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Research Article

Rings of frame maps from $\mathcal{P}(\mathbb{R})$ to frames which vanish at infinity

Ali Akbar Estaji*, Ahmad Mahmoudi Darghadam

Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.

Abstract

Let $\mathcal{F}_{\mathcal{P}}(L)$ be the set of all frame maps from $\mathcal{P}(\mathbb{R})$ to L, which is an f-ring. In this paper, we introduce the subrings $\mathcal{F}_{\mathcal{P}_\infty}(L)$ of all frame maps from $\mathcal{P}(\mathbb{R})$ to L which vanish at infinity and $\mathcal{F}_{\mathcal{P}_K}(L)$ of all frame maps from $\mathcal{P}(\mathbb{R})$ to L with compact support. We prove $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is a subring of $\mathcal{F}_{\mathcal{P}}(L)$ that may not be an ideal of $\mathcal{F}_{\mathcal{P}}(L)$ in general and we obtain necessary and sufficient conditions for $\mathcal{F}_{\mathcal{P}_\infty}(L)$ to be an ideal of $\mathcal{F}_{\mathcal{P}}(L)$. Also, we show that $\mathcal{F}_{\mathcal{P}_K}(L)$ is an ideal of $\mathcal{F}_{\mathcal{P}}(L)$ and it is a regular ring. For $f \in \mathcal{F}_{\mathcal{P}}(L)$, we obtain a sufficient condition for f to be an element of $F_{\mathcal{P}_\infty}(L)$ ($\mathcal{F}_{\mathcal{P}_K}(L)$). Next, we give necessary and sufficient conditions for a frame to be compact. We introduce $\mathcal{F}_{\mathcal{P}}$-pseudocompact and next we establish equivalent condition for an $\mathcal{F}_{\mathcal{P}}$-pseudocompact frame to be a compact frame. Finally, we study when for some frame L with $\mathcal{F}_{\mathcal{P}_\infty}(L) \neq (0)$, there is a locally compact frame M such that $\mathcal{F}_{\mathcal{P}_\infty}(L) \cong \mathcal{F}_{\mathcal{P}_\infty}(M)$ and $\mathcal{F}_{\mathcal{P}_K}(L) \cong \mathcal{F}_{\mathcal{P}_K}(M)$.

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1. Introduction

Let $C(X)$ denote the ring of all real-valued continuous functions on a topological space X; and $C_\infty(X)$ is the subring of all functions $C(X)$ which vanish at infinity. Aliabad et al. in [1] have shown that for every completely regular Hausdorff space X, whenever $C_\infty(X) \neq (0)$, then there exists a locally compact space Y such that $C_\infty(X) \cong C_\infty(Y)$.

Let L be a completely regular frame and $\mathcal{R}L$ be the ring of real-valued continuous functions on L and $\mathcal{R}^*L$ be the ring of bounded real-valued continuous functions on L (see [2,4]). $\mathcal{R}_\infty L$, the family of all functions $f \in \mathcal{R}L$ for which $\uparrow f(\frac{-1}{n}, \frac{1}{n})$ is compact for each $n \in \mathbb{N}$ and $\mathcal{R}_K L$, the family of all functions $f \in \mathcal{R}L$ for which $\uparrow \operatorname{coz}(f)^*$ is compact, were introduced by Dube in [5]. Estaji and Mahmoudi Darghadam in [8] studied when for a frame L with $\mathcal{R}_\infty L \neq (0)$, there is a locally compact frame M such that $\mathcal{R}_\infty L \cong \mathcal{R}_\infty M$ and $\mathcal{R}_K L \cong \mathcal{R}_K M$ (also, see [9]).

The f-ring $\mathcal{F}_{\mathcal{P}}(L) := \mathbf{Frm}(\mathcal{P}(\mathbb{R}), L)$ was introduced by Karimi Feizabadi et al. in [11]. Estaji et al. in [7] showed that for every frame L, there is a zero-dimensional frame M such that $\mathcal{F}_{\mathcal{P}}(L) \cong \mathcal{F}_{\mathcal{P}}(M)$. Hence, for study $\mathcal{F}_{\mathcal{P}}(L)$, we assume that L is a zero-dimensional

*Corresponding Author.
Email addresses: aaestaji@hsu.ac.ir (A.A. Estaji), m.darghadam@yahoo.com (A. Mahmoudi Darghadam)
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frame. Let $C(X, \mathbb{R}_d)$ denote the set of continuous functions from a space X into the discrete space of real-numbers $\mathbb{R}_d$. It is known that $C(X, \mathbb{R}_d) \leq C(X)$. If X is discrete, then

$$C(X, \mathbb{R}_d) = C(X) = \mathbb{R}^X \cong \mathcal{F}_{\mathcal{P}}(\mathcal{P}(X)).$$

In this manner, $\mathcal{F}_{\mathcal{P}}(L)$ is the generalization of the f-ring $C(X, \mathbb{R}_d)$.

In [3] an element $\alpha \in \mathcal{R}L$ is called *locally constant* if there exists a partition P of L, meaning P is a cover of L and its elements are pairwise disjoint, such that $\alpha|a$ is constant for each $a \in P$, where $\alpha|a : \mathcal{L}(\mathbb{R}) \to \downarrow a$ given by $\alpha|a(v) = \alpha(v) \wedge a$ for every $v \in \mathcal{L}(\mathbb{R})$. The set of all locally constant elements of $\mathcal{R}L$ is denoted by $\mathfrak{S}L$. In [3], Banaschewski showed that $\mathcal{F}_{\mathcal{P}}L \cong \mathfrak{S}L$ as f-ring.

In this paper, we introduce the subring $\mathcal{F}_{\mathcal{P}_\infty}(L)$ of all frame maps from $\mathcal{P}(\mathbb{R})$ to L for which vanish at infinity and $\mathcal{F}_{\mathcal{P}_K}(L)$ of all frame maps from $\mathcal{P}(\mathbb{R})$ to L with compact support (see Definition 3.1 and Definition 3.2). We show that $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is a subring of $\mathcal{F}_{\mathcal{P}}(L)$ and is an ideal of $\mathcal{F}^*_{\mathcal{P}}(L)$ (see Proposition 3.6 and Proposition 3.8). We prove that $\mathcal{F}_{\mathcal{P}_\infty}(L)$ may not be regular and an ideal of $\mathcal{F}_{\mathcal{P}}(L)$ in general (see Example 7.7). Also, we give necessary and sufficient conditions for $\mathcal{F}_{\mathcal{P}_\infty}(L)$ to be an ideal of $\mathcal{F}_{\mathcal{P}}(L)$ (see Proposition 4.14). We prove that $\mathcal{F}_{\mathcal{P}_K}(L)$ is an ideal of both $\mathcal{F}_{\mathcal{P}}(L)$ and $\mathcal{F}^*_{\mathcal{P}}(L)$ and also it is a regular ring (see Lemma 3.5). We introduce an $\mathcal{F}_{\mathcal{P}}$-pseudocompact frame and next we establish equivalent condition for an $\mathcal{F}_{\mathcal{P}}$-pseudocompact frame to be a compact frame (see Definition 4.1 and Lemma 4.7). For every frame L with $\mathcal{F}_{\mathcal{P}_\infty}(L) \neq (0)$, there is a locally compact frame M such that $\mathcal{F}_{\mathcal{P}_\infty}(L)$ and $\mathcal{F}_{\mathcal{P}_K}(L)$ are isomorphic with an f-subring of $\mathcal{F}_{\mathcal{P}_\infty}(M)$ and an f-subring $\mathcal{F}_{\mathcal{P}_K}(M)$ respectively, see Lemma 7.3, and if $\mathfrak{c} := \bigvee\{a \in L : \uparrow a^* \text{ is a compact frame}\}$ is complemented then $\downarrow\mathfrak{c}$ is a locally compact frame such that $\mathcal{F}_{\mathcal{P}_\infty}(L) \cong \mathcal{F}_{\mathcal{P}_\infty}(\downarrow\mathfrak{c})$ and $\mathcal{F}_{\mathcal{P}_K}(L) \cong \mathcal{F}_{\mathcal{P}_K}(\downarrow\mathfrak{c})$ (see Propositions 5.6, 7.5 and 7.8).

2. Preliminaries

In this section, we represent several concepts and definitions that are necessary in this paper. Throughout this paper L denotes a zero-dimensional frame, that is, L generated by their complemented elements. An element a of L is called *compact* if, for any subset S of L, $a = \bigvee S$ implies $a = \bigvee T$ for some finite $T \subseteq S$. A frame L is called compact whenever its the top element $\top$ of L is compact. For every $a, b \in L$, we recall from [5] that if $\uparrow a$ and $\uparrow b$ are compact frames then $\uparrow(a \wedge b)$ is a compact frame and also, if $\uparrow a$ is a compact frame and $a \leq b$, then $\uparrow b$ is a compact frame. For general background regarding frames we refer to [12].

For each set X, we can form the set $\mathcal{P}(X)$ of all subsets of X (called the power set of X). Also, $(\mathcal{P}(X), \subseteq)$ is a complete Boolean algebra. Let $\mathcal{F}_{\mathcal{P}}(L)$ be the set of all frame maps from $\mathcal{P}(\mathbb{R})$ to L. Details regarding $\mathcal{F}_{\mathcal{P}}(L)$ can be found in [7, 10, 11]. In [11] the authors showed that, the set $\mathcal{F}_{\mathcal{P}}L$ by operation $\diamond : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a sub-f-ring of $\mathcal{R}L$ in which for all $f, g \in \mathcal{F}_{\mathcal{P}}L$, $f \diamond g : P(\mathbb{R}) \to L$ by

$$(f \diamond g)(X) = \bigvee\{f(Y) \wedge g(Z) : Y \diamond Z \subset X\} = \bigvee\{f(\{y\}) \wedge g(\{z\}) : y \diamond y \in X\},$$

where $\diamond \in \{+, -, \wedge, \vee\}$ and $Y \diamond Z := \{y \diamond z : y \in Y,\, z \in Z\}$. Also, for every $r \in \mathbb{R}$, the corresponding constant function $\mathbf{r} : P(\mathbb{R}) \to L$ such that $\mathbf{r}(X) = \top$ if $r \in X$ and $\mathbf{r}(X) = \bot$ otherwise. According to [11], for every $f \in \mathcal{F}_{\mathcal{P}}L$, $f(\{0\})$ $(f(\mathbb{R} \setminus \{0\}))$ is denoted by $z(f)$ $(\mathrm{coz}(f))$ and is called a *zero-element* (*cozero-element*). We put $Z(A) := \{z(f) : f \in A\}$ and $\mathrm{coz}(A) := \{\mathrm{coz}(f) : f \in A\}$, for every $A \subseteq \mathcal{F}_{\mathcal{P}}(L)$. Also, for every $f \in \mathcal{F}_{\mathcal{P}}L$, $z(f) = \bot$ if and only if f is a unit element of $\mathcal{F}_{\mathcal{P}}L$ (see [10]). The bounded part, in the f-ring sense, of $\mathcal{F}_{\mathcal{P}}L$ is denoted by $\mathcal{F}^*_{\mathcal{P}}(L)$ and is characterized by:

$$f \in \mathcal{F}^*_{\mathcal{P}}(L) \Leftrightarrow f(p, q) = 1 \text{ for some } p, q \in \mathbb{R},$$

where $(p, q) = \{r \in \mathbb{R} : a < r < b\}$.

We recall from [7] that for any set S, an *S-trail* on L is a function $t : S \longrightarrow L$ such that $\bigvee_{x\in\mathbb{R}} t(x) = \top$ and $t(x) \wedge t(y) = \bot$ for any $x, y \in S$ with $x \neq y$ and an $\mathbb{R}$-trail is called *real-trail*. Also, for any S-trail t on a frame L,

$$\begin{aligned} \varphi_t : P(S) &\longrightarrow L \\ X &\longmapsto \textstyle\bigvee_{x\in X} t(x) \end{aligned}$$

is a frame map. Throughout this paper, this notation will be used. Also, if $f \in \mathcal{F}_{\mathcal{P}}L$, then $t_f : \mathbb{R} \longrightarrow L$ by $t_f(r) = f(\{r\})$ is a real-trail on a frame L. The correspondences between real-trails on a frame L and the f-ring $\mathcal{F}_{\mathcal{P}}L$ are powerful tools in the study of $\mathcal{F}_{\mathcal{P}}L$. If a is a complemented element of L, then $t_a : \mathbb{R} \longrightarrow L$ by

$$t_a(x) = \begin{cases} a & \text{if } x = 1 \\ a' & \text{if } x = 0 \\ \bot & \text{if } x \notin \{0, 1\} \end{cases}$$

is a real-trail on L, $\operatorname{coz}(\varphi_{t_a}) = a$, $\varphi_{t_a}^2 = \varphi_{t_a}$ and

$$f\varphi_{t_a}(X) = \begin{cases} a \wedge f(X) & \text{if } 0 \notin X \\ a' \vee f(X) & \text{if } 0 \in X \end{cases}$$

for every $f \in \mathcal{F}_{\mathcal{P}}L$ and every $X \subseteq \mathbb{R}$, throughout this notation will be used (see [10]). It is clear that for S-trail $t : S \to L$ on L, φ_t is a monomorphism frame map if and only if $t(s) \neq \bot$ for any $s \in S$. Let $B(L)$ denote the sublattice of complemented elements of a frame L. Hence,

$$z(\mathcal{F}_{\mathcal{P}}L) = B(L) = \operatorname{coz}(\mathcal{F}_{\mathcal{P}}L)$$

and also, for every $x \in L$, there exists a subset A of $B(L)$ such that $x = \bigvee_{a\in A} \operatorname{coz}(\varphi_{t_a})$.

3. The f-subrings $\mathcal{F}_{\mathcal{P}_\infty}(L)$ and $\mathcal{F}_{\mathcal{P}_K}(L)$ of $\mathcal{F}_{\mathcal{P}}(L)$

In this section, we introduce $\mathcal{F}_{\mathcal{P}_\infty}(L)$ and $\mathcal{F}_{\mathcal{P}_K}(L)$ and prove that $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is the f-subrings of $\mathcal{F}_{\mathcal{P}}(L)$ that may not be both regular ring and an ideal of $\mathcal{F}_{\mathcal{P}}(L)$ in general but is an ideal of $\mathcal{F}^*_{\mathcal{P}}(L)$. We prove that $\mathcal{F}_{\mathcal{P}_K}(L)$ is an ideal of both $\mathcal{F}_{\mathcal{P}}(L)$ and $\mathcal{F}^*_{\mathcal{P}}(L)$ and is a regular f-subring of $\mathcal{F}_{\mathcal{P}}(L)$. Also, we establish several equivalent conditions for the set $\mathcal{F}_{\mathcal{P}_\infty}(L)$ to be an ideal of $\mathcal{F}_{\mathcal{P}}(L)$.

We begin with the following basic definitions.

Definition 3.1. We say $f \in \mathcal{F}_{\mathcal{P}}(L)$ vanishes at infinity if $\uparrow f(-\frac{1}{n}, \frac{1}{n})$ is a compact frame for any $n \in \mathbb{N}$. We denote the family of all $f \in \mathcal{F}_{\mathcal{P}}(L)$ vanishing at infinity with $\mathcal{F}_{\mathcal{P}_\infty}(L)$.

Definition 3.2. We say $f \in \mathcal{F}_{\mathcal{P}}(L)$ has compact support if $\uparrow z(f)$ is a compact frame, or equivalently, $\operatorname{coz}(f)$ is a compact element of L. We denote the family of all $f \in \mathcal{F}_{\mathcal{P}}(L)$ with compact support by $\mathcal{F}_{\mathcal{P}_K}(L)$.

It is obvious that $\mathcal{F}_{\mathcal{P}_K}(L) \subseteq \mathcal{F}_{\mathcal{P}_\infty}(L)$.

Example 3.3. We recall a frame M is called connected, if $B(M) = \{\bot, \top\}$. Let M be a connected frame. Consider $\mathbf{0} \neq f \in F_{\mathcal{P}}(M)$. Then $\operatorname{coz}(f) = \top$ and $z(f) = \bot$, which implies that there exists an $0 \neq r \in \mathbb{R}$ such that $f(\{r\}) \neq \bot$ and so we clearly see that $f = \mathbf{r}$. Therefore, $\mathcal{F}_{\mathcal{P}}(M) = \{\mathbf{r} : r \in \mathbb{R}\} \cong \mathbb{R} \cong \mathcal{F}_{\mathcal{P}}(\mathbf{2})$. Since for every $0 \neq r \in \mathbb{R}$, there is an element n in $\mathbb{N}$ such that $|r| > \frac{1}{n}$, we conclude that $\mathbf{r} \in \mathcal{F}_{\mathcal{P}_\infty}(M)$ if and only if M is a compact frame if and only if $\mathbf{r} \in \mathcal{F}_{\mathcal{P}_K}(M)$. Hence for every connected frame M, the following statements are equivalent.

(1) M is a compact frame
(2) $\mathcal{F}_{\mathcal{P}_\infty}(M) = \mathcal{F}_{\mathcal{P}}(M)$.
(3) $\mathcal{F}_{\mathcal{P}_K}(M) = \mathcal{F}_{\mathcal{P}}(M)$.

Estaji et al. in [7] showed that $\mathcal{F}_{\mathcal{P}}(L)$ is a regular ring. In the following we prove that $\mathcal{F}_{\mathcal{P}_K}(L)$ is a regular ring, too.

Lemma 3.4. *For every $f \in \mathcal{F}_{\mathcal{P}_K}(L)$, $\{x \in \mathbb{R} : f(\{x\}) \neq \bot\}$ is a finite subset of $\mathbb{R}$ and $f \in \mathcal{F}^*_{\mathcal{P}}(L)$.*

Proof. Consider $f \in \mathcal{F}_{\mathcal{P}_K}(L)$. Since $\bigvee_{x\in\mathbb{R}} f(\{0, x\}) = \top$, there are $x_1, x_2, ..., x_n \in \mathbb{R}$ such that $f(\{0, x_1, \ldots, x_n\}) = \top$, and so $f(\mathbb{R} \setminus \{0, x_1, \ldots, x_n\}) = \bot$, which implies that $\{x \in \mathbb{R} : f(\{x\}) \neq \bot\}$ is a finite subset of $\mathbb{R}$ and $f \in \mathcal{F}^*_{\mathcal{P}}(L)$. □

Proposition 3.5. *The following statements hold.*

(1) *The set $\mathcal{F}_{\mathcal{P}_K}(L)$ is an ideal of $\mathcal{F}_{\mathcal{P}}(L)$.*
(2) *The set $\mathcal{F}_{\mathcal{P}_K}(L)$ is an ideal of $\mathcal{F}^*_{\mathcal{P}}(L)$.*
(3) *The set $\mathcal{F}_{\mathcal{P}_K}(L)$ is a regular ring.*

Proof. (1). Let $f, g \in \mathcal{F}_{\mathcal{P}_K}(L)$ and $h \in \mathcal{F}_{\mathcal{P}}(L)$. Since $\uparrow(z(f) \wedge z(g))$ is a compact frame and $z(f+g) \geq z(f)\wedge z(g)$, we conclude that $\uparrow(z(f+g))$ is a compact frame, which implies that $f+g \in \mathcal{F}_{\mathcal{P}_K}(L)$. Also, from $\uparrow z(f)$ is a compact frame and $z(fh) = z(f)\vee z(h) \geq z(f)$, we infer that $\uparrow z(fh)$ is a compact frame, which implies that $fh \in \mathcal{F}_{\mathcal{P}_K}(L)$.

(2). Since, by Lemma 3.4, $\mathcal{F}_{\mathcal{P}_K}(L) \subseteq \mathcal{F}^*_{\mathcal{P}}(L)$, the proof is similar to the first statement.

(3). Consider $f \in \mathcal{F}_{\mathcal{P}_K}(L)$. We define the real-trail $t : \mathbb{R} \to L$ on the frame L by

$$t(x) = \begin{cases} f(\{\frac{1}{x}\}) & \text{if } x \neq 0 \\ f(\{0\}) & \text{if } x = 0. \end{cases}$$

Then

$$f\varphi_t(\{x\}) = \begin{cases} z(f) & \text{if } x = 0 \\ \operatorname{coz}(f) & \text{if } x = 1 \\ \bot & \text{if } x \in \mathbb{R} \setminus \{0, 1\}, \end{cases}$$

which implies that $f^2\varphi_t = f$. Since $\uparrow z(\varphi_t) = \uparrow z(f)$ is a compact frame, we conclude that $\varphi_t \in \mathcal{F}_{\mathcal{P}_K}(L)$, which implies that $\mathcal{F}_{\mathcal{P}_K}(L)$ is a regular ring. □

Proposition 3.6. *The set $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is a subring of $\mathcal{F}_{\mathcal{P}}(L)$.*

Proof. Consider $f, g \in \mathcal{F}_{\mathcal{P}_\infty}(L)$ and $n \in \mathbb{N}$. Since

$$(f + g)((-\frac{1}{n}, \frac{1}{n})) \geq f(-\frac{1}{2n}, \frac{1}{2n}) \wedge g(-\frac{1}{2n}, \frac{1}{2n})$$

and $\uparrow (f(-\frac{1}{2n}, \frac{1}{2n}) \wedge g(-\frac{1}{2n}, \frac{1}{2n}))$ is a compact frame, we conclude that $\uparrow (f + g)(-\frac{1}{n}, \frac{1}{n})$ is a compact frame, which implies that $f + g \in \mathcal{F}_{\mathcal{P}_\infty}(L)$.

Consider $m \in \mathbb{N}$ with $m > [\sqrt{n}]$. From $\uparrow (f(-\frac{1}{m}, \frac{1}{m}) \wedge g(-\frac{1}{m}, \frac{1}{m}))$ is a compact frame and

$$(fg)((-\frac{1}{n}, \frac{1}{n})) \geq f(-\frac{1}{m}, \frac{1}{m}) \wedge g(-\frac{1}{m}, \frac{1}{m}),$$

we infer that $\uparrow (fg)(-\frac{1}{n}, \frac{1}{n})$ is a compact frame, which implies that $fg \in \mathcal{F}_{\mathcal{P}_\infty}(L)$. □

Lemma 3.7. *For every $f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$, the following statements hold.*

(1) *The set $\{x \in \mathbb{R} \setminus (-\frac{1}{n}, \frac{1}{n}) : f(\{x\}) \neq \bot\}$ is finite for every $n \in \mathbb{N}$.*
(2) *$f \in \mathcal{F}^*_{\mathcal{P}}(L)$.*
(3) *The set $\{x \in \mathbb{R} : f(\{x\}) \neq \bot\}$ is an at most countable set.*

Proof. (1). Consider $n \in \mathbb{N}$. Since $\bigvee_{x\in\mathbb{R}\setminus(-\frac{1}{n},\frac{1}{n})} f((-\frac{1}{n}, \frac{1}{n}) \cup \{x\}) = \top$, there are $x_1, x_2, ..., x_m \in \mathbb{R} \setminus (-\frac{1}{n}, \frac{1}{n})$ such that $f((-\frac{1}{n}, \frac{1}{n}) \cup \{x_1, \ldots x_m\}) = \top$, which implies that $f(\mathbb{R} \setminus (\{x_1, x_2, ..., x_m\} \cup (-\frac{1}{n}, \frac{1}{n}))) = \bot$. Hence $\{x \in \mathbb{R} \setminus (-\frac{1}{n}, \frac{1}{n}) : f(\{x\}) \neq \bot\}$ is a finite subset of $\mathbb{R}$.

(2) and (3), by the first statement, are obvious. □

If L is not compact, then $\mathbf{1} \notin \mathcal{F}_{\mathcal{P}_\infty}(L)$, because $\mathbf{1}(-\frac{1}{n}, \frac{1}{n}) = \bot$ and $\uparrow\bot$ is not compact.

Proposition 3.8. *The set $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is an ideal of $\mathcal{F}^*_{\mathcal{P}}(L)$.*

Proof. By Proposition 3.6 and Lemma 3.7, $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is a subring of $\mathcal{F}^*_{\mathcal{P}}(L)$. Now we assume $f \in \mathcal{F}^*_{\mathcal{P}}(L)$ and $g \in \mathcal{F}_{\mathcal{P}_\infty}(L)$. Then $f(-m, m) = \top$ for some $m \in \mathbb{N}$. Hence $\uparrow fg(-\frac{1}{n}, \frac{1}{n})$ is a compact frame, because

$$fg(-\frac{1}{n}, \frac{1}{n}) \geq f(-m, m) \wedge g(-\frac{1}{mn}, \frac{1}{mn}) = g(-\frac{1}{mn}, \frac{1}{mn}),$$

which follows that $fg \in \mathcal{F}_{\mathcal{P}_\infty}(L)$. □

The following example shows that $\mathcal{F}_{\mathcal{P}_\infty}(L)$ may not be an ideal of $\mathcal{F}_{\mathcal{P}}(L)$ in general and also $\mathcal{F}_{\mathcal{P}_\infty}(L)$ my not be a regular ring in general.

Example 3.9. Consider $L = \mathcal{P}(\mathbb{N})$. We define the real-trail $t : \mathbb{R} \to L$ on the frame L by

$$t(x) = \begin{cases} \{\frac{1}{x}\} & \text{if } \frac{1}{x} \in \mathbb{N} \\ \bot & \text{otherwise.} \end{cases}$$

Then $z(\varphi_t) = \bot$ and so φ_t is a unit element of $\mathcal{F}_{\mathcal{P}}(L)$. Since $\mathbf{1} \notin \mathcal{F}_{\mathcal{P}_\infty}(L)$ and $\varphi_t \in \mathcal{F}_{\mathcal{P}_\infty}(L)$, we conclude that $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is not an ideal of $\mathcal{F}_{\mathcal{P}}(L)$. Also, if there is an element f in $\mathcal{F}_{\mathcal{P}_\infty}(L)$ such that $\varphi_t^2 f = \varphi_t$ then $\varphi_t f = \mathbf{1} \in \mathcal{F}_{\mathcal{P}_\infty}(L)$, which is contradiction. Therefore, $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is not a regular ring.

Definition 3.10. Let I be any ideal in $\mathcal{F}_{\mathcal{P}}(L)$. If $\bigvee_{f \in I} \operatorname{coz}(f)$ is the non-top element of L, we call I a *fixed ideal*; if $\bigvee_{f \in I} \operatorname{coz}(f) = \top$, then I is a *free ideal*.

Lemma 3.11. *If c is a compact element of L, then $c \in \operatorname{coz}(I)$ for every free ideal I of $\mathcal{F}_{\mathcal{P}}(L)$ and every $c \in B(L)$.*

Proof. From c is a compact element of L and there exists a subset A of $B(L)$ such that $c = \bigvee_{a \in A} \operatorname{coz}(\varphi_{t_a})$, we conclude that there a finite subset B of A such that $c = \operatorname{coz}(\Sigma_{a \in B} \varphi^2_{t_a}) \in B(L)$. Let I be a free ideal of $\mathcal{F}_{\mathcal{P}}(L)$ and $c = \operatorname{coz}(f)$ for some $f \in \mathcal{F}_{\mathcal{P}}(L)$.

$$c = c \wedge \top = \operatorname{coz}(f) \wedge \bigvee_{g \in I} \operatorname{coz}(g) = \bigvee_{g \in I} \operatorname{coz}(fg),$$

and so, there are $g_1, g_2, \dots g_n \in I$ such that $c = \operatorname{coz}(\Sigma^n_{i=1} (fg_i)^2) \in \operatorname{coz}(I)$. □

Corollary 3.12. *The set of all compact elements of L is a subset of*

$$\bigcap \Big\{ \operatorname{coz}(I) : I \textit{ is a free ideal of } \mathcal{F}_{\mathcal{P}}(L) \Big\}.$$

Proof. By Lemma 3.11, it is clear. □

Definition 3.13. An element a of a frame M is called σ-compact if there exists a family $\{a_n : n \in \mathbb{N}\}$ of compact elements of M such that $a = \bigvee_{n \in \mathbb{N}} a_n$. A frame M is called σ-compact whenever its the top element $\top$ of M is σ-compact.

By Lemma 3.11, if $a \in L$ is a σ-compact element of L, then there exists an ascending sequence $\{a_n\}_{n \in \mathbb{N}}$ of $B(L)$ such that $a = \bigvee_{n \in \mathbb{N}} a_n$ and $\uparrow a'_n$ is compact, for every $n \in \mathbb{N}$.

Proposition 3.14. *The following statements hold.*

(1) *Every element of $\operatorname{coz}(\mathcal{F}_{\mathcal{P}_\infty}(L))$ is a σ-compact element of L.*
(2) *If $B(L)$ is a sub-σ-frame of L and $a \in L$ is a σ-compact element of L then $a \in \operatorname{coz}(\mathcal{F}_{\mathcal{P}_\infty}(L))$.*

Proof. (1). Consider $f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$ and $a = \mathrm{coz}(f)$. We put

$$a_n := f((-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, +\infty)),$$

for every $n \in \mathbb{N}$. Then $\uparrow a'_n = \uparrow f(-\frac{1}{n}, \frac{1}{n})$ is a compact frame and $a = \bigvee_{n\in\mathbb{N}} a_n$, which implies that a is a σ-compact element of L.

(2). Let $\{a_n\}_{n\in\mathbb{N}}$ be an ascending sequence of $B(L)$ such that $a = \bigvee_{n\in\mathbb{N}} a_n$ and $\uparrow a'_n$ is compact for every $n \in \mathbb{N}$. We put $b_1 := a_1$ and $b_n := a_n \wedge a'_{n-1}$ for every $2 \leq n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, $\bigvee_{i=1}^{n} b_i = a_n$, which implies that $a = \bigvee_{i=1}^{\infty} b_i$ and also $b_i \wedge b_j = \perp$ for every $i \neq j$. We define the real-trail $t : \mathbb{R} \to L$ on the frame L by

$$t(x) = \begin{cases} b_n & \text{if there exists an element } n \text{ of } \mathbb{N} \text{ such that } \frac{1}{x} = n \\ a' & \text{if } x = 0 \\ \perp & \text{otherwise.} \end{cases}$$

Since

$$\uparrow\varphi_t(-\frac{1}{n}, -\frac{1}{n}) = \uparrow (a' \vee \bigvee_{i=n+1}^{\infty} b_i) = \uparrow a'_n$$

is a compact frame, we conclude that $\varphi_t \in \mathcal{F}_{\mathcal{P}_\infty}(L)$ and $\mathrm{coz}(\varphi_t) = a$. □

4. Compact and $\mathcal{F}_{\mathcal{P}}$-pseudocompact frames

In this section, we introduce $\mathcal{F}_{\mathcal{P}}$-pseudocompact frame and give several equivalent conditions for it.

