Ocak 2010 Cilt:18 No:1 Kastamonu Eğitim Dergisi 89-98

THE TWO METHOD ON EQUATION WITH RETARDED ARGUMENT

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Abstract

In this paper, we applied two approximate methods for the solution of a boundary value problem for a differential equation with retarded argument:

$$x''(t) + a(t)x(t-\tau(t)) = f(t)$$
$$x(t) = \varphi(t) \quad (\lambda_0 \le t \le 0) \quad x(T) = x_\tau,$$

where $a(t), f(t), \tau(t) \ge 0 \ (0 \le t \le T)$ *and* $\varphi(t) \ (\lambda_0 \le t \le 0)$ *are known as continuous functions.*

Key world: *ordinary differential equations, boundry value problem, succesive approximations method.*

BİR YAVAŞ DEĞİŞKENLİ DENKLEM İÇİN İKİ YAKLAŞIK METOT

Özet

Bu çalışmada

$$x''(t) + a(t)x(t-\tau(t)) = f(t)$$
$$x(t) = \varphi(t) \quad (\lambda_0 \le t \le 0) \quad x(T) = x_\tau,$$

Yavaş değişkenli denklem için konulmuş sınır değer probleminin çözümü için iki yaklaşık metot verilmiştir. Burada $a(t), f(t), \tau(t) \ge 0 \ (0 \le t \le T)$ *ve* $\varphi(t) \ (\lambda_0 \le t \le 0)$ *önceden verilmiş sürekli fonksiyonlardır.*

Key world: *ordinary differential equations, boundry value problem, succesive approximations method*

1. Introduction

A common method used for the analytical solution of the boundary value problems is the integral equation method [1,2]. With this method, we obtain an integral equation that is equivalent to the boundary value problem and the solution of the integral equation is defined as the solution of the boundary value problem. The equivalent integral equation is usually a Fredholm equation in the classical theory. In this study we obtain a Fredholm-Volterra integral equation different from classical theory for the problem

$$x''(t)+a(t)x(t-\tau(t))=f(t)$$
$$x(t)=\varphi(t)\ \ (\lambda_0\le t\le 0),\ \ x(T)=x_\tau, \tag{1}$$

where $0\le t\le T$ and $a(t), f(t), \tau(t)\ge 0\ (0\le t\le T)$ and $\varphi(t)\ (\lambda_0\le t\le 0)$ are known as continuous functions. The Fredholm operator included in the equivalent integral equation is an operator with a degenerated kernel. We applied the modified successive method and consecutive substitution method for problem (1). One of these methods suggested by Ja. D. Mamedov [3] has been problem (1) by Aykut and Yıldız [4].

In this study these methods were applied to the boundary value problem with retarted argument. We investigated the solution for arbitrary continuous function $\tau(t)$.

2. An equivalent integral equation

In problem (1), if we take $\lambda(t)=t-\tau(t)$ then $t_0\in[0,t]$ is a point located at the left side of T such that conditions $\lambda(t_0)=0$ and $\lambda(t)\le 0\quad(0\le t\le t_0)$ are satisfied, where, $\lambda_0=\min_{0\le t\le t_0}\lambda(t)$. We assume that $\lambda(t)$ is a nondecreasing function in the interval $[t_0,t]$ and the equation $\lambda(t)=\sigma$ has differential continuous solution $t=\gamma(\sigma)$ for arbitrary $\sigma\in[0,\lambda(t)]$. It can be seen that if $x^*(t)$ is a solution of the boundary value for problem (1) then $x^*(t)$ is also the solution of the equation

$$x(t)=\hat{h}(t)+\frac{t}{T}\int_0^T(T-s)a(s)x(s-\tau(s))ds-\int_0^t(t-s)a(s)x(s-\tau(s))ds. \tag{2}$$

Here,

$$\hat{h}(t)=\varphi(0)-(x_T-\varphi(0))\frac{t}{T}-\frac{t}{T}\int_0^T(T-s)f(s)ds+\int_0^t(t-s)f(s)ds.$$

Let $\sigma = s - \tau(s)$. Therefore Eq. (2) can be written as follows:

$$x(t) = h(t) + \frac{t}{T}\int_0^{\lambda(T)} (T - \gamma(\sigma)a(\gamma(\sigma))x(\sigma)\gamma'(\sigma)d\sigma$$
$$-\int_0^{\lambda(t)} (t - \gamma(\sigma)a(\gamma(\sigma))x(\sigma)\gamma'(\sigma)d\sigma, \quad (3)$$

where

$$h(t) = \hat{h}(t) + \frac{t}{T}\int_{\lambda_0}^{0} (T - \gamma(\sigma)a(\gamma(\sigma))\varphi(\sigma)\gamma'(\sigma)d\sigma$$
$$-\int_{\lambda_0}^{0} (t - \gamma(\sigma)a(\gamma(\sigma))\varphi(\sigma)\gamma'(\sigma)d\sigma, \quad (4)$$

