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Approximation Properties of Stancu-Type (p,q)-Baskakov Operators

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Abstract
This paper deals with the construction of the Stancu type modification of the Baskakov operators by using (p,q)-integers. The rate of convergence of these operators are obtained by using Peetre's K-functional and modulus of continuity. In addition, the pointwise estimation of the newly constructed operators are examined for functions belong to a Lipschitz space. Finally, the convergence of the constructed operators to some functions is shown with the help of MATLAB.

Keywords: (p,q)-Baskakov-Stancu Operators, Modulus of Continuity, Weighted Korovkin Theorem.

Stancu Tipli (p,q)-Baskakov Operatörlerinin Yaklaşım Özellikleri

Öz
Bu makale (p,q)-tamsayılar kullanılarak oluşturulan Baskakov operatörlerinin Stancu tipli modifikasyonunun inşası ile ilgilenmektedir. Bu operatörlerin yaklaşım derecesi Peetre-K fonksiyonelleri ve süreklilik modülü kullanılarak elde edilmiştir. Buna ek olarak yeni oluşturulan operatörlerin noktasal yaklaşımı bir Lipschitz uzayına ait fonksiyonlar ile incelenmiştir. Sonuç olarak, üretilen operatörlerin bazı fonksiyonlara yakınsaklığı MATLAB yardımıyla elde edilen grafiklerle gösterilmiştir.

Anahtar kelimeler: (p,q)-Baskakov-Stancu Operatörleri, Süreklilik Modülü, Ağırlıklı Korovkin Teoremi.

1. Introduction

Approximation theory has an extensive research area in mathematics. