ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ B- TEORİK BİLİMLER

Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

2020, 8(2), pp. 212 - 222, DOI: 10.20290/estubtdb.625933

STABILITY CRITERIA FOR RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

Ali Fuat YENİÇERİOĞLU *, **Cüneyt YAZICI**

Department of Mathematics, The Faculty of Education, Kocaeli University, 41380, Kocaeli, Turkey

ABSTRACT

In this work, we examine the stability behavior of retarded functional differential equations. The asymptotic behavior of solutions and stability of the zero solution are investigated by using a suitable real root for the characteristic equation. Three examples are also given to illustrate our results.

Keywords: Retarded differential equation, Characteristic equation, Stability, Zero solution

1. INTRODUCTION

Consider the retarded differential equation

$$x'(t) = \int_{-1}^{0} x\big(t - r(\theta)\big) dv(\theta), \quad t \geq 0, \tag{1.1}$$

where $x(t) \in \mathbb{R}$, $r(\theta)$ is a nonnegative real continuous function on $[-1,0]$, $v(\theta)$ is a real function of bounded variation on $[-1,0]$ and v is not a constant on $[-1,0]$. The integral is the Riemann-Stieltjes integral.

Let $\|r\| = max\{r(\theta): -1 \leq \theta \leq 0\}$. Together with the retarded differential equation (1.1), it is customary to specify an *initial condition* of the form

$$x(t) = \phi(t), \quad -\|r\| \leq t \leq 0, \tag{1.2}$$

where the initial function ϕ is a given continuously real-valued function on the initial interval $[-\|r\|, 0]$. Throughout this article, by $C([-\|r\|, 0], \mathbb{R})$ we will denote the set of all continuous real-valued functions on the interval $[-\|r\|, 0]$. This set is a Banach space endowed with the norm $\|\phi\| = \max_{-\|r\| \leq t \leq 0} |\phi(t)|$.

If we look the form $x(t) = e^{\lambda t}$ for $t \in \mathbb{R}$ as a solution of (1.1), it can be seen that λ will be a root of the *characteristic equation*

$$\lambda = \int_{-1}^{0} e^{-\lambda r(\theta)} dv(\theta). \tag{1.3}$$

It will be also considered the relevant class of differential difference equation

*Corresponding Author: fuatyenicerioglu@kocaeli.edu.tr
Received: 27.09.2020 Published: 31.08.2020

$$x'(t) = \sum_{j=1}^{p} a_j x(t - r_j), \tag{1.4}$$

where a_j are nonzero real numbers and each r_j is a positive real number $(j = 1, \dots, p)$. Equation (1.4) can be obtained from (1.1), under the assumption that $v(\theta)$ is a step function with a number p of jump points. More concretely, if $v(\theta)$ is given by

$$v(\theta) = \sum_{j=1}^{p} H(\theta - \theta_j) a_j \, ,$$

we can obtain equation (1.4) it from (1.1). Here, for $-1 < \theta_1 < \cdots < \theta_p < 0$, by H we mean the Heaviside function and r_j are obtained through any function $r(\theta) \in C^+$ which satisfy $r(\theta) = r_j$ for $j = 1, \dots, p$ (see, e.g., [1-5]). The C^+ symbol is the subset of $C([-1, 0], \mathbb{R})$ formed by all nonnegative functions $r(\theta)$.

Furthermore, equation (1.1) for $r(\theta) = -r\theta$ $(r > 0)$ and $\theta \in [-1,0]$ reduces to the class of retarded functional differential equations

$$x'(t) = \int_{-r}^{0} x(t + \theta) \, dq(\theta) \, , \tag{1.5}$$

where $q(\theta) = v(\theta/r)$ is supposed to be atomic at zero. The reader can see the most general linear retarded functional differential equation in [6]. Additionally, they can look at the articles [2, 7-9].

Cooke and Ferreira [1] examined the stability conditions for (1.1). In this article, we applied a different method for the stability of (1.1). In other words, Cooke and Ferreira [1] have achieved the stability of solutions over a cone defined as the region of stability. That is, the authors obtained the set of Banach space of all real valued functions of bounded variation on [-1,0] for which (1.1) is exponentially stable globally in the delays. However, the article in [1] has no information about asymptotic behavior and exponential estimate of solutions. Different from the article in [1], in this article, we obtained the asymptotic behavior of the solutions and then we created a useful exponential estimate for these solutions and finally provided a stability criterion. These results are obtained with a real root of the characteristic equation.