For any element a of a frame M, we have the frame map $M \to \downarrow a$ taking x to $x \wedge a$, and the associated $\theta : \mathcal{F}_{\mathcal{P}}(M) \to \mathcal{F}_{\mathcal{P}}(\downarrow a)$ will be denoted $f \mapsto f|a$, where $f|a(A) = f(A) \wedge a$ for every $A \subseteq \mathbb{R}$. Evidently, this is the counterpart of restricting functions of $\mathbb{R}^X$ on a subset of X. Throughout this paper, this notation will be used.

We begin with the following basic definition.

Definition 4.1. An element a of a frame M is called $\mathcal{F}_{\mathcal{P}}$-pseudocompact if $f|a$ is bounded, for every $f \in \mathcal{F}_{\mathcal{P}}(M)$. If $\top$ is $\mathcal{F}_{\mathcal{P}}$-pseudocompact we say L is an $\mathcal{F}_{\mathcal{P}}$-pseudocompact frame, in fact $\mathcal{F}_{\mathcal{P}}(M) = \mathcal{F}^*_{\mathcal{P}}(M)$.

Proposition 4.2. *L is a compact frame if and only if $\mathcal{F}_{\mathcal{P}_\infty}(L) = \mathcal{F}_{\mathcal{P}}(L)$.*

Proof. *Necessity.* Consider $f \in \mathcal{F}_{\mathcal{P}}(L)$ and $n \in \mathbb{N}$. From $\uparrow\perp = L$ is compact and $\perp \leq f(-\frac{1}{n}, \frac{1}{n})$, we infer that $\uparrow f(-\frac{1}{n}, \frac{1}{n})$ is a compact frame, which implies that $f \in F_{\mathcal{P}_\infty}(L)$. Also, we have, by Lemma 3.7, $\mathcal{F}_{\mathcal{P}_\infty}(L) \subseteq \mathcal{F}^*_{\mathcal{P}}(L) = \mathcal{F}_{\mathcal{P}}(L)$ and this completes the proof.

Sufficiency. It is clear that $L = \uparrow\perp = \uparrow\mathbf{1}(-1, 1)$ is compact, since $\mathbf{1} \in \mathcal{F}_{\mathcal{P}_\infty}(L)$. □

Lemma 4.3. *Let L be a compact frame. If $f \in \mathcal{F}_{\mathcal{P}}(L)$, then there exists a finite subset X of $\mathbb{R}$ such that $f(\mathbb{R} \setminus X) = \perp$.*

Proof. Since $\bigvee_{x\in\mathbb{R}} f(\{x\}) = \top$, we conclude that there are $x_1, x_2, ..., x_n \in \mathbb{R}$ such that $\bigvee_{i=1}^{n} f(\{x_i\}) = \top$, which implies that $f(\mathbb{R} \setminus \{x_1, x_2, ..., x_n\}) = \perp$. □

It is well known that $\mathfrak{t} : \mathcal{R}(\beta M) \to \mathcal{R}^* M$ given by $\mathfrak{t}(f) = j_M f$ is the ring isomorphism for every completely regular frame M, where $j_M : \beta M \to M$ given by $I \mapsto \bigvee I$ (see [6]). We define $\mathfrak{t}_{\mathcal{P}} : \mathcal{F}_{\mathcal{P}}(\beta M) \to \mathcal{F}^*_{\mathcal{P}}(M)$ by $\mathfrak{t}_{\mathcal{P}}(f) = j_M f$ for every $f \in \mathcal{F}_{\mathcal{P}}(\beta M)$. Now, it is natural to ask whether $\mathfrak{t}_{\mathcal{P}}$ is a ring isomorphism. It is clear that $\mathfrak{t}_{\mathcal{P}}$ is a ring monomorphism.

The following example shows that $\mathfrak{t}_{\mathcal{P}}$ is a ring monomorphism, my not be a ring isomorphism.

Example 4.4. Consider $L = \mathcal{P}(\mathbb{N})$. We define the real-trail $t : \mathbb{R} \to L$ on the frame L by

$$t(x) = \begin{cases} \{x\} & \text{if } \frac{1}{x} \in \mathbb{N} \\ \bot & \text{if } \frac{1}{x} \notin \mathbb{N}. \end{cases}$$

Since $\{x \in \mathbb{R} : \varphi_t(x) \neq \bot\}$ is an infinite subset of $\mathbb{R}$, we conclude from Lemma 4.3 that $\varphi_t \notin Im(\mathfrak{t}_{\mathcal{P}})$, which implies that $\mathfrak{t}_{\mathcal{P}}$ is not an isomorphism.

Now, we ask this question: When is $\mathfrak{t}_{\mathcal{P}}$ a ring isomorphism?

Proposition 4.5. *For $\mathfrak{t}_{\mathcal{P}} : \mathcal{F}_{\mathcal{P}}(\beta L) \to \mathcal{F}^*_{\mathcal{P}}(L)$ given by $f \mapsto j_L f$, the following statements hold.*

(1) *If $\mathfrak{t}_{\mathcal{P}}$ is a ring isomorphism then L is a compact frame.*
(2) *If L is a compact frame and $B(L)$ is a sub-σ-frame of L then $\mathfrak{t}_{\mathcal{P}}$ is a ring isomorphism.*

Proof. (1). Consider $f \in \mathcal{F}^*_{\mathcal{P}}(L)$. Then there are $x_1, x_2, ..., x_n \in \mathbb{R}$, such that $\bigvee_{i=1}^n f(\{x_i\}) = \top$. We define the real-trail $\hat{t} : \mathbb{R} \to \beta L$ on the frame βL by $\hat{t}(x) = {\downarrow} f(\{x\})$. Then $\mathfrak{t}_{\mathcal{P}}(\varphi_{\hat{t}}) = f$, which implies that $\mathfrak{t}_{\mathcal{P}}$ is a ring isomorphism.

(2). Let L be not compact and $S \subseteq L$ such that $\bigvee S = \top$ and $\bigvee F \neq \top$ for every finite subset F of S. For every $s \in S$, there is a subset C_s of $B(L)$ such that $s = \bigvee C_s$. Consider $C = \bigcup_{s \in S} C_s$. Therefore $\bigvee F \neq \top$ for every finite subset F of C. Therefore without losing generality we may assume that $\bigvee(C \setminus \{c\}) \neq \top$ for every $c \in C$. Let $B := \{c_{n+1} \in C : n \in \mathbb{N}\}$ be an infinite countable subset of C. Since $B(L)$ is a σ-frame, we conclude that $a = \bigvee B \in B(L)$ has a complement in L. We put $b_n = \bigvee_{i=2}^n c_i$, for every $n \in \mathbb{N} \setminus \{1\}$ and define the real-trail $t : \mathbb{R} \to L$ on L by

$$t(x) = \begin{cases} a' & \text{if } x = 1 \\ b_2 & \text{if } x = \frac{1}{2} \\ b_n \wedge b'_{n-1} & \text{if there is an element } n \text{ of } N \setminus \{1, 2\} \text{ such that } x = \frac{1}{n} \\ \bot & \text{otherwise.} \end{cases}$$

It is clear that $\varphi_t \in \mathcal{F}^*_{\mathcal{P}}(L)$, and by Lemma 4.3, $\varphi_t \notin Im(\mathfrak{t}_{\mathcal{P}})$. Therefore $\mathfrak{t}_{\mathcal{P}}$ is not an isomorphism. □

Proposition 4.6. *The following statements are equivalent.*

(1) *L is compact.*
(2) *Every proper ideal of $\mathcal{F}_{\mathcal{P}}(L)$ ($\mathcal{F}^*_p(L)$) is fixed.*
(3) *Every maximal ideal of $\mathcal{F}_{\mathcal{P}}(L)$ ($\mathcal{F}^*_p(L)$) is fixed.*

Proof. (1) ⇒ (2). Let I be a free proper ideal of $\mathcal{F}_{\mathcal{P}}(L)$. Since, by Lemma 3.11, $\top \in \operatorname{coz}(I)$, we conclude that $I = \mathcal{F}_{\mathcal{P}}(L)$, which is a contradiction.

(2) ⇒ (3). It is clear.

(3) ⇒ (1). Let $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq L$ such that $\top = \bigvee_{\lambda \in \Lambda} a_\lambda$. It is clear that

$$I = \{\varphi \in \mathcal{F}_{\mathcal{P}}(L) : \operatorname{coz}(\varphi) \leq \textstyle\bigvee_{\lambda \in \Lambda'} a_\lambda, \text{ for a finite subset } \Lambda' \text{ of } \Lambda\}$$

is an ideal of $\mathcal{F}_{\mathcal{P}}(L)$. If $I \neq \mathcal{F}_{\mathcal{P}}(L)$, then there exists a maximal ideal M such that $I \subseteq M$ and so

$$\top = \bigvee_{\lambda \in \Lambda} a_\lambda = \bigvee \operatorname{coz}(I) \leq \bigvee \operatorname{coz}(M),$$

which is a contradiction. Therefore $I = \mathcal{F}_{\mathcal{P}}(L)$ and there exists a finite subset Λ' of Λ such that $\top = \operatorname{coz}(\mathbf{1}) = \bigvee_{\lambda \in \Lambda'} a_\lambda$. This completes the proof. □

Proposition 4.7. *The following statements hold.*

(1) *If L is compact then $\mathcal{F}_{\mathcal{P}}(L) = \mathcal{F}^*_{\mathcal{P}}(L)$.*
(2) *If $B(L)$ is a sub-σ-frame of L and $\mathcal{F}_{\mathcal{P}}(L) = \mathcal{F}^*_{\mathcal{P}}(L)$ then L is compact.*

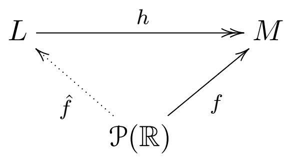
Proof. (1). By Proposition 4.2, it is obvious.

