Trehold. Also Frenet vectors of the curve is showed as $\left\{\overrightarrow{V_1}, \overrightarrow{V_2}, \overrightarrow{V_3}\right\}$. In here $\overrightarrow{V_1} = \overrightarrow{t}$, $\overrightarrow{V_2} = \overrightarrow{n}$, $\overrightarrow{V_3} = \overrightarrow{b}$. Let $\overrightarrow{t}$ be the tangent vector field of the curve α, $\overrightarrow{n}$ be the normal vector field of the curve α and $\overrightarrow{b}$ be the binormal vector field of the curve α.

$$\begin{aligned} \alpha : \quad I \subset M &\rightarrow E^3 \\ s &\rightarrow \alpha(s). \end{aligned}$$

$\alpha : I \rightarrow E^3$ is a curve in E^3 with $\|\alpha\'(s)\| = 1$, then α is called unit velocity. Let $s \in I$ be the arc length parameter of α. In E^3 for a curve α with unit velocity, $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$ Frenet vector fields are calculated as follows ([2]) :

$$\begin{aligned} \overrightarrow{t} &= \alpha^{'}(s), \\ \overrightarrow{n} &= \frac{\alpha^{''}(s)}{\|\alpha''(s)\|}, \\ \overrightarrow{b} &= \overrightarrow{t} \times \overrightarrow{n}. \end{aligned}$$

Strip vector fields of a strip which belong to the curve α are $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$. These vector fields are;
Strip tangent vector field is $\overrightarrow{t} = \overrightarrow{\xi}$
Strip normal vector field is $\overrightarrow{\zeta} = \overrightarrow{N}$
Strip binormal vector field is $\overrightarrow{\eta} = \overrightarrow{\zeta} \Lambda \overrightarrow{\xi}$ ([6]).

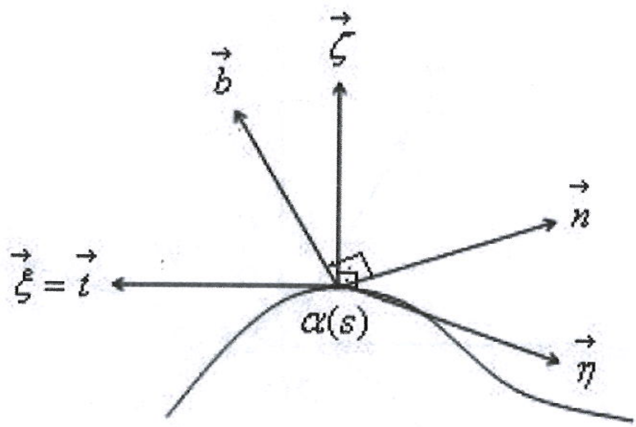

Figure 3 Strip and curve vector fields in E^3

Let α be a curve in $M \subset E^3$. If $\alpha'(s) = \overrightarrow{t}$ $(\overrightarrow{t} = \overrightarrow{\xi})$ and $\overrightarrow{\zeta}$ is a unit strip vector field of a surface M at the point $\alpha(s)$, than we have $\overrightarrow{\eta}\mid_{\alpha(s)} = \overrightarrow{\zeta}\mid_{\alpha(s)} \Lambda \overrightarrow{\xi}\mid_{\alpha(s)}$ ([6]). That is $\overrightarrow{\eta}\mid_{\alpha(s)}$is perpendicular $\overrightarrow{\zeta}\mid_{\alpha(s)}$and also $\overrightarrow{\xi}\mid_{\alpha(s)}$ .So we obtain $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$ orthonormal vector fields system is called strip three-bundle ([6]).

2.4. **Curvatures of a Strip.** Let $k_n = -b$, $k_g = c$, $t_r = a$ be the normal curvature, the geodesic curvature, the geodesic torsion of the strip ([6]).

