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Research Article

Relative bi-ideals and relative quasi ideals in ordered semigroups

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Abstract

In this paper, after introducing the notion of relative bi-ideals and relative quasi ideals in ordered semigroups, some important properties of these bi-ideals and quasi ideals are studied. Then relatively prime and relatively weakly semiprime bi-ideals are defined and some vital results have been proved. We also define relative regularity and relative intra-regularity of an ordered semigroup and prove some results based on the connection among intra-regularity of an ordered semigroup, relative quasi and relative bi-ideals of that ordered semigroup. Finally some important results connecting relative regularity, relatively prime bi-ideals and relatively weakly semiprime bi-ideals of an ordered semigroup have also been obtained.

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1. Introduction and preliminaries

In 1952, Good and Hughes [6], first defined the notion of a bi-ideal of an ordered semigroup. Thereafter, the concept of a quasi-ideal was introduced in 1953 by Steinfeld in [18, 19] for Rings and Semigroups. A.P.J. Van der Walt, in his paper [21], introduced the notions of a prime and a semiprime bi-ideal of an associative ring with unity. Later H J le Roux [15], proved various results by using prime and semiprime bi-ideals of associative rings without unity while N. Kehayopulu [10] derived the notion of regularity of an ordered semigroup. In 1978, S. Lajos and G. Szasz [12, 13] characterized intra-regular semigroups in terms of right and left ideals of semigroups and, in [11], N. Kehayopulu, S. Lajos and M. Singelis derived the ordered version of intra-regularity in terms of left and right ideals.

In 1962, Wallace [22], introduced the notion of relative ideals (H-ideals) on semigroup S. ln 1967, Hrmová [16] generalized the notion of H-ideal by introducing the notion of an (H_1, H_2)-ideal of a semigroup S $(H, H_1, H_2 \subseteq S)$. The notion of prime and weakly prime ideals in semigroups had been considered by Szász in [20] and proved vital results. In 1992, Kehayopulu generalized these results in [8, 9] for ordered semigroups.

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An ordered semigroup is a semigroup with $(S, \leq)$ as an ordered set satisfying

$$(\forall s_1, s_2 \in S)(\forall x \in S)(s_1 \leq s_2 \;\Rightarrow\; s_1 x \leq s_2 x \text{ and } x s_1 \leq x s_2).$$

Following definitions and results have been introduced by M. F. Ali et al. in [1], as a generalization of notions studied by Wallace [22], Hrmová [16] and Kehayopulu [8, 9]. For more details of ordered semigroups and their related notions, the reader is referred to [2–5, 7, 14].

Definition 1.1. Let S be an ordered semigroup and let A, T be any non-empty subsets of S. Then A is said to be a left T-ideal of S if $TA \subseteq A$ and $T \ni x \leq y \in A$ implies $x \in A$. Dually we can define a right T-ideal of S. Further A is said to be a T-ideal of S if it is both a left T-ideal and a right T-ideal of S. If $T = S$, then the notion of a left T-ideal (resp. a right T-ideal, a T-ideal) of S coincides with the notion of a left ideal (resp. a right ideal, an ideal) of S and, thus, shall be called as such in the sequel.

Remark 1.2. An ideal A of an ordered semigroup S is a T-ideal for each subset T of S, but the converse is not true in general.

Example 1.3. Let $S = \{a, b, c, d, \}$. Define a binary operation $(.)$ on S as shown in Table 1 of Section 5. Define an order on S as $\leq = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (a, c)\}$. Clearly S is an ordered semigroup. Let $A = \{a, b\}, B = \{a, d\}$ and $C = \{c, d\}$. It is easy to check that A is a B-ideal of S, but not an ideal of S.

Definition 1.4. Let A and T be any non-empty subsets of an ordered semigroup S. We define

$$(A]_T = \{t \in T \mid t \leq a, \text{ for some } a \in A\}.$$

The following lemma may easily be verified.

Lemma 1.5. *Let S be an ordered semigroup. Then*

(1) $A \subseteq (A]_T$ *for all* $A \subseteq T$.
(2) *If* $A \subseteq B \subseteq T$, *then* $(A]_T \subseteq (B]_T$.
(3) $(A]_T(B]_T \subseteq (AB]_T$.
(4) $((A]_T]_T = (A]_T$.
(5) *For each T-ideal* $A \subseteq T$, *we have* $(A]_T = A$.
(6) *If A, B are T-ideals of S such that* $A \cap B \neq \phi$, *then* $(AB]_T, A \cap B$ *are T-ideals of S.*
(7) *If T is subsemigroup of S, then* $(TaT]_T$ *is T-ideal of S for each* $a \in S$.

Definition 1.6. Let S be an ordered semigroup and let A_1, A_2 be any non-empty subsets of S. A non-empty subset A of S is said to be an (A_1, A_2)-ideal or a relative ideal of S if $A_1 A \subseteq A$, $AA_2 \subseteq A$ and $A_1 \cup A_2 \ni x \leq y$ for some $y \in A$ implies $x \in A$. If $A_1 = \phi$ or $A_2 = \phi$, then the (A_1, A_2)-ideal becomes one sided relative ideal of S. We denote the set of all (A_1, A_2)-ideals of S by $I(A_1, A_2)$.

In Example 1.3, A is a (B, C)-ideal of S.

Remark 1.7. From the definition of the (A_1, A_2)-ideal of S, it is clear that the notion of an (A_1, A_2)-ideal is the generalization of the notions of a left, a right and a two sided T-ideal of S.

The following lemmas may easily be proved.

Lemma 1.8. *Let S be an ordered semigroup. Then the following are true:*

(1) *If* $A_1 \subseteq A_1', A_2 \subseteq A_2'$, *then* $I(A_1', A_2') \subseteq I(A_1, A_2)$.
(2) $I(A_1, A_2) = I(A_1, \phi) \cap I(\phi, A_2)$.
(3) $\phi \in I(A_1, A_2)$ *if and only if* $A_1 = \phi$ *and* $A_2 = \phi$.
(4) $I(\phi, \phi) = \{A \mid A \subseteq S\}$.

Lemma 1.9. *Let S be an ordered semigroup and H_1, H_2 subsemigroups of S. Then the following are true:*

(1) $(H_1a]_H \in I(H_1, \phi)$ *for each* $a \in S$.
(2) $(aH_2]_H \in I(\phi, H_2)$ *for each* $a \in S$.
(3) $(H_1aH_2]_H \in I(H_1, H_2)$ *for each* $a \in S$.
(4) *If* $L \in I(H_1, \phi)$ *and* $R \in I(\phi, H_2)$, *then* $(LR]_H \in I(H_1, H_2)$.
(5) *If* $A, B \in I(H_1, H_2)$ *such that* $A \cap B \neq \phi$, *then* $(AB]_H, A \cap B \in I(H_1, H_2)$.

Definition 1.10. Let S be an ordered semigroup and let H_1, H_2 be any non-empty subsets of S. Then a non-empty subset T of S is said to be an (H_1, H_2)-prime ideal of S if

(1) T is an (H_1, H_2)-ideal of S; and
(2) For any $A, B \subseteq H_1 \cup H_2$ such that $AB \subseteq T$, either $A \subseteq T$ or $B \subseteq T$.

Definition 1.11. Let S be an ordered semigroup and let H_1, H_2 be any non-empty subsets of S. Then a non-empty subset T of S is said to be an (H_1, H_2)-weakly prime ideal of S if

(1) T is an (H_1, H_2)-ideal of S; and
(2) For all (H_1, H_2)-ideals $A, B \subseteq H_1 \cup H_2$ such that $AB \subseteq T$, either $A \subseteq T$ or $B \subseteq T$.

Definition 1.12. Let S be an ordered semigroup and let H_1, H_2 be any non-empty subsets of S. A non-empty subset T of S is said to be an (H_1, H_2)-semiprime ideal of S if

(1) $T \in I(H_1, H_2)$; and
(2) $A \subseteq H_1 \cup H_2$ such that $A^2 \subseteq T$ implies $A \subseteq T$.

2. Relative bi-ideals in ordered semigroups

In this section, we introduce the notion of relatively prime, weakly prime and semiprime bi-ideals in ordered semigroups. We also give some characterizations of relative regular ordered semigroups in terms of aforesaid bi-ideals of ordered semigroups.

Definition 2.1. Let S be an ordered semigroup and let H, T be any non-empty subsets of S. Then T is said to be an H-bi-ideal of S if

(1) $THT \subseteq T$; and
(2) for all $t \in T$, $H \ni h \leq t \Rightarrow h \in T$.

Definition 2.2. Let S be an ordered semigroup and H_1, H_2 be any non-empty subsets of S. Then $T(\neq \phi)$ is said to be an (H_1, H_2)-bi-ideal or a relative bi-ideal of S if

(1) $T(H_1 \cup H_2)T = TH_1T \cup TH_2T \subseteq T$; and
(2) for all $t \in T$, $H_1 \cup H_2 \ni h \leq t \Rightarrow h \in T$.

The set of all relative bi-ideals of S shall be denoted, in whatever follows, by $\mathcal{B}(H_1, H_2)$.

Remark 2.3. It is easy to check that each bi-ideal B of an ordered semigroup S is an (H_1, H_2)-bi-ideal of S for each subset H_1, H_2 of S, but the converse is not true in general.

Example 2.4. Let $S = \{a, b, c, d, \}$. Define a binary operation $(.)$ on S as shown in Table 2 of Section 5. Define an order on S as $\leq = \{(a, a), (b, b), (c, c), (d, d), (a, d)\}$. Clearly S is an ordered semigroup. Let $B = \{a, b\}, H_1 = \{a, c\}$ and $H_2 = \{b, c\}$. Then $H_1 \cup H_2 = \{a, b, c\}$. It is easy to check that B is an (H_1, H_2)-bi-ideal of S, but not a bi-ideal of S.

Definition 2.5. Let S be an ordered semigroup and let H_1, H_2 be any non-empty subsets of S. Then $T(\neq \phi)$ is said to be an (H_1, H_2)-prime bi-ideal of S if

(1) $T \in \mathcal{B}(H_1, H_2)$; and
(2) $h_1(H_1 \cup H_2)h_2 = h_1H_1h_2 \cup h_1H_2h_2 \subseteq T \Rightarrow$ either $h_1 \in T$ or $h_2 \in T$.

Equivalently, $C, D \subseteq H = H_1 \cup H_2$ such that $CHD \subseteq T$ implies either $C \subseteq T$ or $D \subseteq T$.

Definition 2.6. Let S be an ordered semigroup and let H_1, H_2 be any non-empty subsets of S. Then $T(\neq \phi)$ is said to be (H_1, H_2)-semiprime bi-ideal of S if

(1) $T \in \mathcal{B}(H_1, H_2)$; and
(2) $h(H_1 \cup H_2)h = hH_1h \cup hH_2h \subseteq T \Rightarrow h \in T$.

Equivalently, $C \subseteq H = H_1 \cup H_2$ such that $CHC \subseteq T$ implies $C \subseteq T$.

Definition 2.7. Let S be an ordered semigroup and $H \subseteq S$. Then S is called left H-regular (resp. right H-regular) if $\forall a \in H\ \exists\ h \in H$ such $a \leq ha^2$ (resp. $a \leq a^2h$). Equivalently,

(1) $a \in (Ha^2]_H$ (resp. $a \in (a^2H]_H$) $\forall a \in H$; and
(2) $A \subseteq (HA^2]_H$ (resp. $A \subseteq (A^2H]_H$) $\forall A \subseteq H$.

Definition 2.8. Let S be an ordered semigroup and $H \subseteq S$. Then S is called H-regular if $\forall\ a \in H\ \exists\ h \in H$ such that $a \leq aha$. Equivalently,

(1) $a \in (aHa]_H\ \forall a \in H$; and
(2) $A \subseteq (AHA]_H\ \forall A \subseteq H$.

Definition 2.9. Let S be an ordered semigroup and let $H_1, H_2 \subseteq S$. Then S is called (H_1, H_2)-regular if $\forall\ a \in H\ \exists\ h \in (H_1 \cup H_2)$ such that $a \leq aha$. Equivalently,

(1) $a \in (aHa]_H\ \forall a \in H = (H_1 \cup H_2)$; and
(2) $A \subseteq (AHA]_H\ \forall A \subseteq H$.