(2). Let L be not compact and $S \subseteq L$ such that $\bigvee S = \top$ and $\bigvee F \neq \top$ for every finite subset F of S. For every $a \in S$, there is a subset C_a of $B(L)$ such that $a = \bigvee C_a$. Consider $C = \bigcup_{a \in A} C_a$. Then $\bigvee F \neq \top$ for every finite subset F of C. Therefore without losing generality we may assume that $\bigvee(C \setminus \{c\}) \neq \top$ for every $c \in C$. Let $B := \{c_{n+1} \in C : n \in \mathbb{N}\}$ be an infinite countable subset of C. Since $B(L)$ is a σ-frame, we conclude that $\bigvee B \in B(L)$ has a complement in L, say c_1. We put $b_n = \bigvee_{i=1}^{n} c_i$ for every $n \in \mathbb{N}$, and define the real-trail $t : \mathbb{R} \to L$ on L by

$$t(x) = \begin{cases} b_1 & \text{if } x = 1 \\ b_x \wedge b'_{x-1} & \text{if } x \in N \setminus \{1\} \\ \bot & \text{otherwise.} \end{cases}$$

It is clear that $\varphi_t \in \mathcal{F}_{\mathcal{P}}(L) \setminus \mathcal{F}^*_{\mathcal{P}}(L)$, which is a contradiction. $\square$

Definition 4.8. A onto frame map $h : L \to M$ is called *$\mathcal{F}_{\mathcal{P}}$-quotient* if for every $f \in \mathcal{F}_{\mathcal{P}}(M)$, there is an element $\hat{f}$ in $\mathcal{F}_{\mathcal{P}}(L)$ such that $h\hat{f} = f$, i.e., the following diagram commutes.



Also, an onto frame map $h : L \to M$ is called *$coz_{\mathcal{F}_{\mathcal{P}}}$-onto* if for every $c \in \mathrm{coz}(\mathcal{F}_{\mathcal{P}}(M))$, there is an element $\hat{c}$ in $\mathrm{coz}(\mathcal{F}_{\mathcal{P}}(L))$ such that $h(\hat{c}) = c$.

Corollary 4.9. *A frame map $h : L \to M$ is $coz_{\mathcal{F}_{\mathcal{P}}}$-onto if and only if it is $\mathcal{F}_{\mathcal{P}}$-quotient.*

Proof. It is obvious. $\square$

Any frame map $h : M \to N$ between frames gives rise to an f-ring homomorphism

$$\mathcal{F}_{\mathcal{P}}h : \mathcal{F}_{\mathcal{P}}(M) \to \mathcal{F}_{\mathcal{P}}(N)$$
$$f \mapsto h \circ f,$$

and this results in a variant functor $F_{\mathcal{P}}$ from the category **Frm** of frames and frame maps to **AfR** from archimedean f-rings, and morphisms which are f-ring homomorphisms, for if $\diamond \in \{+, ., \vee, \wedge\}$ and $f, g \in \mathcal{F}_{\mathcal{P}}(M)$, then

$$\begin{aligned} \mathcal{F}_{\mathcal{P}}h(f \diamond g)(\{a\}) &= h((f \diamond g)(\{a\})) \\ &= h\Big(\bigvee\Big\{f\{x\} \wedge g(\{y\}) : x \diamond y = a\Big\}\Big) \\ &= \bigvee\Big\{h(f\{x\}) \wedge h(g(\{y\})) : x \diamond y = a\Big\}, \quad \text{since } h \text{ is the frame map} \\ &= \mathcal{F}_{\mathcal{P}}h(f) \diamond \mathcal{F}_{\mathcal{P}}h(g)(\{a\}), \end{aligned}$$

for every $a \in \mathbb{R}$, which implies that $\mathcal{F}_{\mathcal{P}}h(f \diamond g) = \mathcal{F}_{\mathcal{P}}h(f) \diamond \mathcal{F}_{\mathcal{P}}h(g)$. Hence we have

Proposition 4.10. *If $\mathcal{F}_{\mathcal{P}}$-quotient map $h : M \to N$ is codense then the f-ring homomorphism $\mathcal{F}_{\mathcal{P}}h : \mathcal{F}_{\mathcal{P}}(M) \to \mathcal{F}_{\mathcal{P}}(N)$ given by $f \mapsto h \circ f$ is an f-ring isomorphism. Also, h is $coz_{\mathcal{F}_{\mathcal{P}}}$-onto.*

Proof. If $f \in \ker(\mathcal{F}_{\mathcal{P}}h)$, then $\mathcal{F}_{\mathcal{P}}h(f) = \mathbf{0}$, which implies that $\mathcal{F}_{\mathcal{P}}h(f)(\{0\}) = h(f(\{0\})) = \top$ and so $z(f) = \top$, i.e., $f = \mathbf{0}$. It is clear that $\mathcal{F}_{\mathcal{P}}h$ is onto. $\square$

Lemma 4.11. *Let L be σ-compact and not compact. Then $L \cong \mathcal{P}(\mathbb{N})$ and there is an f-ring isomorphism $\eta : \mathcal{F}_{\mathcal{P}}(P(\mathbb{N})) \to \mathcal{F}_{\mathcal{P}}(L)$ such that*

(1) *$f \in \mathcal{F}^*_{\mathcal{P}}(\mathcal{P}(\mathbb{N}))$ if and only if $\eta(f) \in \mathcal{F}^*_{\mathcal{P}}(L)$.*
(2) *$f \in \mathcal{F}_{\mathcal{P}_\infty}(\mathcal{P}(\mathbb{N}))$ if and only if $\eta(f) \in \mathcal{F}_{\mathcal{P}_\infty}(L)$.*
(3) *$f \in \mathcal{F}_{\mathcal{P}_K}(\mathcal{P}(\mathbb{N}))$ if and only if $\eta(f) \in \mathcal{F}_{\mathcal{P}_K}(L)$.*

Proof. Similar to the proof of Proposition 4.7, there exists an infinite countable subset $\{c_n : n \in \mathbb{N}\}$ of $B(L)$ such that $c'_1 = \bigvee_{\substack{n\in\mathbb{N}\\ n\neq 1}} c_n$ and $\bigvee F \neq \top$ for every finite subset F of $\{c_n : n \in \mathbb{N}\setminus\{1\}\}$. We put $b_n = \bigvee_{i=1}^n c_i$ for every $n \in \mathbb{N}$, and define the real-trail $t : \mathbb{R} \to L$ on L by

$$t(x) = \begin{cases} b_1 & \text{if } x = 1 \\ b_x \wedge b'_{x-1} & \text{if } x \in N \setminus \{1\} \\ \bot & \text{otherwise.} \end{cases}$$

It is clear that $\varphi_t \in \mathcal{F}_{\mathcal{P}}(L) \setminus \mathcal{F}^*_{\mathcal{P}}(L)$. We define the $\mathbb{N}$-trail $\bar{t} : \mathbb{N} \to L$ on L by

$$\bar{t}(x) = \begin{cases} \varphi_t((-\infty, 1]) & \text{if } x = 1 \\ \varphi_t((x-1, x]) & \text{if } x \in \mathbb{N} \setminus \{1\} \end{cases}$$

Hence $\varphi_{\bar{t}} : \mathcal{P}(\mathbb{N}) \to L$ given by $\varphi_{\bar{t}}(X) = \bigvee_{x\in X} \bar{t}(x)$ is an isomorphism $\mathcal{F}_{\mathcal{P}}$-quotient map. By Proposition 4.10, $\eta = \mathcal{F}_{\mathcal{P}}\varphi_{\bar{t}} : \mathcal{F}_{\mathcal{P}}(\mathcal{P}(\mathbb{N})) \to \mathcal{F}_{\mathcal{P}}(L)$ given by $f \mapsto \varphi_{\bar{t}} \circ f$ is an f-ring isomorphism. □

Proposition 4.12. *For every $c \in B(L)$, there exists an f-ring monomorphism $\theta : \mathcal{F}_{\mathcal{P}}(\downarrow c) \to \mathcal{F}_{\mathcal{P}}(L)$ such that*

(1) *$f \in \mathcal{F}_{\mathcal{P}_\infty}(\downarrow c)$ if and only if $\theta(f) \in \mathcal{F}_{\mathcal{P}_\infty}(L)$.*
(2) *$f \in \mathcal{F}_{\mathcal{P}_K}(\downarrow c)$ if and only if $\theta(f) \in \mathcal{F}_{\mathcal{P}_K}(L)$.*

Proof. We define $\theta : \mathcal{F}_{\mathcal{P}}(\downarrow c) \to \mathcal{F}_{\mathcal{P}}(L)$ by $\theta(f) = \overline{f}$, where $\overline{f} : \mathcal{P}(\mathbb{R}) \to L$ give by

$$\overline{f}(X) = \begin{cases} f(X) & \text{if } 0 \notin X \\ f(X) \vee c' & \text{if } 0 \in X \end{cases}$$

is a frame map. Consider $f, g \in \mathcal{F}_{\mathcal{P}}(\downarrow c)$ and $\diamond \in \{+, ., \vee, \wedge\}$. Then we have

$$\begin{aligned} \theta(f) \diamond \theta(g)(\{0\}) &= \bigvee \Big\{ f(\{x\}) \wedge g(\{y\}) : x \diamond y = 0,\ x \neq 0 \text{ or } y \neq 0 \Big\} \vee c' \\ &= (f \diamond g)(\{0\}) \vee c' \\ &= \theta(f \diamond g)(\{0\}). \end{aligned}$$

Consider $0 \neq x \in \mathbb{R}$. Since for every $r \in \mathbb{R}$,

$$f(\{r\}) \wedge (g(\{0\}) \vee c') = f(\{r\}) \wedge g(\{0\})$$

and

$$(f(\{0\}) \vee c') \wedge g(\{r\}) = f(\{0\}) \wedge g(\{r\}),$$

we conclude that

$$\theta(f) \diamond \theta(g)(\{x\}) = \theta(f \diamond g)(\{x\}).$$

Therefore, θ is an f-ring homomorphism. Let f be an element of $\ker(\theta)$. From $f(\{0\}) \vee c' = \theta(f)(\{0\}) = \mathbf{0}(\{0\}) = \top$ and $f(\{0\}) \wedge c' \leq c \wedge c' = \bot$, we infer that $f(\{0\}) = c$ and since for every $0 \neq x \in \mathbb{R}$, $f(\{x\}) = \theta(f)(\{x\}) = \mathbf{0}(\{x\}) = \bot$, we conclude that $f = \mathbf{0}$. Therefore, θ is an f-ring monomorphism. □

We recall from [7] that a proper ideal I in $\mathcal{F}_{\mathcal{P}}L$ is called a $z_{F_{\mathcal{P}}}$-ideal if $z(f) = z(g)$ and $f \in I$ implies that $g \in I$. We will also need the following results which appear in [7], for the proof of the following proposition.