Let $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$ be the strip vector fields on α. Then we have

$$\begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$

or,

$$\begin{aligned} \xi' &= c\eta - b\zeta \\ \eta' &= -c\xi + a\zeta \\ \zeta' &= b\xi - a\eta \end{aligned} \; .$$

2.5. **Some Relations between Frenet Vector Fields of a Curve and Strip Vector Fields of a Strip.** Let $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$, $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$ and φ be the unit strip vector fields, the unit Frenet vector fields and the angle between $\overrightarrow{\eta}$ and $\overrightarrow{n}$ on α.

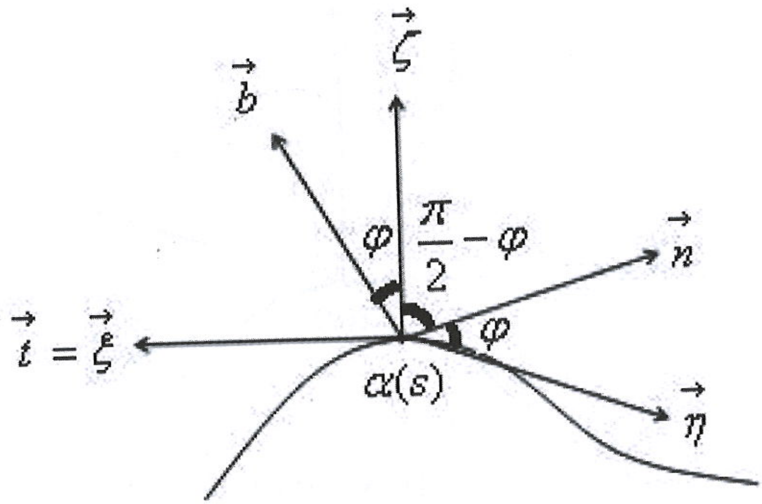

Figure 4 Strip and curve vector fields and the angle φ between $\vec{\eta}$ and $\vec{n}$ in E^3

We can see that $\vec{\eta}, \vec{\zeta}, \vec{n}, \vec{b}$ vectors are in the same surface from the Figure 4. then we obtain the following equations

$$\begin{aligned} \left\langle \vec{t}, \vec{\zeta} \right\rangle &= 0 \\ \left\langle \vec{t}, \vec{n} \right\rangle &= 0 \\ \left\langle \vec{t}, \vec{b} \right\rangle &= 0 \\ \left\langle \vec{t}, \vec{\eta} \right\rangle &= 0. \end{aligned}$$

2.5.1. *The Equations of the Strip Vector Fields in type of the Frenet vector Fields:*

Let $\left\{ \vec{t}, \vec{n}, \vec{b} \right\}$, $\left\{ \vec{\xi}, \vec{\eta}, \vec{\zeta} \right\}$ and φ be the Frenet Vector fields, strip vector fields and the angle between $\vec{\eta}$ and $\vec{n}$. We can write the following equations by the Figure 4

$$\begin{aligned} \vec{\xi} &= \vec{t} \\ \vec{\eta} &= \cos\varphi \; \vec{n} - \sin\varphi \; \vec{b} \\ \vec{\zeta} &= \sin\varphi \; \vec{n} + \cos\varphi \; \vec{b} \end{aligned}$$

or in matrix form

$$\begin{bmatrix} \overrightarrow{\xi} \\ \overrightarrow{\eta} \\ \overrightarrow{\zeta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix}.$$

2.5.2. *The Equations of the Frenet Vector Fields in type of the Strip Vector Fields:*

By the help of the Figure 4 we can write

$$\begin{aligned} \overrightarrow{t} &= \overrightarrow{\xi} \\ \overrightarrow{n} &= \cos\varphi\ \overrightarrow{\eta} + \sin\varphi\ \overrightarrow{\zeta} \\ \overrightarrow{b} &= -\sin\varphi\ \overrightarrow{\eta} + \cos\varphi\ \overrightarrow{\zeta} \end{aligned}$$

or in matrix form

$$\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \overrightarrow{\xi} \\ \overrightarrow{\eta} \\ \overrightarrow{\zeta} \end{bmatrix}.$$

2.5.3. *Some Relations between a, b, c invariants (Curvatures of a Strip) and κ, τ invariants (Curvatures of a Curve):*