The following example shows that an (H_1, H_2)-regular ordered semigroup is not regular in general.

Example 2.10. Let $S = \{a, b, c, d, e\}$. Define a binary operation (.) on S as shown in Table 3 of Section 5. Define an order relation on S as $\leq = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, d), (a, e)\}$. Clearly S is an ordered semigroup. The covering relation and the figure of S (as shown in Figure 1 of Section 5) is given as follows: $\preccurlyeq$: {(a,b),(a,c),(a,d),(a,e)}.

As for $e \in S$, $\nexists$ any $x \in S$ such that $e \leq exe$, S is not regular. But, for $H_1 = \{a, b\}$ and $H_2 = \{c, d\}$, it may be easily checked that S is an (H_1, H_2)-regular ordered semigroup.

Theorem 2.11. *Let S be an ordered semigroup and let H be a subsemigroup of S. For any H-ideal T of S, the following are equivalent:*

(1) *T is H-weakly prime.*
(2) *If $a, b \in H$ such that $(aHb]_H \subseteq T$, then either $a \in T$ or $b \in T$.*
(3) *If $a, b \in H$ such that $I_R(a)I_R(b) \subseteq T$, then either $a \in T$ or $b \in T$.*
(4) *If A, B are left H-ideals of S such that $AB \subseteq T$, then either $A \subseteq T$ or $B \subseteq T$.*
(5) *If A, B are right H-ideals of S such that $AB \subseteq T$, then either $A \subseteq T$ or $B \subseteq T$.*
(6) *If A is a right H-ideal and B is a left H-ideal of S such that $AB \subseteq T$, then either $A \subseteq T$ or $B \subseteq T$.*

Proof. (1) $\Rightarrow$ (2) Let T be H-weakly prime. Take any $a, b \in H$ such that $(aHb]_H \subseteq T$. Then

$$\begin{aligned}(HaH]_H(HbH]_H &\subseteq (HaH^2bH]_H \\ &\subseteq (H(aHb)H]_H \\ &\subseteq (H(aHb]_HH]_H \\ &\subseteq (HTH]_H \\ &\subseteq (T]_H = T.\end{aligned}$$

Since T is H-weakly prime, either $(HaH]_H \subseteq T$ or $(HbH]_H \subseteq T$. Let $(HaH]_H \subseteq T$. Then

$$\begin{aligned}(I_R(a))^3 &= (a \cup Ha \cup aH \cup HaH]_H^3 \\ &\subseteq ((a \cup Ha \cup aH \cup HaH)^2]_H(a \cup Ha \cup aH \cup HaH]_H \\ &\subseteq (Ha \cup HaH]_H(a \cup Ha \cup aH \cup HaH]_H \\ &\subseteq ((Ha \cup HaH)(a \cup Ha \cup aH \cup HaH)]_H \\ &\subseteq (HaH]_H \subseteq T.\end{aligned}$$

So, we have

$$((I_R(a))^2]_H I_R(a) = ((I_R(a))^2]_H (I_R(a)]_H \subseteq ((I_R(a))^3]_H \subseteq (T]_H = T.$$

Since T is H-weakly prime and $((I_R(a))^2]_H$ is an H-ideal of S, either $((I_R(a))^2]_H \subseteq T$ or $I_R(a) \subseteq T$. If $I_R(a) \subseteq T$, then $a \in I_R(a) \subseteq T$. Let $((I_R(a))^2]_H \subseteq T$. Then $(I_R(a))^2 \subseteq T$. Since T is H-semiprime, $I_R(a) \subseteq T$ and, so, $a \in T$. Simlarly we may prove that if $(HbH]_H \subseteq T$, then $b \in T$.

(2) $\Rightarrow$ (3) Take any $a, b \in H$ such that $I_R(a)I_R(b) \subseteq T$. Then

$$(a]_H)(Hb]_H \subseteq ((a \cup Ha \cup aH \cup HaH]_H)((b \cup Hb \cup bH \cup HbH]_H) \subseteq T.$$

and so

$$(aHb]_H \subseteq (((a]_H)(Hb]_H]_H \subseteq (T]_H = T.$$

By (2), we have either $a \in T$ or $b \in T$, as required.

(3) $\Rightarrow$(4) Let $A, B \subseteq H$ and A, B be right H-ideals of S such that $AB \subseteq T$ and $A \nsubseteq T$. Let $a \in A, a \notin T$ and $b \in B$. Then

$$\begin{aligned}I_R(a) &= (a \cup Ha \cup aH \cup HaH]_H \\ &\subseteq (A \cup HA \cup AH \cup HAH]_H \\ &\subseteq (A \cup HA]_H.\end{aligned}$$

Similarly $I_R(b) \subseteq (B \cup HB]_H$. Now

$$\begin{aligned}I_R(a)I_R(b) &\subseteq ((A \cup HA]_H)((B \cup HB]_H) \\ &\subseteq ((A \cup HA)(B \cup HB)]_H \\ &= (AB \cup AHB \cup HAB \cup HAHB]_H \\ &\subseteq (AB \cup HAB]_H \\ &\subseteq (T \cup HT]_H \subseteq (T]_H = T.\end{aligned}$$

By (3), either $I_R(a) \subseteq T$ or $I_R(b) \subseteq T$ which implies that $b \in T$ and, hence, $B \subseteq T$.

(3) $\Rightarrow$ (5) The proof follows on the lines similar to the above proof.

(3) $\Rightarrow$ (6) Take any right H-ideal A and any left H-ideal B of S such that $AB \subseteq T$, but $A \nsubseteq T$. Take any $a \in A$ such that $a \notin T$. For any $b \in B$, as $I_R(a) \subseteq (A \cup HA]_H$ and $I_R(b) \subseteq (B \cup HB \cup BH \cup HBH]_H \subseteq (B \cup BH]_H$, we have

$$\begin{aligned}I_R(a)I_R(b) &\subseteq ((A \cup HA)(B \cup BH)]_H \\ &= (AB \cup ABH \cup HAB \cup HABH]_H \\ &\subseteq (T \cup TH \cup HT \cup HTH]_H \\ &\subseteq (T]_H = T.\end{aligned}$$

By (3), either $I_R(a) \subseteq T$ or $I_R(b) \subseteq T$ which implies that $b \in T$ and, hence, $B \subseteq T$.