Let $K_1(\sigma) = [T - \gamma(\sigma)]a(\gamma(\sigma))\gamma'(\sigma)$ and $K(t,\sigma) = [t - \gamma(\sigma)]a(\gamma(\sigma))\gamma'(\sigma)$. Therefore we write

$$x(t) = h(t) + \frac{t}{T}\int_0^{\lambda(T)} K_1(\sigma)x(\sigma)d\sigma - \int_0^{\lambda(t)} K(t,\sigma)x(\sigma)d\sigma \quad (5)$$

or

$$x(t) + h(t) + \frac{t}{T}F_\lambda x + V_\lambda x \quad (6)$$

where

$$F_\lambda x \equiv \int_0^{\lambda(T)} K_1(\sigma)x(\sigma)d\sigma$$

is the Fredholm operator,

$$V_\lambda x \equiv -\int_0^{\lambda(t)} K(t,\sigma)x(\sigma)d\sigma$$

is the Volterra operator. Eq. (6) is a Fredholm-Volterra integral equation and it is equivalent to problem (1)

3. Modified successive approximations method

Now, we will define the modified successive approximations for the integral equation (5), which are different from the ordinary successive approximations, as follows:

$$x_n(t) = h(t) + \frac{t}{T} F_\lambda x_n + V_\lambda x_{n-1} \qquad (n = 1,2,...).$$
(7)

In order to obtain a solution of problem (1) using this method, we will consider the following theorem.

Theorem 1. *If*

$$\alpha = 1 - \frac{1}{T} \int_0^{\lambda(T)} K_1(\sigma)\, \sigma\, d\sigma \neq 0,$$

$$q = \left(\frac{1}{|\alpha|} \int_0^{\lambda(T)} |K_1(\sigma)|\, d\sigma + 1 \right) \int_0^{\lambda(T)} |K_1(T,\sigma)|\, d\sigma < 1,$$

then the limit of the modified successive approximations

$$x_n(t) = h(t) + \frac{t}{T\alpha} F_\lambda h + V_\lambda x_{n-1} + \frac{t}{T\alpha} F_\lambda V_\lambda x_{n-1}$$

converges to the solution of the problem (1) and the convergence speed is determined by

$$\| x_n - x \| \leq q^n \| x - x_0 \|$$

or

$$\| x_n - x \| \leq \frac{q^n}{1-q} \| x_1 - x_0 \|.$$

Proof. In order to obtain the approximation of $x_n(t)$ we will use the auxiliary equation

$$y(t) = \hat{h}(t) + \frac{t}{T} F_\lambda y. \tag{8}$$

Let $F_\lambda y = c$.

Then,

$$y(t) = \hat{h}(t) + \frac{t}{T} c. \tag{9}$$

If we use Eq. (9) in Eq. (8), then we obtain

$$\left(1-\frac{1}{T}F_{\lambda}t\right)c=F_{\lambda}\hat{h}$$

for c. By hypothesis we know that $\alpha=1-(1/T)F_{\lambda}t\neq 0,$ therefore $c=(1/\alpha)F_{\lambda}\hat{h}$ and we can write

$$y=\hat{h}(t)+\frac{t}{T\alpha}F_{\lambda}\hat{h}.$$

If we consider $\hat{h}(t)=h(t)+V_{\lambda}x_{n-1}$ and use Eq. (7), then we have

$$x_n(t)=h(t)+\frac{t}{T\alpha}F_{\lambda}h+V_{\lambda}x_{n-1}+\frac{t}{T\alpha}F_{\lambda}V_{\lambda}x_{n-1} \quad (10)$$

Now, we will show that the approximations $\{x_n(t)\}$, which are defined by Eq. (10), converge to the solution of integral equation (5). Let us write

$$\Omega x=\left(\frac{t}{T\alpha}F_{\lambda}V_{\lambda}+V_{\lambda}\right)x.$$

Therefore, we write the approximations of Eq. (10) as

$$x_n(t)=h+\frac{t}{T\alpha}F_{\lambda}h+\Omega x_{n-1} \qquad (n=1,2,3,...). \quad (11)$$