Proposition 4.13. *Every proper ideal in $\mathcal{F}_{\mathcal{P}}L$ is a $z_{F_{\mathcal{P}}}$-ideal.*

Proposition 4.14. *Let $B(L)$ be a sub-σ-frame of L. The following statements are equivalent.*

(1) *$\mathcal{F}_{\mathcal{P}_\infty}(L)$ is an ideal of $\mathcal{F}_{\mathcal{P}}(L)$.*
(2) *Every σ-compact element a of L is $\mathcal{F}_{\mathcal{P}}$-pseudocompact.*
(3) *Every σ-compact element of L is compact.*
(4) *If $\{a_n\}_{n\in\mathbb{N}}$ is a family of compact elements of L such that*

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots ,$$

then there exists an element k of $\mathbb{N}$ such that $a_k = a_{k+i}$ for all $i \in \mathbb{N}$.
(5) *$\mathcal{F}_{\mathcal{P}_\infty}(L) = \mathcal{F}_{\mathcal{P}_K}(L)$.*
(6) *$\mathcal{F}_{\mathcal{P}_\infty}(L)$ is a regular ring.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). Let a be a σ-compact element of L. Then, by Proposition 3.14, there is an element f of $\mathcal{F}_{\mathcal{P}_\infty}(L)$ such that $\mathrm{coz}(f) = a$. Since $\mathrm{coz}(\varphi_{t_a}) = a = \mathrm{coz}(f)$, we conclude from Proposition 4.13 that $\varphi_{t_a} \in \mathcal{F}_{\mathcal{P}_\infty}(L)$, which implies that $\uparrow\varphi_{t_a}(-\frac{1}{n}, \frac{1}{n}) = \uparrow a'$ is compact for any $n \in \mathbb{N}$ and so a is compact. Therefore, by Lemma 4.7, $\downarrow a$ is an $\mathcal{F}_{\mathcal{P}}$-pseudocompact frame.

(2) $\Leftrightarrow$ (3). By Lemma 4.7, it is clear.

(3) $\Rightarrow$ (4). It is clear.

(4) $\Rightarrow$ (5). Consider $f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$. Since for every $n \in \mathbb{N}$, $f((-\infty, -\frac{1}{n}] \vee [\frac{1}{n}, +\infty))$ is compact, we conclude from the fourth statement that there exists an element m of $\mathbb{N}$ such that

$$\mathrm{coz}(f) = \bigvee_{n\in\mathbb{N}} f((-\infty, -\frac{1}{n}] \vee [\frac{1}{n}, +\infty)) = f((-\infty, -\frac{1}{m}] \vee [\frac{1}{m}, +\infty)),$$

which implies that $\mathrm{coz}(f)$ is compact and so $f \in \mathcal{F}_{\mathcal{P}_K}(L)$. Therefore, $\mathcal{F}_{\mathcal{P}_\infty}(L) = \mathcal{F}_{\mathcal{P}_K}(L)$.

(5) $\Rightarrow$ (1) and (5) $\Rightarrow$ (6). By Proposition 3.5, it is clear.

(6) $\Rightarrow$ (2). Let a be a σ-compact element of L and not a compact element of L. Let t and φ_t be the same in Proposition 3.14. Because $\varphi_t \in \mathcal{F}_{\mathcal{P}_\infty}(L)$ and $\mathcal{F}_{\mathcal{P}_\infty}(L)$ is the regular ring, there exists an element f of $\mathcal{F}_{\mathcal{P}_\infty}(L)$ such that $\varphi_t = \varphi_t^2 f$. Since for every $x \in \mathbb{R}\setminus\{0\}$,

$$\begin{aligned}
\varphi_t(\{x\}) &= \varphi_t(\{x\}) \wedge \varphi_t^2 f(\{x\}) \\
&= \varphi_t(\{x\}) \wedge \bigvee\{\varphi_t\{y\} \wedge f\varphi_t(\{y'\}) : yy' = x\} \\
&= \bigvee\{\varphi_t(\{x\}) \wedge \varphi_t(\{y\}) \wedge f\varphi_t(\{y'\}) : yy' = x\} \\
&= \varphi_t(\{x\}) \wedge f\varphi_t(\{1\}),
\end{aligned}$$

we infer that $\mathrm{coz}(\varphi_t) \leq f\varphi_t(\{1\}) \leq \mathrm{coz}(f\varphi_t) \leq \mathrm{coz}(\varphi_t)$ and hence $\mathrm{coz}(f) \geq \mathrm{coz}(\varphi_t)$. Since $\mathrm{coz}(\varphi_t|a) = a = \top_{\downarrow a}$, we conclude that $\varphi_t|_a$ is a unit element of $\mathcal{F}_{\mathcal{P}}(\downarrow a)$ and $\varphi_t|af|a = \mathbf{1}$, which implies that $f|_a(\{n\}) = \varphi_t|_a(\{\frac{1}{n}\}) = b_n \neq \bot$ for every $n \in \mathbb{N}$. Therefore $f|_a \notin \mathcal{F}^*_{\mathcal{P}}(\downarrow a)$, which is a contradiction. □

It is clear that if I is an ideal of the f-ring $\mathcal{F}_{\mathcal{P}}(L)$, then $\mathrm{coz}(I)$ is an ideal of $B(L)$.

Corollary 4.15. *For every $f, g \in \mathcal{F}_{\mathcal{P}}(L)$, if $\mathrm{coz}(f) \leq \mathrm{coz}(g)$ then there exists an element h of $\mathcal{F}_{\mathcal{P}}(L)$ such that $f = gh$.*

Proof. Consider $f, g \in \mathcal{F}_{\mathcal{P}}(L)$ and I is the ideal generated by g. Since $\mathrm{coz}(I)$ is an ideal of $B(L)$ and $\mathrm{coz}(f) \leq \mathrm{coz}(g) \in \mathrm{coz}(I)$, we conclude from Proposition 4.13 that $f \in I$, which implies that there exists an element h of $\mathcal{F}_{\mathcal{P}}(L)$ such that $f = gh$. □

If A is an ideal of frame L then $coz^{\leftarrow}(A) := \{f \in \mathcal{F}_{\mathcal{P}}(L) : \mathrm{coz}(f) \in A\}$ is an ideal of $\mathcal{F}_{\mathcal{P}}(L)$.

Proposition 4.16. *If I is a free proper ideal in $\mathcal{F}_{\mathcal{P}}(L)$ then $f(-\frac{1}{n}, \frac{1}{n}) \notin \operatorname{coz}(I)$ for every $f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$ and every $n \in \mathbb{N}$.*

Proof. Consider $f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$ and $n \in \mathbb{N}$. From

$$\top = \bigvee \operatorname{coz}(I) = \bigvee \Big\{ \operatorname{coz}(g) \vee f(-\frac{1}{n}, \frac{1}{n}) : g \in I \Big\}$$

and $\uparrow f(-\frac{1}{n}, \frac{1}{n})$ is compact, we conclude that there exists an element g of I such that $\top = \operatorname{coz}(g) \vee f(-\frac{1}{n}, \frac{1}{n})$. If $f(-\frac{1}{n}, \frac{1}{n}) \in \operatorname{coz}(I)$, then $\top \in \operatorname{coz}(I)$, i.e., $I = L$, which is a contradiction. Hence $f(-\frac{1}{n}, \frac{1}{n}) \notin \operatorname{coz}(I)$. □

5. Locally compact frames

In this section, we consider $\mathfrak{C} := \{a \in L : \uparrow a^* \text{ is a compact frame}\}$ and $\mathfrak{c} := \bigvee \mathfrak{C}$. We show that if $\mathcal{F}_{\mathcal{P}_\infty}(L) \neq (0)$, then $\downarrow \mathfrak{c}$ is a locally compact frame and

$$\bigvee_{\varphi \in \mathcal{F}_{\mathcal{P}_\infty}(L)} coz(\varphi) = \mathfrak{c} = \bigvee_{\varphi \in \mathcal{F}_{\mathcal{P}_K}(L)} coz(\varphi).$$

Next, we prove that $\mathcal{F}_{\mathcal{P}_\infty}(L) \cong \mathcal{F}_{\mathcal{P}_\infty}(\downarrow \mathfrak{c})$ if $\mathfrak{c}$ is complemented.

Proposition 5.1. *The following statements hold.*

(1) $\mathfrak{c} = \bigvee \operatorname{coz}(\mathcal{F}_{\mathcal{P}_\infty}(L))$.
(2) *If $\mathcal{F}_{\mathcal{P}_\infty}(L) \neq (0)$ then $\mathcal{F}_{\mathcal{P}_K}(L) \neq (0)$ and $\mathfrak{c} = \bigvee \operatorname{coz}(\mathcal{F}_{\mathcal{P}_K}(L))$.*

Proof. (1). Consider $f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$. For every $n \in \mathbb{N}$, we put $v_n = f(-\infty, \dfrac{-1}{n}] \vee \varphi[\dfrac{1}{n}, +\infty)$. From $f(\dfrac{-1}{n}, \dfrac{1}{n}) = v_n'$ and $\uparrow f(\dfrac{-1}{n}, \dfrac{1}{n})$ is a compact frame, we conclude that $v_n \in \mathfrak{C}$ for every $n \in \mathbb{N}$. Then $\operatorname{coz}(f) = \bigvee_{n \in \mathbb{N}} v_n \leq \mathfrak{c}$, it implies that $\bigvee_{f \in \mathcal{F}_{\mathcal{P}_\infty}(L)} \operatorname{coz}(f) \leq \mathfrak{c}$. Now, assume that $a \in \mathfrak{C}$ and $\{f_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{F}_{\mathcal{P}}(L)$ with $a = \bigvee_{\lambda \in \Lambda} \operatorname{coz}(f_\lambda)$. From $a^* \leq \operatorname{coz}(f_\lambda)^*$ and $\uparrow a^*$ is a compact frame, we conclude that $\uparrow z(f_\lambda)$ is a compact frame for every $\lambda \in \Lambda$. Hence, $\{f_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{F}_{\mathcal{P}_K}(L) \subseteq \mathcal{F}_{\mathcal{P}_\infty}(L)$ and $a \leq \bigvee coz(\mathcal{F}_{\mathcal{P}_\infty}(L))$, which implies that $\mathfrak{c} \leq \bigvee_{f \in \mathcal{F}_{\mathcal{P}_\infty}(L)} \operatorname{coz}(f)$, and hence $\mathfrak{c} = \bigvee_{f \in \mathcal{F}_{\mathcal{P}_\infty}(L)} \operatorname{coz}(f)$.