(4),(5) and (6) $\Rightarrow$ (1) are obvious. □

Theorem 2.12. *Let S be an ordered semigroup and let H be a subsemigroup of S. An H-ideal of S is H-weakly semiprime if and only if one of the four equivalent conditions holds in S.*

(1) *For every* $a \in H$ *such that* $(aHa]_H \subseteq T$, *we have* $a \in T$.
(2) *For* $a \in H$ *such that* $(I_R(a))^2 \subseteq T$, *we have* $a \in T$.
(3) *For right* H*-ideal* A *of* S *such that* $A^2 \subseteq T$, *we have* $A \subseteq T$.
(4) *For left* H*-ideal* B *of* S *such that* $B^2 \subseteq T$, *we have* $B \subseteq T$.

We shall, in the followings, extend the results proved, in [15], for an associative ring without unity and, in [17], for an ordered semigroup.

Proposition 2.13. *Let* S *be an ordered semigroup and let* H_1, H_2 *be subsemigroups of* S *such that* $H_2H_1 \subseteq H_1 \cup H_2$. *Let* $T \in I(H_1, H_2)$ *and* $T \in \mathcal{B}(H_1, H_2)$, *Then the* (H_1, H_2)*-bi-ideal* T *of* S *is* (H_1, H_2)*-prime if and only if* $RL \subseteq T$ *with* $R \in I(\phi, H_2), L \in I(H_1, \phi)$ *and* $R, L \subseteq H_1 \cup H_2$ *implies either* $R \subseteq T$ *or* $L \subseteq T$.

Proof. Let T be an (H_1, H_2)-prime bi-ideal of the ordered semigroup S and $RL \subseteq T$. Suppose $R \nsubseteq T$. For all $l \in L$ and $y \in R \backslash T$, we have $y(H_1 \cup H_2)l = yH_1l \cup yH_2l \subseteq RH_1L \cup RH_2L \subseteq RL \cup RL \subseteq RL \subseteq T$. As T is an (H_1, H_2)-prime bi-ideal and $y \notin T$, we have $l \in T$ for all $l \in L$. Therefore $L \subseteq T$.
Conversely suppose $RL \subseteq T \Rightarrow$ either $R \subseteq T$ or $L \subseteq T$ for any $R \in I(\phi, H_2)$ and $L \in I(H_1, \phi)$. Let $h_1, h_2 \in H = H_1 \cup H_2$ such that $h_1Hh_2 \subseteq T$. Then $(h_1H_2]_H(H_1h_2]_H \subseteq (h_1H_2H_1h_2]_H \subseteq (h_1Hh_2]_H \subseteq (T]_H = T$. Since $(h_1H_2]_H$ is in $I(\phi, H_2)$ and $(H_1h_2]_H \in I(H_1, \phi)$, we have $(h_1H_2]_H \subseteq T$ or $(H_1h_2]_H \subseteq T$. As $h_1, h_2 \in H$, following cases arise.

Case 1. Suppose $h_2 \in H_1$ and $h_1 \in H_2$. Consider $(h_1H_2]_H \subseteq T$. Then $h_1^2 \in T$. Then $H_1(h_1)$ and $H_2(h_1)$ are (H_1, ϕ)-ideal and (ϕ, H_2)-ideal of S generated by h_1 respectively. Now

$$\begin{aligned} H_1(h_1)H_2(h_1) &= (h_1 \cup H_1h_1]_H(h_1 \cup h_1H_2]_H \\ &\subseteq ((h_1 \cup H_1h_1)(h_1 \cup h_1H_2)]_H \\ &= (h_1^2 \cup h_1^2H_2 \cup H_1h_1^2 \cup H_1h_1^2H_2]_H \\ &\subseteq (T \cup TH_2 \cup H_1T \cup H_1TH_2]_H \\ &\subseteq (T \cup T \cup T \cup T]_H \subseteq (T]_H = T. \end{aligned}$$

By hypothesis, either $H_1(h_1) \subseteq T$ or $H_2(h_1) \subseteq T$. Hence $h_1 \in T$. If $(H_1h_2]_H \subseteq T$, then $h_2^2 \in T$. In a similar manner we have $h_2 \in T$. Hence T is an (H_1, H_2)-prime bi-ideal of S.

Case 2. Suppose $h_1 \in H_1$ and $h_2 \in H_2$. Then clearly $h_1h_2 \in T$. Since $H_2(h_1)$ and $H_1(h_2)$ are $I(\phi, H_2)$ and $I(H_1, \phi)$ ideals of S respectively, we have

$$\begin{aligned} H_2(h_1)H_1(h_2) &= (h_1 \cup h_1H_2]_H(h_2 \cup H_1h_2]_H \\ &\subseteq ((h_1 \cup h_1H_2)(h_2 \cup H_1h_2)]_H \\ &= (h_1h_2 \cup h_1H_1h_2 \cup h_1H_2h_2 \cup h_1H_2H_1h_2]_H \\ &\subseteq (h_1h_2 \cup h_1Hh_2 \cup h_1Hh_2 \cup h_1Hh_2]_H \\ &\subseteq (T \cup T \cup T \cup T]_H \subseteq (T]_H = T. \end{aligned}$$

By hypothesis, either $H_2(h_1) \subseteq T$ or $H_1(h_2) \subseteq T$. Thus either $h_1 \in T$ or $h_2 \in T$. Hence T is an (H_1, H_2)-prime bi-ideal of S.

Case 3. Suppose $h_1, h_2 \in H_1$ or $h_1, h_2 \in H_2$. Then, by combining the previous cases, we may show that either $h_1 \in T$ or $h_2 \in T$. Hence T is an (H_1, H_2)-prime bi-ideal of S. □

Proposition 2.14. *Let* S *be an ordered semigroup and let* H_1, H_2 *be subsemigroups of* S *such that* $H_1H_2 \subseteq H = H_1 \cup H_2$ *and* $H_2H_1 \subseteq H$. *Then an* (H_1, H_2)*-prime bi-ideal of* S *is either* (ϕ, H_2)*-prime ideal or* (H_1, ϕ)*-prime ideal of* S.