We know that a curve α has two curvatures κ and τ. A curve has a strip and a strip has three curvatures k_n, k_g and t_r ([4], [6], [7]).From the derivative equations we can write

$$\xi' = c\eta - b\zeta.$$

If we substitude $\overrightarrow{\xi} = \overrightarrow{t}$ in last equation, we obtain

$$\xi' = \kappa n$$

and

$$\begin{aligned} b &= -\kappa \sin\varphi \\ c &= \kappa \cos\varphi \end{aligned}$$

([4], [6], [7]). From last two equations we obtain

$$\kappa^2 = b^2 + c^2.$$

This equation is a relation between the curvature κ of a curve α and normal curvature and geodesic curvature of a strip ([4], [6], [7]).

By using similar operations, we obtain a new equation as follows

$$\tau = -a + \frac{b'c - bc'}{b^2 + c^2}$$

([4], [6]). This equation is a relation between τ (torsion or second curvature of α) and a, b, c curvatures of a strip that belongs to the curve α.

And also we can write

$$a = \varphi' + \tau.$$

The special case: If φ =constant, then $\varphi' = 0$. So the equation is $a = \tau$. That is, if the angle is constant, then torsion of the strip is equal to torsion of the curve.

Definition 2.6. Let α be a curve in $M \subset E^3$. If the geodesic curvature (torsion) of the curve α is equal to zero, then the curve-surface pair (α, M) is called a curvature strip ([6]).

3. Harmonic Curvatures of a Strip in E^3

Definition 3.1. Let α be a curve in E^3 and V_1 be the first Frenet vector field of $\alpha.U \in \chi(E^3)$ be a constant unit vector field. If

$$\langle V_1, U \rangle = \cos\varphi \quad \text{(constant)}$$

α, φ and Sp$\{U\}$ is called an general helix, the slope angle and the slope axis ([2]).

Definition 3.2. A regular curve is called general helix if its first and second curvatures κ, τ are not constant but $\frac{\varkappa}{\tau}$ is constant ([2]).

Definition 3.3. A curve is called a general helix or cylindrical helix if its tangent makes a constant angle with a fixed line in space. A curve is a general helix if and only if the ratio $\frac{\varkappa}{\tau}$ is constant ([2]).

Definition 3.4. If the curve α is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be constant. The ratio $\frac{\tau}{\kappa}$ is called first Harmonic curvature of the curve and is denoted by H_1 or H ([2]).

Theorem 3.5. *A regular curve $\alpha \subset E^3$ is a general helix if and only if $H(s) = \frac{k_1}{k_2}$ =constant for $\forall s \in I$ ([2]).*

Proof. ($\Rightarrow$) Let α be a general helix. The slope axis of the curve α is showed as $Sp\{U\}$. Note that

$$\left\langle \alpha'(s), U \right\rangle = \cos\varphi = \text{constant}.$$

If the Frenet trehold is $\{V_1(s), V_2(s), V_3(s)\}$ at the point $\alpha(s)$, then we have

$$\langle V_1(s), U \rangle = \cos\varphi.$$

If we take derivative of the both sides of the last equation, then we have

$$\langle k_1(s)V_2(s), U \rangle = 0 \Rightarrow \langle V_2(s), U \rangle = 0.$$

Hence

$$U \in Sp\{V_1(s), V_3(s)\}.$$

Therefore

$$U = \cos\varphi\ V_1(s) + \sin\varphi\ V_3(s).$$

U is the linear combination of $V_1(s)$ and $V_3(s)$. By differentiating the equation $\langle V_2(s), U\rangle = 0$, we obtain

$$\begin{aligned} \langle -k_1(s)V_1(s) + k_2(s)V_3(s), U\rangle &= 0 \\ -k_1(s)\langle V_1(s), U\rangle + k_2(s)\langle V_3(s), U\rangle &= 0 \\ -k_1(s)\cos\varphi + k_2(s)\sin\varphi &= 0. \end{aligned}$$