The stability theory of delay differential equations is important for applied mathematics. It can be seen the textbooks [6,10-15] and references therein. Pedro [3-5] established the oscillatory criteria for retarded functional equations of form (1.1). Ford, Yan and Malique [16] used the numerical method in equation (1.1). Malique [17] completed his doctorate on the oscillation and numerical field related to equation (1.1).

In this paper, we are interested in the asymptotic behavior and stability for retarded differential equations. Our results are investigated by using a suitable real root for (1.3). The techniques which used in our results, are take placed in a combination of the methods in references [9, 18-20].

By a *solution* x to the retarded differential equation (1.1), we mean a continuous real-valued function, defined on $[-\|r\|, +\infty)$, which is continuously differentiable on $[0, +\infty)$ and satisfies (1.1).

In this section, some definitions will be given (see, e.g., [6]). The zero solution of (1.1) is said to be *stable* if for every $\varepsilon > 0$, there exists a number $\ell = \ell(\varepsilon) > 0$ such that, for any initial function ϕ with $\|\phi\| < \ell$, the solution x of (1.1)-(1.2) satisfies

$$|x(t)| < \varepsilon \quad \text{for all } t \in [-\|r\|, \infty) \, .$$

Otherwise, the zero solution of (1.1) is said to be *unstable*. Furthermore, the zero solution of (1.1) is called *asymptotically stable* if it is stable in the sense described above and there also exists a number $\ell_0 > 0$ such that, for any initial function ϕ with $\|\phi\| < \ell_0$, the solution x of (1.1)-(1.2) satisfies

$$\lim_{t\to\infty} x(t) = 0 .$$

In this paper, we will show the total variation function of v with $V(v)$. Moreover, it must be noted that $V(v)$ is not identically zero in the interval $[-1,0]$. It is assumed that the reader knows the theory of Riemann-Stieltjes integration and the theory of functions of bounded variation (see [21, Chapter 12]).

2. STATEMENT OF THE MAIN RESULTS

Our main results in this article consist of two theorems. The first theorem deals with the asymptotic behavior of the solutions of the equation (1.1). The second theorem gives the exponential estimate and stability criterion of the solutions. The proofs of these theorems are given in Section 3 and Section 4.

Theorem 2.1. *Let λ_0 be a real root of (1.3). Assume that*

$$\mu(\lambda_0) = \int_{-1}^{0} r(\theta) e^{-\lambda_0 r(\theta)}\, dV(v)(\theta) < 1 \tag{2.1}$$

and set

$$\beta(\lambda_0) = \int_{-1}^{0} r(\theta) e^{-\lambda_0 r(\theta)} dv(\theta) . \tag{2.2}$$

Then, for every $\phi \in C([-\|r\|, 0], \mathbb{R})$, the solution x of (1.1)-(1.2) satisfies

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t} x(t)\right] = \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} ,$$

where

$$L(\lambda_0;\phi) = \phi(0) + \int_{-1}^{0} e^{-\lambda_0 r(\theta)} \left(\int_{-r(\theta)}^{0} e^{-\lambda_0 s}\phi(s) ds \right) dv(\theta) . \tag{2.3}$$

Note: *Property (2.1) guarantees that* $1 + \beta(\lambda_0) > 0$.

It immediately appears that $\lambda_0 = 0$ is the root of the characteristic equation (1.3) with properties (2.1) if and only if

$$\int_{-1}^{0} dv(\theta) = 0 \quad and \quad \int_{-1}^{0} r(\theta)\, dV(v)(\theta) < 1,$$

i.e.,

$$v(0) = v(-1) \quad and \quad \int_{-1}^{0} r(\theta)\, dV(v)(\theta) < 1 . \tag{2.4}$$

So, the following corollary is related to an application of Theorem 2.1 with $\lambda_0 = 0$.