(2). The proof is similar to the part (1). □

From the Proposition 5.1, we conclude the following corollary.

Corollary 5.2. $\mathcal{F}_{\mathcal{P}_\infty}(L) \neq (0)$ *if and only if* $\mathfrak{C} \neq \{\bot\}$ *if and only if* $\mathcal{F}_{\mathcal{P}_K}(L) \neq (0)$.

Remark 5.3. Consider $a \in \mathfrak{C}$ and $f \in \mathcal{F}_{\mathcal{P}}(L)$. Since $\uparrow a^*$ is a compact frame and

$$\top = \bigvee_{p,q \in \mathbb{Q}} f(p,q) = \bigvee_{p,q \in \mathbb{Q}} f(p,q) \vee a^*,$$

we conclude that there exist $p, q \in \mathbb{Q}$ such that $f(p,q) \vee a^* = \top$, which follows that $a \prec f(p,q)$. Therefore, for any $a \in \mathfrak{C}$ and any $f \in \mathcal{F}_{\mathcal{P}}(L)$ there exist $p, q \in \mathbb{Q}$ such that $a \prec\!\!\prec f(p,q)$.

Remark 5.4. Let J be a free ideal of $\mathcal{F}_{\mathcal{P}}(L)$ and $a \in \mathfrak{C}$. Since $\uparrow a^*$ is a compact frame and

$$\top = \bigvee_{f \in J} \operatorname{coz}(f) = \bigvee_{f \in J} \operatorname{coz}(f) \vee a^*,$$

we conclude that there exists an element f of J such that $\operatorname{coz}(f) \vee a^* = \top$. Hence, if J is a free ideal of $\mathcal{F}_{\mathcal{P}}(L)$ or $\mathcal{F}^*_{\mathcal{P}}(L)$, then for every $a \in \mathfrak{C}$, there exists an element f of J such that $a \prec\!\!\prec \operatorname{coz}(f)$.

Lemma 5.5. *The following statements hold.*

(1) $\mathfrak{C}$ *is an ideal of* L.
(2) *If* $x \prec a$ *then* $x \ll a$ *for every* $(x, a) \in L \times \mathfrak{C}$.
(3) *For any* $a \in \mathfrak{C}$, $a = \bigvee_{x \ll a} x$.

Proof. (1). Consider $a, b \in L$ such that $b \leq a$ and $a \in \mathfrak{C}$. From $\uparrow a^*$ is a compact frame and $a^* \leq b^*$, we conclude that $\uparrow b^*$ is a compact frame, which implies that $b \in \mathfrak{C}$. Hence M is a down set in L. Also, for $a, b \in \mathfrak{C}$, $\uparrow(a \vee b)^* = \uparrow a^* \wedge b^*$ is a compact frame, which implies that $a \vee b \in \mathfrak{C}$, that implies $\mathfrak{C}$ is an up directed subset of L, Therefore, $\mathfrak{C}$ is an ideal of L.

(2). Consider $(x, a) \in L \times \mathfrak{C}$ with $x \prec a$. If $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq L$ such that $a \leq \bigvee \{a_\lambda\}_{\lambda \in \Lambda}$, then

$$\bigvee_{\lambda \in \Lambda} (x^* \vee a_\lambda) = x^* \vee (\bigvee_{\lambda \in \Lambda} a_\lambda) = x^* \vee a = \top.$$

From the first statement we conclude $x \in \mathfrak{C}$, and hence $\uparrow x^*$ is a compact frame. Since $\{(x^* \vee a_\lambda\}_{\lambda \in \Lambda} \subseteq \uparrow x^*$, we infer that there are $\lambda_1, \lambda_2 ... \lambda_n \in \Lambda$ such that $\top = x^* \vee (\bigvee_{i=1}^k a_{\lambda_i})$, which implies that $x \leq (\bigvee_{i=1}^k a_{\lambda_i})$. Hence $x \ll a$.

(3). Consider $a \in \mathfrak{C}$. Since L is a completely regular frame, we conclude from the statement (2) that $a = \bigvee_{x \prec a} x = \bigvee_{x \ll a} x$ and so, the proof is now complete. □

Proposition 5.6. *If* $\mathcal{F}_{\mathcal{P}_\infty}(L) \neq (0)$, *then* $\downarrow \mathfrak{c}$ *is a locally compact frame.*

Proof. Consider $a \in \downarrow \mathfrak{c}$. Then $a = \bigvee_{m \in \mathfrak{C}} (a \wedge m)$. By Lemma 5.5, $a \wedge m \in \mathfrak{C}$ and $a \wedge m = \bigvee_{x \ll a \wedge m} x \leq a$ for every $m \in \mathfrak{C}$. Hence $a = \bigvee_{x \ll a} x$. This completes the proof. □

Consider $S \subseteq \mathfrak{C}$ and $a \in \mathfrak{C}$ is an upper bound of S. Since $\bigvee S \leq a$, we conclude that $\bigvee S \in \mathfrak{C}$. Therefore, if $S \subseteq \mathfrak{C}$ is bounded in $\mathfrak{C}$, then $\bigvee S \in \mathfrak{C}$.

6. The relation between the generated subframe by $\mathrm{coz}(\mathcal{F}_{\mathcal{P}_\infty}(L))$ and $\mathrm{coz}(\mathcal{F}_{\mathcal{P}_K}(L))$ in L

In this section, we show that $\mathrm{coz}(\mathcal{F}_{\mathcal{P}_K}(L))$ and $\mathrm{coz}(\mathcal{F}_{\mathcal{P}_\infty}(L))$ are the bases of $\downarrow\mathfrak{c}$.

Lemma 6.1. *If* $\mathcal{F}_{\mathcal{P}_\infty}(L) \neq (0)$ *then the following statements hold.*

(1) *For any* $f \in F_{\mathcal{P}}(L)$, *if* $\mathrm{coz}(f) \leq \mathfrak{c}$ *then there is a subset* $\{f_\lambda\}_{\lambda \in \Lambda}$ *of* $\mathcal{F}_{\mathcal{P}_K}(L)$ *such that* $\mathrm{coz}(f) = \bigvee_{\lambda \in \Lambda} \mathrm{coz}(f_\lambda)$.
(2) *For any* $f \in \mathcal{F}_{\mathcal{P}}(L)$, *if* $\mathrm{coz}(f) \leq \mathfrak{c}$ *then there is a subset* $\{f_\lambda\}_{\lambda \in \Lambda}$ *of* $\mathcal{F}_{\mathcal{P}_\infty}(L)$ *such that* $\mathrm{coz}(f) = \bigvee_{\lambda \in \Lambda} \mathrm{coz}(f_\lambda)$.

Proof. (1). Consider $f \in \mathcal{F}_{\mathcal{P}}(L)$ with $\mathrm{coz}(f) \leq \mathfrak{c}$. we have

$$\begin{aligned}
\mathrm{coz}(f) &= \mathrm{coz}(f) \wedge \mathfrak{c} \\
&= \mathrm{coz}(f) \wedge (\bigvee_{g \in \mathcal{F}_{\mathcal{P}_K}(L)} \mathrm{coz}(g)), && \text{by Proposition 5.1} \\
&= \bigvee_{g \in \mathcal{F}_{\mathcal{P}_K}(L)} (\mathrm{coz}(f) \wedge \mathrm{coz}(g)) \\
&= \bigvee_{g \in \mathcal{F}_{\mathcal{P}_K}(L)} \mathrm{coz}(fg).
\end{aligned}$$

Since, by Lemma 3.5, $\mathcal{F}_{\mathcal{P}_K}(L)$ is an ideal of $\mathcal{F}_{\mathcal{P}}(L)$, we conclude that $fg \in \mathcal{F}_{\mathcal{P}_K}(L)$ for every $g \in \mathcal{F}_{\mathcal{P}_K}(L)$ and every $f \in \mathcal{F}_{\mathcal{P}}(L)$.

(2). By the first statement, it is clear. □

A base B of a frame L is a subset of L such that every element of L is a join of elements of B.

Proposition 6.2. *If $\mathcal{F}_{\mathcal{P}_\infty}(L) \neq (0)$ then the following statements hold.*

(1) $\operatorname{coz}(\mathcal{F}_{\mathcal{P}_K}(L))$ *is a base of* $\downarrow\mathfrak{c}$.
(2) $\operatorname{coz}(\mathcal{F}_{\mathcal{P}_\infty}(L))$ *is a base of* $\downarrow\mathfrak{c}$.

Proof. (1). Consider $x \leq \mathfrak{c}$ and $\{f_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{F}_{\mathcal{P}}(L)$ with $x = \bigvee_{\lambda\in\Lambda} \operatorname{coz}(f_\lambda)$. Since $\operatorname{coz}(f_\lambda) \leq x \leq \mathfrak{c}$. Lemma 6.1 implies that there exists a subset B_λ of $\mathcal{F}_{\mathcal{P}_K}(L)$ such that $\operatorname{coz}(f_\lambda) = \bigvee_{g\in B_\lambda} \operatorname{coz}(g)$ for every $\lambda \in \Lambda$. We put $B = \bigcup_{\lambda\in\Lambda} B_\lambda$ then $B \subseteq \mathcal{F}_{\mathcal{P}_K}(L)$ and $x = \bigvee_{g\in B} \operatorname{coz}(g)$. The proof is now complete.