By using the last equation, we see that

$$H = \text{constant}.$$

($\Leftarrow$) Let $H(s)$ be constant for $\forall s \in I$, and $\lambda = \tan\varphi$, then we obtain

$$U = \cos\varphi\ V_1(s) + \sin\varphi\ V_3(s).$$

1) If U is a constant vector,then we have

$$D_{\overset{\bullet}{\alpha}}U = (k_1(s)\cos\varphi - \sin\varphi\ k_2(s))V_2(s).$$

By substituting $H(s) = \tan\varphi$ is in the last equation, we see that

$$k_1(s)\cos\varphi - k_2(s)\sin\varphi = 0,$$

and so

$$U = \text{constant}.$$

2) If α is an inclined curve with slope axis $Sp\{U\}$. Since

$$\begin{aligned} \left\langle \alpha'(s), U\right\rangle &= \langle V_1(s), \cos\varphi\ V_1(s) + \sin\varphi\ V_3(s)\rangle \\ &= \cos\varphi\ \langle V_1(s), V_1(s)\rangle + \sin\varphi\ \langle V_1(s), V_3(s)\rangle \end{aligned}$$

we obtain

$$\left\langle \alpha'(s), U\right\rangle = \cos\varphi = \text{constant ([2])}.$$

□

Definition 3.6. Let $\bar{H}$ and $\bar{H}'$be the harmonic curvature of a strip in E^3.Then the harmonic curvature of the strip is the ratio of the torsion to the normal curvature of the strip and also the ratio of the torsion of the strip to the geodesic curvature of the strip, respectively.

3.1. **In First case:** If we take first curvature of a strip $k_n = -b$, then we obtain harmonic curvature of a strip is as follows;

$$\bar{H} = \frac{t_r}{k_n} = \frac{a}{-b}. \tag{3.1}$$

3.1.1. ***Relations between Harmonic Curvature of a Strip and Harmonic Curvature of a Curve when we take first curvature of the strip is*** $k_n = -b$ ***and torsion of the strip is*** $t_r = a$:

If we write the equations of k_n, t_r and H into the harmonic curvature of the strip, then we obtain the equation

$$\begin{aligned} \bar{H} &= \frac{t_r}{k_n} = \frac{a}{-b} = \frac{\varphi' + \tau}{\kappa \sin\varphi} \\ \bar{H} &= \frac{\varphi'}{\kappa \sin\varphi} + \frac{\tau}{\kappa \sin\varphi} \\ \bar{H} &= \frac{\varphi'}{\kappa \sin\varphi} + \frac{\tau}{\kappa}\frac{1}{\sin\varphi} \\ \bar{H} &= \frac{\varphi'}{\kappa \sin\varphi} + H\frac{1}{\sin\varphi} \\ \bar{H} &= \frac{\varphi'}{\kappa \sin\varphi} + H\csc\varphi \end{aligned} \tag{3.2}$$

The last equation is the relation between harmonic curvature of the strip and harmonic curvature of a curve in first case (when we take first curvature of a strip $k_n = -b$ and torsion of the strip is $t_r = a$).

3.1.2. ***Special case:***

Theorem 3.7. *Let $\bar{H}$ be the harmonic curvature of the strip. If the angle φ between normal vector field of the surface and binormal vector field of the curve is constant, $k_n = -b$ and $t_r = a$ is the first curvature of the strip and the torsion of the strip in E^3, then we give the relation between harmonic curvatures of the strip and the curve,*

$$\begin{aligned} \bar{H} &= \frac{\varphi'}{\kappa \sin\varphi} + H\csc\varphi \\ \bar{H} &= \frac{0}{\kappa \sin\varphi} + H\csc\varphi \\ \bar{H} &= H\csc\varphi. \end{aligned} \tag{3.3}$$

3.2. **In Second case:** Let $\bar{H}$be the harmonic curvature of the strip. If we take first curvature of the strip is $k_g = c$ and torsion of the strip is $t_r = a$, then we obtain

$$\begin{aligned}\bar{H}' &= \frac{t_r}{k_g} = \frac{a}{c} \qquad (3.4)\\ \bar{H}' &= \frac{\varphi'}{\kappa\cos\varphi} + \frac{\tau}{\kappa\cos\varphi}\\ \bar{H}' &= \frac{\varphi'}{\kappa\cos\varphi} + \frac{\tau}{\kappa}\frac{1}{\cos\varphi}\\ \bar{H}' &= \frac{\varphi'}{\kappa\cos\varphi} + H\frac{1}{\cos\varphi}\\ \bar{H}' &= \frac{\varphi'}{\kappa\cos\varphi} + H\sec\varphi.\end{aligned}$$