Corollary 2.2. Let condition (2.4) be satisfied. Then, for every $\phi \in C([-\|r\|, 0], \mathbb{R})$, the solution x of (1.1)-(1.2) satisfies

$$\lim_{t\to\infty} x(t) = \frac{\phi(0) + \int_{-1}^{0}\left(\int_{-r(\theta)}^{0}\phi(s)ds\right)dv(\theta)}{1+\int_{-1}^{0} r(\theta)\,dv(\theta)}.$$

Due to the second assumption of (2.4) we get

$$1 + \int_{-1}^{0} r(\theta)\,dv(\theta) > 0.$$

Theorem 2.3. *Let λ_0 be a real root of (1.3) and consider $\beta(\lambda_0)$ as in (2.2). Suppose that condition (2.1) be satisfied. Then, for every $\phi \in C([-\|r\|, 0], \mathbb{R})$, the solution x of (1.1)-(1.2) satisfies*

$$|x(t)| \le \left[\frac{\left(1+\mu(\lambda_0)\right)^2}{1+\beta(\lambda_0)} + \mu(\lambda_0)\right] N(\lambda_0;\phi)e^{\lambda_0 t} \;\; for\ all\ \ t \ge 0,$$

where

$$N(\lambda_0;\phi) = \max_{-\|r\|\le t\le 0}\left|e^{-\lambda_0 t}\phi(t)\right|. \tag{2.5}$$

In addition, the zero solution of (1.1) is stable if $\lambda_0 = 0$, asymptotically stable if $\lambda_0 < 0$, and unstable if $\lambda_0 > 0$.

3. PROOF OF THEOREM 2.1

Firstly, let us define $\mu(\lambda_0)$ and $\beta(\lambda_0)$ as in (2.1) and (2.2), respectively. Property (2.1) implies

$$0 < \mu(\lambda_0) < 1. \tag{3.1}$$

We have

$$|\beta(\lambda_0)| = \left|\int_{-1}^{0} r(\theta)e^{-\lambda_0 r(\theta)}dv(\theta)\right| \le \int_{-1}^{0} r(\theta)e^{-\lambda_0 r(\theta)}dV(v)(\theta).$$

That is $|\beta(\lambda_0)| \le \mu(\lambda_0)$. So, it holds $|\beta(\lambda_0)| < 1$. We obtain $1+\beta(\lambda_0) > 0$.

Consider now any initial function ϕ in $C([-\|r\|, 0], \mathbb{R})$ and let x be the solution of (1.1)-(1.2). Define

$$y(t) = e^{-\lambda_0 t}x(t) \quad \text{for } \ t \in [-\|r\|, \infty).$$

Then, we get for every $t \ge 0$

$$y'(t) + \lambda_0 y(t) = \int_{-1}^{0} e^{-\lambda_0 r(\theta)}y\big(t-r(\theta)\big)dv(\theta). \tag{3.2}$$

Furthermore, the initial condition (1.2) can be equivalently written

$$y(t) = e^{-\lambda_0 t}\phi(t) \quad \text{for } \ t \in [-\|r\|, 0]. \tag{3.3}$$

By using λ_0 which is a root of (1.3) and taking into consideration (3.3), we can confirm that (3.2) is equivalent to

$$y(t) - y(0) + \lambda_0 \int_0^t y(s)ds = \int_0^t \left(\int_{-1}^0 e^{-\lambda_0 r(\theta)} y\big(s - r(\theta)\big) dv(\theta) \right) ds \,,$$

$$y(t) = \phi(0) - \lambda_0 \int_0^t y(s)ds + \int_{-1}^0 e^{-\lambda_0 r(\theta)} \left(\int_0^t y\big(s - r(\theta)\big) ds \right) dv(\theta) \,,$$

$$y(t) = \phi(0) - \left(\int_{-1}^0 e^{-\lambda_0 r(\theta)} dv(\theta) \right) \int_0^t y(s)ds + \int_{-1}^0 e^{-\lambda_0 r(\theta)} \left(\int_{-r(\theta)}^{t-r(\theta)} y(s)ds \right) dv(\theta) \,,$$

$$y(t) = \phi(0) - \int_{-1}^0 e^{-\lambda_0 r(\theta)} \left(\int_0^t y(s)ds \right) dv(\theta)$$