(2). By the first statement, it is clear. □

By Proposition 6.2, we have the following Corollary.

Corollary 6.3. *The subframes produced by* $\operatorname{coz}(\mathcal{F}_{\mathcal{P}_\infty}(L))$ *and* $\operatorname{coz}(\mathcal{F}_{\mathcal{P}_K}(L))$ *in* L *are the same.*

7. The relationship between $\mathcal{F}_{\mathcal{P}_\infty}(L)$ and $\mathcal{F}_{\mathcal{P}_\infty}(\downarrow \mathfrak{c})$

In this section, we assume that $\mathcal{F}_{\mathcal{P}_\infty}(L) \neq (0)$ and $\mathfrak{c} = \bigvee \mathfrak{C}$.

Lemma 7.1. *The map* $\theta : \mathcal{F}_{\mathcal{P}}(L) \to \mathcal{F}_{\mathcal{P}}(\downarrow\mathfrak{c})$ *given by* $\theta(f) = f|\mathfrak{c}$ *is an* f*-ring homomorphism.*

Proof. Straightforward. □

Lemma 7.2. *If* $f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$ *then* $f(r,s) \vee \mathfrak{c} = \top$ *for every* $r, s \in \mathbb{R}$ *with* $r < 0 < s$.

Proof. Consider $f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$ and $r, s \in \mathbb{R}$ with $r < 0 < s$. There exists an element n of $\mathbb{N}$ such that $(\frac{-1}{n}, \frac{1}{n}) \leq (r,s)$. Since $\uparrow\ f(\frac{-1}{n}, \frac{1}{n})$ is a compact frame, we infer that $f(-\infty, \dfrac{-1}{n}] \vee f[\dfrac{1}{n}, +\infty) \in \mathfrak{C}$, which implies that

$$f(r,s) \vee \mathfrak{c} \geq f(\frac{-1}{n}, \frac{1}{n}) \vee \mathfrak{c} \geq f(\frac{-1}{n}, \frac{1}{n}) \vee f(-\infty, \frac{-1}{n}] \vee f[\frac{1}{n}, +\infty) = \top.$$

The proof is now completed. □

For every $a, b \in L$, we put $[a,b] := \{x \in L : a \leq x \leq b\}$. Consider $0 \neq f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$, $r, s \in \mathbb{R}$ with $r < 0 < s$ and $S \subseteq [f(r,s) \wedge \mathfrak{c}, \mathfrak{c}]$ with $\bigvee S = \mathfrak{c}$. By the Lemma 7.2,

$$\top = \mathfrak{c} \vee f(r,s) = \bigvee_{x\in S} (x \vee f(r,s)).$$

Consider $n \in \mathbb{N}$ such that $(\frac{-1}{n}, \frac{1}{n}) \leq (r,s)$. From $\uparrow f(\frac{-1}{n}, \frac{1}{n})$ is a compact frame, we conclude that $\uparrow f(r,s)$ is a compact frame, it implies that there exist $x_1, x_2, ..., x_k \in S$ such that $\top = f(r,s) \vee \bigvee_{i=1}^{k} x_i$. Since $x_i \in S \subseteq [f(r,s) \wedge \mathfrak{c}, \mathfrak{c}]$, we have

$$\mathfrak{c} = (\mathfrak{c} \wedge f(r,s)) \vee (\bigvee_{i=1}^{k} (\mathfrak{c} \wedge x_i)) = \bigvee_{i=1}^{k} x_i \leq \bigvee S = \mathfrak{c}.$$

Therefore $[f(r,s) \wedge \mathfrak{c}, \mathfrak{c}]$ is a compact frame. Hence $f|\mathfrak{c} \in \mathcal{F}_{\mathcal{P}_\infty}(\downarrow\mathfrak{c})$, which implies that

$$\theta_\infty = \theta|_{\mathcal{F}_{\mathcal{P}_\infty}(L)} : \mathcal{F}_{\mathcal{P}_\infty}(L) \to \mathcal{F}_{\mathcal{P}_\infty}(\downarrow\mathfrak{c})$$

is an f-ring homomorphism. If $f \in \ker\theta_\infty$, then $f|_{\mathfrak{c}}(-\frac{1}{n}, \frac{1}{n}) = f(-\frac{1}{n}, \frac{1}{n}) \wedge \mathfrak{c} = \mathfrak{c}$, therefore $f(-\frac{1}{n}, \frac{1}{n}) \geq \mathfrak{c}$ for any $n \in \mathbb{N}$. By Lemma 7.2, $f(-\frac{1}{n}, \frac{1}{n}) = f(-\frac{1}{n}, \frac{1}{n}) \vee \mathfrak{c} = \top$ for any $n \in \mathbb{N}$. We show that $f = \mathbf{0}$. C $0 \neq x \in \mathbb{R}$, there is an element m in $\mathbb{N}$, such that $x \notin (-\frac{1}{m}, \frac{1}{m})$, we infer that

$$f(\{x\}) = f(\{x\}) \wedge \top = f(\{x\}) \wedge f(-\frac{1}{m}, \frac{1}{m}) = \bot.$$

We infer that $f = \mathbf{0}$. Hence, we have

Proposition 7.3. *The map*

$$\theta_\infty := \theta|_{\mathcal{F}_{\mathcal{P}_\infty}(L)} : \mathcal{F}_{\mathcal{P}_\infty}(L) \to \mathcal{F}_{\mathcal{P}_\infty}(\downarrow\mathfrak{c})$$

is an f-ring monomorphism.

In what follows, for every $f \in \mathcal{F}_{\mathcal{P}}(\downarrow\mathfrak{c})$, we define the real-trail $\hat{t}_f : \mathbb{R} \longrightarrow L$ on L by

$$\hat{t}_f(x) = \begin{cases} f(\{x\}) \vee \mathfrak{c}^* & \text{if } x = 0 \\ f(\{x\}) & \text{if } x \neq 0. \end{cases}$$

Lemma 7.4. *If $\mathfrak{c}$ is complemented and $f \in \mathcal{F}_{\mathcal{P}}(\downarrow\mathfrak{c})$ then the following statements hold.*

(1) $\mathrm{coz}(\varphi_{\hat{t}_f}) = coz(f)$ *and* $z(\varphi_{\hat{t}_f}) = z(f) \vee \mathfrak{c}'$.
(2) $\varphi_{\hat{t}_f}|_{\mathfrak{c}} = f$.
(3) $f \in \mathcal{F}_{\mathcal{P}_\infty}(\downarrow\mathfrak{c})$ *if and only if* $\varphi_{\hat{t}_f} \in \mathcal{F}_{\mathcal{P}_\infty}(L)$.

Proof. (1) and (2) are clear.
(3). If $f \in \mathcal{F}_{\mathcal{P}_\infty}(\downarrow\mathfrak{c})$, then $[f(-\frac{1}{n}, \frac{1}{n}), \mathfrak{c}]$ is compact, for every $n \in \mathbb{N}$. Hence $\uparrow (f(-\frac{1}{n}, \frac{1}{n}) \vee \mathfrak{c}') = \uparrow\varphi_{\hat{t}_f}(-\frac{1}{n}, \frac{1}{n})$ is compact for every $n \in \mathbb{N}$, therefore $\varphi_{\hat{t}_f} \in \mathcal{F}_{\mathcal{P}_\infty}(L)$. Conversely, if $\varphi_{\hat{t}_f} \in \mathcal{F}_{\mathcal{P}_\infty}(L)$ then, by the second statement and Proposition 7.3, $\varphi_{\hat{t}_f}|_{\mathfrak{c}} = f \in \mathcal{F}_{\mathcal{P}_\infty}(L)$. □

Proposition 7.5. *If $\mathfrak{c}$ is complemented, then*

$$\theta_\infty := \theta|_{\mathcal{F}_{\mathcal{P}_\infty}(L)} : \mathcal{F}_{\mathcal{P}_\infty}(L) \to \mathcal{F}_{\mathcal{P}_\infty}(\downarrow\mathfrak{c})$$

is an f-ring isomorphism.

Proof. By Proposition 7.3 and lemma 7.4, θ_∞ is an f-ring isomorphism. □

Proposition 7.6. *If $\mathfrak{c}$ is complemented, then there is a locally compact frame L' such that $\mathcal{F}_{\mathcal{P}_\infty}(L) \cong \mathcal{F}_{\mathcal{P}_\infty}(L')$.*

Proof. We consider $L' = \downarrow\mathfrak{c}$, by Propositions 5.6 and 7.5, it is obvious. □

Lemma 7.7. *If $\mathfrak{c}$ is complemented, then $f \in \mathcal{F}_{\mathcal{P}_K}(\downarrow\mathfrak{c})$ if and only if $\varphi_{\hat{t}_f} \in \mathcal{F}_{\mathcal{P}_K}(L)$.*

Proof. $f \in \mathcal{F}_{\mathcal{P}_K}(\downarrow\mathfrak{c})$ if and only if $[z(f), \mathfrak{c}]$ is compact if and only if $\uparrow(z(f) \vee \mathfrak{c}')$ is compact if and only if $\uparrow z(\varphi_{\hat{t}_f})$ is compact, by Lemma 7.4, if and only if $\varphi_{\hat{t}_f} \in \mathcal{F}_{\mathcal{P}_K}(L)$. □

Proposition 7.8. *If $\mathfrak{c}$ is complemented, then*

$$\theta_K := \theta|_{\mathcal{F}_{\mathcal{P}_K}(L)} : \mathcal{F}_{\mathcal{P}_K}(L) \to \mathcal{F}_{\mathcal{P}_K}(\downarrow\mathfrak{c})$$

is an f-ring isomorphism.

Proof. By Proposition 7.5 and Lemma 7.7, θ_K is an f-ring isomorphism. □

Proposition 7.9. *If $\mathfrak{c}$ is complemented, then there is a locally compact frame L' such that $\mathcal{F}_{\mathcal{P}_K}(L) \cong \mathcal{F}_{\mathcal{P}_K}(L')$.*

Proof. Put $L' = \downarrow\mathfrak{c}$. □

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