3.2.1. *Special case:*

Theorem 3.8. *Let $k_g = c$, $t_r = a$ and the constant angle φ be the first curvature of the strip, the torsion of the strip and the angle between normal vector field of the surface and binormal vector field of the curve. If the angle φ is constant, then $\varphi' = 0$. thus we take*

$$\begin{aligned}\bar{H}' &= \frac{\varphi'}{\kappa\cos\varphi} + H\sec\varphi \qquad (3.5)\\ \bar{H}' &= \frac{0}{\kappa\cos\varphi} + H\sec\varphi\\ \bar{H}' &= H\sec\varphi.\end{aligned}$$

Theorem 3.9. *Let the angle φ between normal vector field of the surface and binormal vector field of the curve be constant and (α, M) be strip in E^3. When the strip $(\alpha, M)'s$ curvatures a, b, c are not constant but hormonic curvature of strip is constant, the strip is called inclined strip $\Longleftrightarrow \bar{H}^2 + \bar{H}'^2 = cst.$*

Proof.

($\Rightarrow$) Let $\bar{H}$ and $\bar{H}'$ be the harmonic curvatures of (α, M). We should show that $\bar{H}^2 + \bar{H}'^2$ must be constant.

We know that harmonic curvatures of the strip $\overline{H}=\frac{t_r}{k_n}=\frac{a}{-b}$ ve $\overline{H}'=\frac{t_r}{k_g}=\frac{a}{c}$. If we use these equations, we take

$$\begin{aligned}\overline{H}^2+\overline{H}'^2 &= \left(\frac{a}{-b}\right)^2+\left(\frac{a}{c}\right)^2\\ &= \left(\frac{\varphi'+\tau}{\kappa\sin\varphi}\right)^2+\left(\frac{\varphi'+\tau}{\kappa\cos\varphi}\right)^2.\end{aligned}$$

In here, φ is constant. So,

$$\varphi'=0.$$

Since φ' is equal to zero, we obtain

$$\begin{aligned}\overline{H}^2+\overline{H}'^2 &= \left(\frac{\tau}{\kappa\sin\varphi}\right)^2+\left(\frac{\tau}{\kappa\cos\varphi}\right)^2\\ &= \frac{\tau^2}{\kappa^2\sin^2\varphi}+\frac{\tau^2}{\kappa^2\cos^2\varphi}\\ &= \frac{\tau^2\cos^2\varphi+\tau^2\sin^2\varphi}{\kappa^2\sin^2\varphi\cos^2\varphi}\\ &= \frac{\tau^2(\cos^2\varphi+\sin^2\varphi)}{\kappa^2\sin^2\varphi\cos^2\varphi}\\ &= \left(\frac{\tau}{\kappa}\right)^2\frac{1}{(\sin\varphi\cos\varphi)^2}.\end{aligned}$$

We know $\frac{\tau}{\kappa}$ and φ is constant. So $\overline{H}^2+\overline{H}'^2$ must be constant.

$\Leftarrow$Let $\overline{H}^2+\overline{H}'^2$ be constant. In this case is (α,M) a helix strip? If we observe $\overline{H}^2+\overline{H}'^2$, then we take

$$\begin{aligned}\overline{H}^2+\overline{H}'^2 &= \left(\frac{\tau}{\kappa\sin\varphi}\right)^2+\left(\frac{\tau}{\kappa\cos\varphi}\right)^2\\ &= \frac{\tau^2}{\kappa^2\sin^2\varphi}+\frac{\tau^2}{\kappa^2\cos^2\varphi}\\ &= \frac{\tau^2\cos^2\varphi+\tau^2\sin^2\varphi}{\kappa^2\sin^2\varphi\cos^2\varphi}\\ &= \frac{\tau^2(\cos^2\varphi+\sin^2\varphi)}{\kappa^2\sin^2\varphi\cos^2\varphi}\\ &= \left(\frac{\tau}{\kappa}\right)^2\frac{1}{(\sin\varphi\cos\varphi)^2}.\end{aligned}$$