$$+ \int_{-1}^0 e^{-\lambda_0 r(\theta)} \left(\int_{-r(\theta)}^0 e^{-\lambda_0 s} \phi(s)ds \right) dv(\theta) + \int_{-1}^0 e^{-\lambda_0 r(\theta)} \left(\int_0^{t-r(\theta)} y(s)ds \right) dv(\theta),$$

$$y(t) = L(\lambda_0; \phi) - \int_{-1}^0 e^{-\lambda_0 r(\theta)} \left(\int_{t-r(\theta)}^t y(s)ds \right) dv(\theta), \tag{3.4}$$

where $L(\lambda_0; \phi)$ is defined as in Theorem 2.1. Next, we set for $t \geq -\|r\|$

$$z(t) = y(t) - \frac{L(\lambda_0; \phi)}{1 + \beta_{\lambda_0}} \,.$$

If we use above transformation, we can obtain following equation which equivalent (3.4) for $t \geq 0$

$$z(t) = - \int_{-1}^0 e^{-\lambda_0 r(\theta)} \left(\int_{t-r(\theta)}^t z(s)ds \right) dv(\theta) \,. \tag{3.5}$$

Moreover, (3.3) is written as for $t \in [-\|r\| \,, 0]$

$$z(t) = e^{-\lambda_0 t} \phi(t) - \frac{L(\lambda_0; \phi)}{1 + \beta_{\lambda_0}} \,. \tag{3.6}$$

By the definition of y and z, what we have to prove is that

$$\lim_{t \to \infty} z(t) = 0 \,. \tag{3.7}$$

We will see that (3.7) is correct at the end of the proof. Put

$$M(\lambda_0; \phi) = \max_{t \in [-\|r\| \,, 0]} \left| e^{-\lambda_0 t} \phi(t) - \frac{L(\lambda_0; \phi)}{1 + \beta_{\lambda_0}} \right| .$$

Then, in view of (3.6), we have

$$|z(t)| \leq M(\lambda_0; \phi) \quad \text{for} \ -\|r\| \leq t \leq 0 \,. \tag{3.8}$$

We will obtain that $M(\lambda_0;\phi)$ is a bound of z on the interval $[-\|r\|,\infty)$, namely that

$$|z(t)| \le M(\lambda_0;\phi) \quad \text{for all} \quad t \ge -\|r\| . \tag{3.9}$$

For an any number $\epsilon > 0$, we claim that

$$|z(t)| < M(\lambda_0;\phi) + \epsilon \quad \text{for every} \quad t \ge -\|r\| . \tag{3.10}$$

Otherwise, because of (3.8), there exists a point $t_0 > 0$ such that

$$|z(t)| < M(\lambda_0;\phi) + \epsilon \quad \text{for} \quad -\|r\| \le t < t_0 \quad \text{and} \quad |z(t_0)| = M(\lambda_0;\phi) + \epsilon .$$

Then, by taking into consideration condition (2.1), from (3.5) we obtain

$$\begin{aligned} M(\lambda_0;\phi) + \epsilon = |z(t_0)| &= \left| - \int_{-1}^{0} e^{-\lambda_0 r(\theta)} \left(\int_{t_0 - r(\theta)}^{t_0} z(s)ds \right) dv(\theta) \right| \\ &\le \int_{-1}^{0} e^{-\lambda_0 r(\theta)} \left(\int_{t_0 - r(\theta)}^{t_0} |z(s)|ds \right) dV(v)(\theta) \\ &\le [M(\lambda_0;\phi) + \epsilon] \int_{-1}^{0} r(\theta) e^{-\lambda_0 r(\theta)} dV(v)(\theta) \\ &= [M(\lambda_0;\phi) + \epsilon]\mu(\lambda_0) < [M(\lambda_0;\phi) + \epsilon] . \end{aligned}$$

So, due to (2.1), we obtain a contradiction. From here, (3.10) holds for all numbers $\epsilon > 0$. Hence, we see (3.9) is correct. Now, by using (3.9), from (3.5) it is obtained for $t \ge 0$

$$\begin{aligned} |z(t)| &\le \int_{-1}^{0} e^{-\lambda_0 r(\theta)} \left(\int_{t - r(\theta)}^{t} |z(s)|ds \right) dV(v)(\theta) \\ &\le M(\lambda_0;\phi) \int_{-1}^{0} r(\theta) e^{-\lambda_0 r(\theta)} dV(v)(\theta) . \end{aligned}$$