Thus, Eq. (5) can be written as

$$x=h+\frac{t}{T\alpha}F_{\lambda}h+\Omega x. \quad (12)$$

Hence, the modified succesive approximations of Eq. (10) are similar to the ordinary successive approximations (12) which are equivalent to the integral equation (5). In order to show the convergence of approximations (10) to Eq. (5) we must prove the convergence of approximations (11) to Eq. (12). If the condition

$$\|\Omega x\|\leq\|x\| \quad (q<1),$$

is satisfied then ordinary successive approximations (11) converge to the solution of Eq. (12) and the convergence speed is determined by

$$\|x_n-x\|\leq q^n\|x-x_0\|$$

or

$$\|x_n-x\|\leq\frac{q^n}{1-q}\|x_1-x_0\|.$$

By this hypothesis we have $q<1$ and

$$|\Omega x| = \frac{1}{|\alpha|}\int_0^{\lambda(T)} |K_1(\sigma)| \left(\int_0^{\lambda(\sigma)} |K(\sigma,\tau)|\, x(\tau)\, |d\tau\right) d\sigma + \int_0^{\lambda(T)} |K(T,\sigma)|\, x(\sigma)\, d\sigma$$

$$\leq \left[\frac{1}{|\alpha|}\int_0^{\lambda(T)} |K_1(\sigma)|\, d\sigma + 1\right]\int_0^{\lambda(T)} |K(T,\tau)|\, d\tau \| x \|,$$

so we write

$$\| \Omega x \| \leq q \| x \| .$$

4. The consecutive substitution method

If we substitue right side of equation (6) instead of x in the operator $V_\lambda x$ to equation (6) then we obtain

$$x(t) = h(t) + V_\lambda h + \left[\frac{t}{T} + V_\lambda \frac{t}{T}\right] F_\lambda x + V_\lambda^2 x \quad (13)$$

If we rewrite the right side of equation (6) instead of x in the operator $V_\lambda^2 x$ to equation (13) then we have

$$x(t) = h(t) + V_\lambda h + V^2{}_\lambda h + \left[\frac{t}{T} + V_\lambda \frac{t}{T} + V^2{}_\lambda \frac{t}{T}\right] F_\lambda x + V_\lambda^3 x$$

(14)

Whene this operation is applied of n time we have

$$x = \sum_{i=0}^{n} V_\lambda^i h + \sum_{i=0}^{n} V_\lambda^i \frac{t}{T} F_\lambda x + V_\lambda^{n+1} x$$

If we choose

$$h_n(t) = \sum_{i=0}^{n} V_\lambda^i h, \quad a_n(t) = \sum_{i=0}^{n} V_\lambda^i \frac{t}{T}$$

then it becomes

$$x = h_n(t) + a_n(t) F_\lambda x + V_\lambda^{n+1} x \quad (15)$$

Now we can proof that the formula it is true

$$|V_\lambda^n x| \leq \frac{[K\lambda(T)]^n}{n!} \quad (n \in N) \quad (16)$$

for the operator

$$V_\lambda x \equiv -\int_0^{\lambda(t)} K(t,s)x(s)ds \qquad (0 \le t \le T) \qquad (17)$$

As a result of this we neclect the operator $|V_\lambda^{n+1} x|$ in equation (15) for n's which are big enough. Thus the consecutive approximations are formed by taking the Volterra operator into consideration.

$$x_n(t) = h_n(t) + a_n(t) F_\lambda x_n \qquad (18)$$

Theorem 2: Let $\tau(t) \ge 0, a(t), f(t) \quad (0 \le t \le T)$ be the functions given in problem (1) and

$$\alpha_n = 1 - \int_0^{\lambda(T)} K_1(s) h_n(s) ds \ne 0 .$$

Also

such that
$$\lim_{n\to\infty} A_n \frac{[K\lambda(T)]^n}{n!} = 0$$

$$A_n = 1 + \frac{\| a_n \|}{|\alpha_n|} \int_0^{\lambda(T)} |K_1(s)| d\, s$$

Then. the limit of the approximations

$$x_n(t) = h_n(t) + \frac{a_n(t)}{\alpha_n} \int_0^{\lambda(T)} K_1(s) h_n(s) ds$$

converges to the solution of problem (1) and the speed of the convergence is determined by

$$|x_n(t) - x(t)| \le A_n \frac{[K\lambda(T)]^n}{n!}$$

Proof: Eq. (18) is the Fredholm integral equation with a degerenated kernel.The solution of

$$x_n(t) = h_n(t) + \int_0^{\lambda(T)} a_n(t) K_1(s) x_n(s) ds \qquad (19)$$

is the same as the solution of eq. (15) and problem (1). Now, let us find the solution of equation (19). So, we use auxiliary equation,

$$y(t) = h_n(t) + a_n(t) F_\lambda y$$

where $F_\lambda y$ is shown as $F_\lambda y = c_n$.Thus $y(t)$ is like that

$$y(t) = h_n(t) + a_n(t)\,c_n \qquad (20)$$

Therefore,

$$c_n = \int_0^{\lambda(T)} K_1(s)\,h_n(s)\,ds + c_n \int_0^{\lambda(T)} K_1(s)\,a_n(s)\,ds$$

When $\alpha_n = 1 - \int_0^{\lambda(T)} K_1(s)\,h_n(s)\,ds \neq 0$ is given c_n is found as follows

$$c_n = \frac{1}{\alpha_n}\int_0^{\lambda(T)} K_1(s)\,h_n(s)\,ds \qquad (21)$$

If we use equation (21) in (20), for $n = 1,2,...$ then

$$x_n(t) = h_n(t) + \frac{a_n(t)}{\alpha_n}\int_0^{\lambda(T)} K_1(s)\,h_n(s)\,ds \qquad (22)$$

This operation is the approximate solution of problem(1) that is, the limit of $x_n(t)$ converges to the solution of problem (1).

Now let us determine the error of the approximate solution of eq. (22). Using (15) and (18), we reach

$$x - x_n = a_n(t)\,F_\lambda\,(x - x_n) + V_\lambda^{n+1} x$$

If it is acceted that ε is $\varepsilon = x - x_n$, then we obtain the Fredholm integral equation with a degenerated kernel

$$\varepsilon = a_n(t)\,F_\lambda \varepsilon + V_\lambda^{n+1} x \qquad (23)$$

It was proven that solution of equation (23) is being found by using the following formula:

$$\varepsilon = V_\lambda^{n+1} x + \frac{a_n(t)}{\alpha_n}\int_0^{\lambda(T)} K_1(s)\,V_\lambda^{n+1} x(s)\,ds$$

Thus, we write

$$|\,\varepsilon\,| \le [1 + \frac{\|\,a_n\,\|}{|\,\alpha_n\,|}\int_0^{\lambda(T)} |\,K_1(s)\,|\,ds]\,\|\,V_\lambda^{n+1} x(s)\,\|$$

Then, by the hypothesis,

$$A_n = 1 + \frac{\|\,a_n\,\|}{|\,\alpha_n\,|}\int_0^{\lambda(T)} |\,K_1(s)\,|\,ds$$

and we have

$$|x_n(t)-x(t)| \le A_n \frac{[K\lambda(T)]^n}{n!} \|x\| \qquad (24)$$

Example 1. Let us consider the boundary value problem:

$$x''(t)+tx(t-\frac{1}{2}\sqrt{t})=t^2-\frac{1}{2}t^{\frac{3}{2}} \quad 0\le t\le 1), \qquad (25)$$
$$x(t)=0 \quad (-1/16\le t\le 0), \quad x(1)=1$$

This equation can be written as the Fredholm-Volterra integral equation

$$x(t)=\frac{409}{420}t+\frac{1}{12}t^4-\frac{2}{35}t^{\frac{7}{2}}+\frac{t}{16}\int_0^{1/2}\left[3+4\sigma-16\sigma^2+\frac{3+28\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)\,d\sigma$$
$$-\frac{1}{16}\int_0^{t-\sqrt{t}/2}\left[(4t-1)+(16t-12)\sigma-16\sigma^2+\frac{(4t-1)+(48t-20)\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)d\sigma. \qquad (26)$$

Let

$$h(t)=\frac{409}{420}t+\frac{1}{12}t^4-\frac{2}{35}t^{\frac{7}{2}},$$

$$K_1(\sigma)=\frac{1}{16}\left[3+4\sigma-16\sigma^2+\frac{3+28\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right],$$

$$K(t,\sigma)=\frac{1}{16}\left[(4t-1)+(16t-12)\sigma-16\sigma^2+\frac{(4t-1)+(48t-20)\sigma-80\sigma^2}{\sqrt{1+16\sigma}}\right]$$

and

$$F_\lambda x\equiv\int_0^{1/2}K_1(\sigma)\,x(\sigma)\,d\sigma, \qquad V_\lambda x\equiv-\int_0^{t-\sqrt{t}/2}K(t,\sigma)x(\sigma)d\sigma.$$

Therefore, the integral equation (19) can be written as

$$x(t)=h(t)+tF_\lambda x+V_\lambda x \qquad (27)$$

and this equation is equivalent to problem (25). Some values of the solution of this equation are obtained by using the method of modified successive approximations and the method of consecutive approximations of order two which are given in Table 1, where the first approximation is $x_0(t) = (409/420)t$.

Table 1. Values at some point inthe interval $[0,1]$

t_i	$x(t_i)$	$x_2^{(2)}(t_i)$	$x_2^{(3)}(t_i)$	$\varepsilon_1(t_i)$	$\varepsilon_2(t_i)$
0.00	0.00	0.0000000	0.0000000	0.000000	0.000000
0.25	0.25	0.2501132	0.2501078	0.000113	0.000107
0.50	0.50	0.5000794	0.5000684	0.000079	0.000068
1.00	1.00	1.0000013	0.9999046	0.000001	-0.000095

$x_2^{(2)}$: The modified successive method; $x_2^{(3)}$: The concecutive substitution method. ε_1 and ε_2 are speed of the modified successive method and the concecutive substitution method, respectively.

5. References

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