Proof. Let T be any (H_1, H_2)-prime bi-ideal of S. We only need to show that $T \in I(\phi, H_2)$ or $T \in I(H_1, \phi)$. Clearly $(TH_2]_H(H_1T]_H \subseteq (TH_2H_1T]_H \subseteq (THT]_H \subseteq (T]_H = T$. Since $(TH_2]_H \in I(\phi, H_2)$, $(H_1T]_H \in I(H_1, \phi)$ and $(TH_2]_H$, $(H_1T]_H \subseteq H$, by Proposition 2.13, either $(TH_2]_H \subseteq T$ or $(H_1T]_H \subseteq T$. Thus either $TH_2 \subseteq T$ or $H_1T \subseteq T$. Now suppose that $h \in H = (H_1 \cup H_2)$ and $t \in T$ be such that $h \leq t$. Since $T \in \mathcal{B}(H_1, H_2)$, we have $h \in T$, as required. □

Let S be an ordered semigroup and $T \in \mathcal{B}(H_1, H_2)$, where $H_1, H_2 \subseteq S$. Let $L(T) = \{t \in T | H_1 t \subseteq T\}$ and $M(T) = \{x \in L(T) | xH_2 \subseteq L(T)\}$.

Lemma 2.15. *Let S be an ordered semigroup and let H_1, H_2 be subsemigroups of S. If $T \in \mathcal{B}(H_1, H_2)$, then $L(T) \in I(H_1, \phi)$.*

Proof. Let $t \in L(T)$ and $h \in H_1$. Then $ht \in H_1 t \subseteq T$ and $H_1(ht) \subseteq H_1H_2t \subseteq H_1t \subseteq T \Rightarrow ht \in L(T)$. Now choose $u \in L(T) \subseteq T$ such that $H_1 \ni h_1 \leq u$. Then $h_1 \in T$ as $T \in \mathcal{B}(H_1, H_2)$. As $h_1 \leq u \Rightarrow kh_1 \leq ku \ \forall \ k \in H_1$. So $kh_1 \leq ku \in H_1 u \subseteq (H_1u]_H \subseteq (T]_H = T \Rightarrow kh_1 \in T \ \forall \ k \in H_1$. Thus $H_1h_1 \subseteq T$ implies $h_1 \in L(T)$. Hence $L(T) \in I(H_1, \phi)$. □

Proposition 2.16. *Let S be an ordered semigroup and let H_1, H_2 be subsemigroups of S. If $T \in \mathcal{B}(H_1, H_2)$, then $M(T)$ is the (unique) largest (H_1, H_2)-ideal of S contained in T.*

Proof. It is clear that $M(T) \subseteq L(T) \subseteq T$. Take any $a \in M(T)$, $h_1 \in H_1$ and $h_2 \in H_2$. Then $a \in T, a \in L(T)$, $H_1 a \subseteq T$ and $aH_2 \subseteq L(T)$. Clearly $h_1 a \in H_1 a \subseteq T \Rightarrow h_1 a \in T$. Further, $H_1(h_1a) \subseteq H_1H_1a \subseteq H_1a \subseteq T$ implies $h_1a \in L(T)$. Also $ah_2 \in aH_2 \subseteq L(T) \Rightarrow ah_2 \in L(T)$. Now we show that $h_1 a \in M(T)$ and $ah_2 \in M(T)$. As $(ah_2)H_2 \subseteq aH_2H_2 \subseteq aH_2 \subseteq L(T)$, we have $ah_2 \in M(T)$. Also $(h_1a)H_2 \subseteq H_1aH_2 \subseteq H_1L(T) \subseteq L(T) \Rightarrow ah_2 \in M(T)$.

Now let $b \in M(T)$, $H_1 \cup H_2 = H \ni h \leq b$. Then $h \in L(T)$ as $M(T) \subseteq L(T)$ and $L(T) \in I(H_1, \phi)$. Since $h \leq b$ and $h \in H_1$ or $h \in H_2$, we have $hk \leq bk \ \forall \ k \in H_2$. So $hk \leq bk \in bH_2 \subseteq L(T) \Rightarrow hk \in L(T) \ \forall \ k \in H_2$. Thus $hH_2 \subseteq L(T)$ implies $h \in M(T)$. Hence $M(T) \in I(H_1, H_2)$.

Now let G be any (H_1, H_2)-ideal of S and $G \subseteq T$. For any $g \in G$, $g \in T$ and $H_1 g \subseteq G \subseteq T$ implies $G \subseteq L(T)$. Also for $g \in L(T)$, as $gH_2 \subseteq G \subseteq L(T)$, we have $g \in M(T)$. Hence $G \subseteq M(T)$, as required. □

Proposition 2.17. *Let S be an ordered semigroup and let H be a subsemigroup of S. If T is an H-prime bi-ideal of S, then $M(T)$ is an H-weakly prime H-ideal of S.*

Proof. Let T be any H- prime bi-ideal of S. Since T is H-prime bi-ideal of S, $M(T) \in I(H, H)$. It remains to show that $M(T)$ is H-weakly prime. For this take any $a, b \in H$ such that $I_R(a)I_R(b) \subseteq M(T)$. Then, by Theorem 2.11, either $I_R(a) \subseteq T$ or $I_R(b) \subseteq T$. As $M(T)$ is the unique largest H-ideal in T, we get either $I_R(a) \subseteq M(T)$ or $I_R(b) \subseteq M(T)$. Thus either $a \in M(T)$ or $b \in M(T)$. Hence, by Theorem 2.11, $M(T)$ is an H-weakly prime H-ideal of S. □

Proposition 2.18. *Let S be an ordered semigroup and let H_1, H_2 be any non-empty subsets of S. If T is an (H_1, H_2)-semiprime bi-ideal of S, then $A^2 \subseteq T$ implies $A \subseteq T$ for each $A \in I(H_1, H_2)$.*