Consequently, by the definition of $\mu(\lambda_0)$, we have

$$|z(t)| \le \mu(\lambda_0) M(\lambda_0;\phi) \quad \text{for every } t \ge 0 . \tag{3.11}$$

Using (3.5) and by taking into consideration condition (2.1), in addition to (3.9) and (3.11), it can be shown, by an easy induction, that z satisfies

$$|z(t)| \le [\mu(\lambda_0)]^n M(\lambda_0;\phi) \quad \text{for all} \quad t \ge n\|r\| - \|r\| \quad (n = 0,1,2,\dots) . \tag{3.12}$$

Because of (3.1), we have $\lim_{n\to\infty} [\mu(\lambda_0)]^n = 0$. So, from (3.12) it follows that $\lim_{t\to\infty} z(t) = 0$, i.e., (3.7) is obtained. The proof of Theorem 2.1 is complete.

4. PROOF OF THEOREM 2.3

Consider any function ϕ in $C([-\|r\|,0],\mathbb{R})$ and let x be the solution of (1.1)-(1.2). Let y and z be defined as in the proof of Theorem 2.1, i.e.

$$y(t) = e^{-\lambda_0 t}x(t) \quad \text{and} \quad z(t) = y(t) - \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)} \quad \text{for } t \geq -\|r\| ,$$

where $L(\lambda_0;\phi)$ is defined as (2.3). Furthermore, let $M(\lambda_0;\phi)$ be defined as in the proof of Theorem 2.1, i.e.

$$M(\lambda_0;\phi) = \max_{t\in[-\|r\|,0]} \left| e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{1+\beta_{\lambda_0}} \right| .$$

Then, it can be shown that z satisfies (3.11). From (3.11) it follows that for $t \geq 0$

$$|y(t)| \leq \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)} + \mu(\lambda_0)M(\lambda_0;\phi) . \tag{4.1}$$

By using (2.1) and (2.5), from (2.3) we obtain

$$|L(\lambda_0;\phi)| \leq |\phi(0)| + \int_{-1}^{0} e^{-\lambda_0 r(\theta)} \left(\int_{-r(\theta)}^{0} \left|e^{-\lambda_0 s}\phi(s)\right| ds \right) dV(v)(\theta)$$

$$\leq \left(1 + \int_{-1}^{0} r(\theta)e^{-\lambda_0 r(\theta)} dV(v)(\theta) \right) N(\lambda_0;\phi) = \big(1+\mu(\lambda_0)\big)N(\lambda_0;\phi) .$$

On the other hand, from the definition of $M(\lambda_0;\phi)$ we get

$$M(\lambda_0;\phi) \leq \max_{t\in[-\|r\|,0]} \left|e^{-\lambda_0 t}\phi(t)\right| + \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)} = N(\lambda_0;\phi) + \frac{|L(\lambda_0;\phi)|}{1+\beta(\lambda_0)}$$

$$\leq N(\lambda_0;\phi) + \frac{\big(1+\mu(\lambda_0)\big)N(\lambda_0;\phi)}{1+\beta(\lambda_0)} = \left(1 + \frac{1+\mu(\lambda_0)}{1+\beta(\lambda_0)}\right) N(\lambda_0;\phi) .$$

So, (4.1) gives

$$|y(t)| \leq \frac{\big(1+\mu(\lambda_0)\big)N(\lambda_0;\phi)}{1+\beta(\lambda_0)} + \mu(\lambda_0)\left(1 + \frac{1+\mu(\lambda_0)}{1+\beta(\lambda_0)}\right) N(\lambda_0;\phi)$$

$$= \left[\frac{\big(1+\mu(\lambda_0)\big)^2}{1+\beta(\lambda_0)} + \mu(\lambda_0) \right] N(\lambda_0;\phi) .$$

Finally, due to the definition of y, we get

$$|x(t)| \leq \left[\frac{\big(1+\mu(\lambda_0)\big)^2}{1+\beta(\lambda_0)} + \mu(\lambda_0) \right] N(\lambda_0;\phi)e^{\lambda_0 t} \quad \text{for all } t \geq 0 . \tag{4.2}$$

This completes the proof of the first part of the Theorem 2.3. It remains to show the proof of the second part of the Theorem 2.3, i.e., the stability criterion contained in the Theorem 2.3.