Since $\bar{H}^2 + \bar{H}'^2$ is constant, $\left(\frac{\tau}{\kappa}\right)^2 \frac{1}{(\sin\varphi\cos\varphi)^2}$ must be constant. We know φ is constant. So $\frac{\tau}{\kappa}$ must be constant. this means that the curve α is helix and the strip (α, M) is a helix strip.

□

Theorem 3.10. *Let (α, M), and $\bar{H}$, $\bar{H}'$ be a curve-surface pair (strip) and harmonic curvatures of (α, M) in E^3. Let the angle φ between normal vector field of the surface and binormal vector field of the curve is constant. The strip (α, M) is inclined strip$\Leftrightarrow \frac{\bar{H}'}{\bar{H}} = \tan\varphi =$constant.*

Proof. $(\Rightarrow)$ We know $\bar{H}' = \frac{H}{\cos\varphi}$ and $\bar{H} = \frac{H}{\sin\varphi}$ by the Theorems 2.2 and 2.3. Thus

$$\begin{aligned}\frac{\bar{H}'}{\bar{H}} &= \frac{\frac{H}{\cos\varphi}}{\frac{H}{\sin\varphi}} \\ &= \tan\varphi.\end{aligned}$$

Since φ is constant, $\frac{\bar{H}'}{\bar{H}}$ is constant.

$(\Leftarrow)$Let $\frac{\bar{H}'}{\bar{H}}$ be constant. Thus $\tan\varphi$ must be constant.So φ is constant. Since φ is constant, (α, M) is inclined strip. □

4. Total Curvature of the Curve in type Curvatures of the Strip in E^3

Definition 4.1. Let k_1 and k_2 be the first and second curvature of the curve α (Hacısalihoğlu 2000) and $-b, c, a$ are the normal curvature, geodesic curvature and the geodesic torsion of the strip (α, M) respectively ([4], [6]). We know that the total curvature of the curve is $\sqrt{k_1^2 + k_2^2}$ ([5]).

$$\text{Total curvature of the curve} = \sqrt{k_1^2 + k_2^2}. \tag{4.1}$$

We may write $k_1^2 = b^2 + c^2$ and $k_2^2 = (-a + \frac{b'c - bc'}{b^2+c^2})^2$,so we find the equation as follows:

$$\text{Total curvature of the curve} = \sqrt{b^2 + c^2 + (-a + \frac{b'c - bc'}{b^2 + c^2})^2}. \tag{4.2}$$

The last equation is the total curvature of the curve in type strip's curvatures.

If we calculate $b'c - bc'$, we obtain $b'c - bc' = -k_1^2\varphi'$.From this equation we can write $\frac{b'c-bc'}{b^2+c^2} = -\varphi'$. So

$$\begin{aligned}\text{Total curvature of the curve} = &\quad \sqrt{b^2+c^2+(-a+-\varphi')^2} \qquad (4.3)\\ = &\quad \sqrt{b^2+c^2+(a+\varphi')^2}.\end{aligned}$$

4.1. Special case:

Theorem 4.2. *If the angle φ is constant, then $\varphi' = 0$. Thus the total curvature of the curve is obtained as follows:*

$$\begin{aligned}\textit{Total curvature of the curve} = &\quad \sqrt{b^2+c^2+a^2} \qquad (4.4)\\ = &\quad \sqrt{a^2+b^2+c^2}.\end{aligned}$$

ÖZET: Bu çalışmada helis şeridi adını verdiğimiz yeni bir tanım verilmiş, harmonik eğrilik fonksiyonları yardımıyla bir şeridin harmonik eğrilik fonksiyonları incelenmiş ve helis şeridinin harmonik eğriliği ve total eğriliklerinin bazı karakterizasyonları verilmiştir.

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