Proof. Let T be any (H_1, H_2)-semiprime bi-ideal of S such that $A^2 \subseteq T$. On contrary suppose that $A \nsubseteq T$, then $\exists \ a \in A$ such that $a \notin T$. As $A \in I(H_1, H_2)$, we have $a(H_1 \cup H_2)a \subseteq A(H_1 \cup H_2)A = AH_1A \cup AH_2A \subseteq A^2 \cup A^2 = A^2 \subseteq T$. Since T is (H_1, H_2)-semiprime, we have $a \in T$, which is a contradiction. Hence $A \subseteq T$, as required. □

Proposition 2.19. *Let S be an ordered semigroup and let H be a subsemigroup of S. If T is an H-bi-ideal of S, then $M(T)$ is an H-weakly semiprime ideal of S.*

Proof. Let T be any H-bi-ideal of S. By Proposition 2.16, we have $M(T)$ is an H-ideal of S. It remains to show that $M(T)$ is an H-weakly semiprime. For this, take any $a \in H$ such that $(I_R(a))^2 \subseteq M(T)$. By Theorem 2.12, $I_R(a) \subseteq T$ as $(I_R(a))^2 \subseteq T$. As $M(T)$

is the unique largest H-ideal of T, we get $I_R(a) \subseteq M(T)$ which implies that $a \in M(T)$. Hence, by Theorem 2.12, $M(T)$ is an H-weakly semiprime ideal of S. □

Proposition 2.20. *Let S be an ordered semigroup and let H_1, H_2 be subsemigroups of S such that $H_2H_1 \subseteq H_1 \cup H_2$. Then each (H_1, H_2)-semiprime bi-ideal of S is an (H_1, H_2)-quasi ideal of S.*

Proof. Let T be any (H_1, H_2)-semiprime bi-ideal of S. Suppose $h \in TH_2 \cap H_1T$. Then $h \in TH_2$ and $h \in H_1T$. Now $h(H_1 \cup H_2)h \subseteq (TH_2)(H_1 \cup H_2)(H_1T) = (TH_2H_1^2 \cup TH_2^2H_1) \subseteq TH_2H_1T \cup TH_2H_1T \subseteq T(H_1 \cup H_2)T \subseteq T$. Since T is an (H_1, H_2)-semiprime bi-ideal of S, we have $h \in T$. Hence $(TH_2 \cap H_1T) \subseteq T$.

Further, let $t \in T, (H_1 \cup H_2) \ni h \leq t$. Then, as $T \in \mathcal{B}(H_1, H_2),\ h \in T,$. Hence T is an (H_1, H_2)-quasi ideal of S. □

Proposition 2.21. *Let S be an Ordered semigroup and let $H_1, H_2 \subseteq S$ be such that $H_2H_1, H_1H_2 \subseteq H = H_1 \cup H_2$. Then S is (H_1, H_2)-regular if and only if each (H_1, H_2)-bi-ideal of S is (H_1, H_2)-semiprime.*

Proof. Let S be an (H_1, H_2)-regular ordered semigroup and $T \in \mathcal{B}(H_1, H_2)$. Suppose $aHa \subseteq T$ for $a \in H$. Then, by (H_1, H_2)-regularity of S, there exists $h \in H$ such that $a \leq aha$. But $aha \in aHa \subseteq T$. Now, as $H \ni a \leq aha$ and $T \in \mathcal{B}(H_1, H_2)$, $a \in T$. Hence T is (H_1, H_2)-semiprime.

Conversely assume that every (H_1, H_2)-bi-ideal of S is (H_1, H_2)-semiprime. Let $a \in H$. Clearly $B = (aHa]_H \in \mathcal{B}(H_1, H_2)$. Therefore either $a \in H_1$ or $a \in H_2$. Let $a \in H_1$. We show that $BHB = BH_1B \cup BH_2B \subseteq B$. Now $BH_1B = (aHa]_H H_1 (aHa]_H \subseteq ((aH_1a \cup aH_2a)H_1(aH_1a \cup aH_2a)]_H = (aH_1aH_1aH_1a \cup aH_1aH_1aH_2a \cup aH_2aH_1aH_1a \cup aH_2aH_1aH_2a]_H \subseteq (aH_1^3a \cup aH_1^2H_2a \cup aH_2H_1^2a \cup aH_2H_1H_2a]_H \subseteq (aH_1a \cup aHa \cup aHa \cup aHa]_H \subseteq (aHa]_H = B$. Clearly $((aHa]_H]_H = (aHa]_H$. By hypothesis $(aHa]_H$ is (H_1, H_2)-semiprime for any $a \in H$. Since $aHa \subseteq (aHa]_H \Rightarrow a \in (aHa]_H$ implies that $a \leq aha$ for some $h \in H$ and, hence, S is (H_1, H_2)-regular. □

Proposition 2.22. *Let S be a commutative ordered semigroup and let H_1, H_2 be subsemigroups of S such that $H_2H_1 \subseteq H_1 \cup H_2$. Then S is (H_1, H_2)-regular if and only if each (H_1, H_2)-ideal of S is (H_1, H_2)-semiprime.*

Proof. Let S be an (H_1, H_2)-regular commutative ordered semigroup and $T \in I(H_1, H_2)$. Suppose $h^2 \in T$ for some $h \in H = H_1 \cup H_2$. Then $\exists\ k \in H$ such that $h \leq hkh$. For $k \in H_1$, we have $h \leq hkh = (hk)h = k(hh) = kh^2 \in H_1T \subseteq T$ which implies that $h \in T$ as $T \in I(H_1, H_2)$. If $k \in H_2$, then we have $h \leq hkh = h(kh) = h(hk) = h^2k \in TH_2 \subseteq T \Rightarrow h \in T$. Hence T is (H_1, H_2)-semiprime.

Conversely suppose that each (H_1, H_2)-ideal of S is (H_1, H_2)-semiprime. Let $a \in H = H_1 \cup H_2$. As $(a^2H]_H \in I(H_1, H_2)$, by hypothesis, $(a^2H]_H$ is (H_1, H_2)-semiprime. Since $a^4 \in (a^2H]_H \Rightarrow a^2 \in (aHa]_H \Rightarrow a \in (a^2H]_H$. Thus $a \leq a^2h$ for some $h \in H$ which implies that $a \leq aah = aha$ for some $h \in H$ and, hence, S is (H_1, H_2)-regular. □

3. On relative intra-regular ordered semigroups

Definition 3.1. Let S be an ordered semigroup and $H \subseteq S$. Then S is called H-intra-regular (or relative intra-regular) if for every $a \in H$, there exist $h, k \in H$ such that $a \leq ha^2k$. Equivalently, for all $A \subseteq H,\ A \subseteq (HA^2H]_H$.