Let us assume that $\lambda_0 \leq 0$. Let $\phi \in C([-\|r\|,0],\mathbb{R})$ be any initial function and let x be the solution of (1.1)-(1.2). Then (4.2) holds and therefore

$$|x(t)| \leq \left[\frac{\big(1+\mu(\lambda_0)\big)^2}{1+\beta(\lambda_0)} + \mu(\lambda_0) \right] N(\lambda_0;\phi) \quad \text{for all } t \geq 0 .$$

Since $\frac{\left(1+\mu(\lambda_0)\right)^2}{1+\beta(\lambda_0)} > 1$, it follows that

$$|x(t)| \leq \left[\frac{(1+\mu(\lambda_0))^2}{1+\beta(\lambda_0)} + \mu(\lambda_0)\right] N(\lambda_0; \phi) \quad \text{for all } t \geq -\|r\| .$$

From above the last inequality, it can be confirmed that zero solution of (1.1) is stable. Furthermore, if $\lambda_0 < 0$, then (4.2) guarantess that

$$\lim_{t\to\infty} x(t) = 0 .$$

Thus, for $\lambda_0 < 0$ the zero solution of (1.1) is asymptotically stable.

Finally, we suppose that $\lambda_0 > 0$ and it will be shown that the zero solution of (1.1) is unstable. Then it can be choosen a number $\delta > 0$ such that, for each $\phi \in C([-\|r\|, 0], \mathbb{R})$ with $\|\phi\| < \delta$, the solution x of (1.1)-(1.2) satisfies

$$|x(t)| < 1 \quad \text{for all } t \geq -\|r\| . \tag{4.3}$$

Set

$$\phi_0(t) = e^{\lambda_0 t} \quad \text{for } t \in [-\|r\|, 0] .$$

We see that $\phi_0 \in C([-\|r\|, 0], \mathbb{R})$ and, from (2.3) we have

$$\begin{aligned} L(\lambda_0; \phi_0) &\equiv \phi_0(0) + \int_{-1}^{0} e^{-\lambda_0 r(\theta)} \left(\int_{-r(\theta)}^{0} e^{-\lambda_0 s} \phi_0(s) ds \right) dv(\theta) \\ &= 1 + \int_{-1}^{0} e^{-\lambda_0 r(\theta)} \left(\int_{-r(\theta)}^{0} e^{-\lambda_0 s} e^{\lambda_0 s} ds \right) dv(\theta) \\ &= 1 + \int_{-1}^{0} r(\theta) e^{-\lambda_0 r(\theta)} dv(\theta) = 1 + \beta(\lambda_0) > 0 , \end{aligned} \tag{4.4}$$

where $\beta(\lambda_0)$ is defined as (2.2). Next, it is considered a number δ_0 with $0 < \delta_0 < \delta$ and let

$$\phi = \frac{\delta_0}{\|\phi_0\|} \phi_0 . \tag{4.5}$$

Clearly, it is in $C([-\|r\|, 0], \mathbb{R})$ and $\|\phi\| = \delta_0 < \delta$. Therefore, for (4.5), the solution x of (1.1)-(1.2) satisfies (4.3). Also, from Theorem 2.1 and taking into consideration (4.4), in addition to the linearity of the operator $L(\lambda_0; .)$, it is obtained

$$\lim_{t\to\infty} \left[e^{-\lambda_0 t} x(t) \right] = \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} = \frac{(\delta_0 / \|\phi_0\|) L(\lambda_0; \phi_0)}{1 + \beta(\lambda_0)} = \frac{\delta_0}{\|\phi_0\|} > 0 .$$

But, since $\lambda_0 > 0$, from (4.3) it follows that

$$\lim_{t\to\infty} \left[e^{-\lambda_0 t} x(t) \right] = 0 .$$

We obtain a contradiction. The proof of Theorem 2.3 is now complete.

5. EXAMPLES

Example 5.1. Consider the equation (1.1) for $r(\theta) = \theta^2$ and $v(\theta) = \theta^2 + \theta$. In this example, if we replace theese functions in (1.3), it is obtained the following characteristic equation

$$\lambda = \int_{-1}^{0} e^{-\lambda\theta^2} d(\theta^2 + \theta) \tag{5.1}$$

and we see that $\lambda = 0$ is a root of (5.1). Therefore, for $\lambda_0 = 0$ the condition of the Theorem 2.3 is provided. That is, since v is decreasing on $\left[-1, -\frac{1}{2}\right]$ and v is increasing on $\left[-\frac{1}{2}, 0\right]$,

$$\mu(\lambda_0) = \mu(0) = \int_{-1}^{0} \theta^2 dV(\theta^2 + \theta) = \int_{-1}^{-\frac{1}{2}} \theta^2 dV(\theta^2 + \theta) + \int_{-\frac{1}{2}}^{0} \theta^2 dV(\theta^2 + \theta)$$

$$\leq \left[\max_{-1 \leq \theta \leq -\frac{1}{2}} (\theta^2)\right] V(\theta^2 + \theta)\left(-1, -\frac{1}{2}\right) + \left[\max_{-\frac{1}{2} \leq \theta \leq 0} (\theta^2)\right] V(\theta^2 + \theta)\left(-\frac{1}{2}, 0\right)$$

$$= 1 \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{5}{16} < 1 .$$

Since $\lambda_0 = 0$, the zero solution of (1.1) is stable.

Example 5.2. Consider the equation (1.1) for $r(\theta) = \frac{\theta+1}{2}$ and $v(\theta) = \frac{\theta}{2(1-\sqrt{e})}$. Then, the characteristic equation (1.3) is

$$\lambda = \int_{-1}^{0} e^{-\lambda(\frac{\theta+1}{2})} d\left(\frac{\theta}{2(1-\sqrt{e})}\right) = \frac{1}{2(1-\sqrt{e})} \int_{-1}^{0} e^{-\lambda(\frac{\theta+1}{2})} d\theta \tag{5.2}$$

and we see that $\lambda = -1$ is a root of (5.2). Therefore, $\lambda_0 = -1$ is a root, and the condition of Theorem 2.3 is satisfied. That is, since v is decreasing on $[-1,0]$,

$$\mu(\lambda_0) = \mu(-1) = \int_{-1}^{0} \left(\frac{\theta+1}{2}\right) e^{\frac{\theta+1}{2}} dV\left(\frac{\theta}{2(1-\sqrt{e})}\right)$$

$$\leq \left[\max_{-1 \leq \theta \leq 0} \left(\frac{\theta+1}{2}\right) e^{\frac{\theta+1}{2}}\right] V\left(\frac{\theta}{2(1-\sqrt{e})}\right)(-1,0) = \frac{\sqrt{e}}{2} \cdot \frac{1}{2(\sqrt{e}-1)} \cong 0.64 < 1 .$$

Since $\lambda_0 = -1$, the zero solution of (1.1) is asymptotically stable.

Example 5.3. Consider the equation (1.1) for $r(\theta) = -\theta$ and $v(\theta) = \frac{2}{1-e^{-2}} e^{\theta}$. Then, the characteristic equation (1.3) is

$$\lambda = \int_{-1}^{0} e^{-\lambda(-\theta)} d\left(\frac{2}{1-e^{-2}} e^{\theta}\right) = \frac{2}{1-e^{-2}} \int_{-1}^{0} e^{\theta(\lambda+1)} d\theta \tag{5.3}$$

and we see that $\lambda = 1$ is a root of (5.3). Therefore, for $\lambda_0 = 1$ the condition of Theorem 2.3 is satisfied. That is, since v is increasing on $[-1,0]$,

$$\mu(\lambda_0) = \mu(1) = \int_{-1}^{0} e^{\theta}(-\theta)dV(\frac{2}{1-e^{-2}}e^{\theta}) \leq \left[\max_{-1\leq\theta\leq 0} e^{\theta}(-\theta)\right] V\left(\frac{2}{1-e^{-2}}e^{\theta}\right)(-1,0)$$

$$= \frac{1}{e}\cdot\frac{2}{1-e^{-2}}(1-e^{-1}) = \frac{2}{e+1} < 1\,.$$

Since $\lambda_0 = 1$, the zero solution of (1.1) is unstable.