Definition 3.2. Let S be an ordered semigroup and $H_1, H_2 \subseteq S$. Then S is called (H_1, H_2)-intra-regular if for every $a \in H_1 \cup H_2$, there exist $h, k \in H = H_1 \cup H_2$ such that $a \leq ha^2k$. Equivalently, for all $A \subseteq H = H_1 \cup H_2,\ A \subseteq (HA^2H]_H$.

If $H_1 = \phi$ or $H_2 = \phi$, then S is called (ϕ, H_2)-intra regular or (H_1, ϕ)-intra-regular. Clearly relative intra-regularity of S does not imply intra-regularity of S.

Definition 3.3. Let S be an ordered semigroup and let $H_1, H_2 \subseteq S$. A non-empty subset Q of S is called an (H_1, H_2)-quasi ideal of S if

(1) $(QH_2]_H \cap (H_1Q]_H \subseteq Q$, where $H = H_1 \cup H_2$; and
(2) $q \in Q, H \ni h \leq q$ implies $h \in Q$.

An (H_1, H_2)-bi-ideal $B_R(a)$ and (H_1, H_2)-quasi ideal $Q_R(a)$ of S generated by an element a of S are given by $B_R(a) = (a \cup a^2 \cup aHa]_H$ and $Q_R(a) = (a \cup ((aH_2]_H \cap (H_1a]_H)]_H$ respectively, where $H = H_1 \cup H_2$.

4. Main theorems

N. Kehayopulu, S. Lajos and M. Tsingelis, in [11], proved various characterizations of the intra-regular ordered semigroups. In this section, we give some new characterizations of the relative intra-regular ordered semigroups in terms of relative bi-ideals, relative quasi ideals, relative left and right ideals of ordered semigroups.

Theorem 4.1. *Let S be an ordered semigroup and let H_1, H_2 be subsemigroups of S such that $H_1H_2 \subseteq H_1 \cup H_2$ and $H_2H_1 \subseteq H_1 \cup H_2 (= H)$. Then*

(i) *S is (H_1, H_2)-intra-regular if and only if for an (H_1, H_2)-bi-ideal B contained in H and an (H_1, H_2)-quasi ideal Q of S implies $B \cap Q \subseteq (H_1BQH_2]_H$.*
(ii) *S is (H_1, H_2)-intra-regular if and only if for an (H_1, H_2)-bi-ideal B contained in H and an (H_1, H_2)-quasi ideal Q of S implies $B \cap Q \subseteq (H_1QBH_2]_H$.*

Proof. (i) Let $t \in B \cap Q \subseteq H_1 \cup H_2$. As S is (H_1, H_2)-intra-regular, $\exists\, h \in H_1, k \in H_2$ such that $t \leq ht^2k$. Now $t \leq ht^2k \leq ht(ht^2k)k = h(tht)tk^2 \in H_1(BH_1B)QH_2 \subseteq H_1BQH_2$. Hence $B \cap Q \subseteq (H_1BQH_2]_H$.
Conversely take any $t \in H$. As $t \in H$, either $t \in H_1$ or $t \in H_2$. Suppose $t \in H_1$. Then, as $B_R(t)$ and $Q_R(t)$ are (H_1, H_2)-bi-ideal and (H_1, H_2)-quasi ideal generated by t respectively, we have

$$\begin{aligned} t &\in B_R(t) \cap Q_R(t) \subseteq (H_1B_R(t)Q_R(t)H_2]_H \\ &= (H_1(t \cup t^2 \cup tH_1t \cup tH_2t]_H(t \cup ((tH_2]_H \cap (H_1t]_H)]_HH_2]_H \\ &\subseteq ((H_1t \cup H_1t^2 \cup H_1tH_1t \cup H_1tH_2t]_H(t \cup (tH_2]_H]_HH_2]_H \\ &\subseteq ((H_1t \cup H_1H_2t]_H(t \cup (tH_2]_H]_HH_2]_H \\ &\subseteq ((Ht \cup Ht]_H(tH_2 \cup (tH_2^2]_H]_H \subseteq ((Ht]_H(tH \cup (tH]_H]_H \\ &\subseteq ((Ht^2H]_H \cup (Ht^2H]_H]_H \\ &\subseteq ((Ht^2H]_H]_H = (Ht^2H]_H. \end{aligned}$$

Similarly we may prove if $t \in H_2$. Therefore S is (H_1, H_2)-intra-regular.
(ii) Let $t \in B \cap Q \subseteq H_1 \cup H_2$. As S is (H_1, H_2)-intra-regular, $\exists\, h \in H_1, k \in H_2$ such that $t \leq ht^2k$. Now $t \leq ht^2k \leq h(ht^2k)tk = h^2t(tkt)k \in H_1Q(BH_2B)H_2 \subseteq H_1QBH_2$. Hence $B \cap Q \subseteq (H_1QBH_2]_H$.
Conversely suppose that $B_R(t)$ and $Q_R(t)$ are (H_1, H_2)-bi-ideal and (H_1, H_2)-quasi ideal generated by t in H. Then either $t \in H_1$ or $t \in H_2$. Let $t \in H_2$. Then

$$\begin{aligned} t &\in B_R(t) \cap Q_R(t) \subseteq (H_1Q_R(t)B_R(t)H_2]_H \\ &= (H_1(t \cup ((tH_2]_H \cap (H_1t]_H)]_H(t \cup t^2 \cup tH_1t \cup tH_2t]_HH_2]_H \\ &\subseteq (H_1(t \cup (H_1t]_H)]_H((tH_2 \cup t^2H_2 \cup tH_1tH_2 \cup tH_2tH_2]_H \\ &\subseteq ((H_1t \cup (H_1^2t]_H]_H(tH_2 \cup tH_1H_2]_H]_H \\ &\subseteq ((Ht]_H(tH]_H]_H \\ &\subseteq ((Ht^2H]_H]_H = (Ht^2H]_H. \end{aligned}$$

If $t \in H_1$, then we may prove in a similar way. Therefore S is (H_1, H_2)-intra-regular. $\square$

Theorem 4.2. *Let S be an ordered semigroup and let H_1 and H_2 be subsemigroups of S such that $H_1H_2 \subseteq H$ and $H_2H_1 \subseteq H(= H_1 \cup H_2)$. Then*

(i) *S is (H_1, H_2)-intra-regular if and only if for an (H_1, ϕ)-ideal $L \subseteq H_1$ and an (H_1, H_2)-bi-ideal B of S implies $L \cap B \subseteq (LBH_2]_H$.*

(ii) *S is (H_1, H_2)-intra-regular if and only if for an (ϕ, H_2)-ideal $R \subseteq H_2$ and an (H_1, H_2)-bi-ideal B of S implies $B \cap R \subseteq (H_1BR]_H$.*

Proof. (i) Let $b \in L \cap B \subseteq H_1 \subseteq H_1 \cup H_2$. As S is (H_1, H_2)-intra-regular, $\exists\, h \in H_1,\ k \in H_2$ such that $b \leq hb^2k$. Now $b \leq hb^2k \leq h(hb^2k)bk = h^2b(bkb)k \in H_1L(BH_2B)H_2 \subseteq LBH_2$. Hence $L \cap B \subseteq (LBH_2]_H$.
Conversely let $B_R(b)$ and $H_1(b) = (b \cup H_1b]_H$ (H, ϕ) be (H_1, H_2)-bi-ideal and (H_1, ϕ)-ideal of S generated by $b \in H$. Now, as $b \in H$, either $b \in H_1$ or $b \in H_2$. First suppose that $b \in H_1$. Then

$$\begin{aligned} b &\in H_1(b) \cap B_R(b) \subseteq (H_1(b)B_R(b)H_2]_H \\ &= ((b \cup H_1b]_H(b \cup b^2 \cup bH_1b \cup bH_2b]_HH_2]_H \\ &\subseteq ((b \cup H_1b]_H(bH_2 \cup b^2H_2 \cup bH_1bH_2 \cup bH_2bH_2]_H]_H \\ &\subseteq ((b \cup H_1b]_H(bH_2 \cup bH_1H_2 \cup bH_1H_2 \cup bH_2H_1H_2]_H]_H \\ &\subseteq ((b \cup Hb]_H(bH]_H]_H = (b^2H \cup Hb^2H]_H. \end{aligned}$$

Therefore $b \leq u$ for some $u \in b^2H \cup Hb^2H$. If $u \in b^2H$, then $b \leq b^2h$ for some $h \in H$. As, either $h \in H_1$ or $h \in H_2$, $b \leq b^2h \leq b(b^2h)h = bb^2h^2 \in Hb^2H_1 \ \Rightarrow\ b \in (Hb^2H_1]_H$ or $b \in (Hb^2H_2]_H$. Thus $b \in (Hb^2H]_H$. In the other case when $u \in Hb^2H$, $\ b \in (Hb^2H]_H$. Hence S is (H, H)-intra-regular. The case when $b \in H_2$ is similar.

(ii) Let $b \in B \cap R \subseteq H_2 \subseteq H_1 \cup H_2$. As S is (H_1, H_2)-intra-regular, $\exists\, h \in H_1,\ k \in H_2$ such that $b \leq hb^2k \leq hb(hb^2k)k = h(bhb)bk^2 \in H_1(BH_1B)RH_2 \subseteq H_1BR$. Hence,$B \cap R \subseteq (H_1BR]_H$.

Conversely take $B_R(b)$ and $H_2(b) = (b \cup bH_2]_H$, the (H_1, H_2)-bi-ideal and the (ϕ, H_2)-ideal respectively generated by $b \in H$. As $b \in H$, either $b \in H_1$ or $b \in H_2$. Suppose first that $b \in H_1$. Now

$$\begin{aligned} b &\in B_R(b) \cap H_2(b) \subseteq (H_1B_R(b)H_2(b)]_H \\ &= (H_1(b \cup b^2 \cup bH_1b \cup bH_2b]_H(b \cup bH_2]_H]_H \\ &\subseteq ((H_1b \cup H_1b^2 \cup H_1bH_1b \cup H_1bH_2b]_H(b \cup bH]_H]_H \\ &\subseteq ((H_1b \cup H_1H_2b]_H(b \cup bH_2]_H]_H \\ &\subseteq ((Hb]_H(b \cup bH]_H]_H \\ &\subseteq ((Hb^2 \cup Hb^2H]_H]_H \\ &= (Hb^2 \cup Hb^2H]_H. \end{aligned}$$

Therefore $b \leq u$ for some $u \in (Hb^2 \cup Hb^2H)$. If $u \in Hb^2$, then $b \leq hb^2$ for some $h \in H$. So either $h \in H_1$ or $h \in H_2$. Now $b \leq hb^2 \leq h(hb^2)b = h^2b^2b \in H_1b^2H \ \Rightarrow\ b \in (H_1b^2H]_H$ or $b \in (H_2b^2H]_H$. Thus $b \in (Hb^2H]_H$. On the other hand if $u \in Hb^2H$, then obviously $b \in (Hb^2H]_H$. Hence, in either case, $b \in (Hb^2H]_H$ i.e. S is H-intra-regular. Similarly we may prove when $b \in H_2$. □

5. Figures and tables

Figure 1. Hasse diagram of the poset S

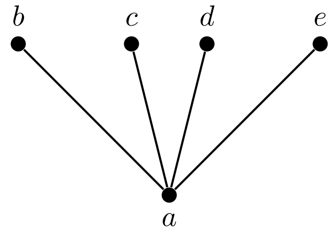

Table 1. Caylay table of the binary operation (.)

$\cdot$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	a	a	b	a

Table 2. Caylay table of the binary operation (.)

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	d
c	a	b	b	d
d	a	d	d	d

Table 3. Caylay table of the binary operation (.)

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

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