6. CONCLUSIONS

In this study, firstly, a basic asymptotic result for the solution of the equation (1.1) is proved. Secondly, we obtained a useful exponential boundary for solutions and the stability of zero solutions were shown. These results were obtained using a suitable real root for the characteristic equation. Finally, three examples were given for stability.

ACKNOWLEDGEMENTS

The authors thank the authors which listed in References for many useful support.

REFERENCES

[1] Cooke KL and Ferreira JM. Stability Conditions for Linear Retarded Functional Differential Equations, Journal of Mathematical Analysis and Applications, 1983; 96, 480-504.

[2] Ferreira JM and Györi I. Oscillatory Behavior in Linear Retarded Functional Differential Equations, Journal of Mathematical Analysis and Applications, 1987; 128, 332-346.

[3] Pedro AM. Oscillation and Nonoscillation Criteria for Retarded Functional Differential Equations, Proceedings of Equadiff-11, 2005; pp.353–362.

[4] Pedro AM. Nonoscillation Criteria for Retarded Functional Differential Systems, International Journal of Pure and Applied Mathematics, Volume 31 No. 1 2006, 47-71.

[5] Pedro AM. Oscillation Criteria for Retarded Functional Differential Systems, International Journal of Pure and Applied Mathematics, 2010; Volume 63 No. 1 75-84.

[6] Hale JK and Verduyn Lunel SM. Introduction to Functional Differential Equations, Springer, Berlin, Heidelberg, New York, 1993.

[7] d'Albis H, Augeraud-Véron E and Hupkes HJ. Multiple Solutions in Systems of Functional Differential Equations, Journal of Mathematical Economics, 2014; 52 50–56.

[8] Kong Q, Oscillation for Systems of Functional Differential Equations, Journal of Mathematical Analysis and Applications, 1996; 198, 608-619.

[9] Philos Ch. G and Purnaras IK. Asymptotic Properties, Nonoscillation, and Stability for Scalar First Order Linear Autonomous Neutral Delay Differential Equations, Electron. J. Differential Equations, 2004; No. 03, pp. 1-17.

[10] Burton TA. Volterra Integral and Differential Equations, Academic Press, New York, 1983.

[11] Corduneanu C, Integral Equations and Applications, Cambridge University Press, New York, 1991.

[12] Kolmanovski V, Myshkis A, Applied Theory of Functional Differential Equations, Kluver Academic, Dordrecht, 1992.

[13] Kuang Y, Delay Differential Equations with Applications in Population Dynamics, Academic Press, San Diego, 1993.

[14] Lakshmikantham V, Wen L and Zhang B, Theory of Differential Equations with Unbounded Delay, Kluwer Academic Publishers, London, 1994.

[15] Niculescu SI., Delay Effects on Stability, Springer-Verlag London Limited, 2001.

[16] Ford NJ, Yan Y, Malique Md.A. Numerical Treatment of Oscillatory Functional Differential Equations, Journal of Computational and Applied Mathematics, 2010; 234 2757–2767.

[17] Malique Md.A. Numerical Treatment of Oscillatory Delay and Mixed Functional Differential Equations Arising in Modeling, University of Liverpool (University of Chester), Unpublished Doctoral Dissertation, 2012.

[18] Kordonis I-G E, Niyianni NT and Philos Ch. G, On the Behavior of the Solutions of Scalar First Order Linear Autonomous Neutral Delay Differential Equations, Arch. Math. 1998; (Basel) 71, 454-464.

[19] Kordonis I-GE and Philos Ch.G, The Behavior of Solutions of Linear Integro-Differential Equations with Unbounded Delay, Computers & Mathematics with Applications, 1999; 38, 45-50.

[20] Philos Ch.G and Purnaras IK. Periodic first order linear neutral delay differential equatiıons, Applied Mathematics and Computation, 2001; 117 203-22

[21] Protter MH. MorreyCB. A first course in Real Analysis, Springer Verlag, New York, 1991.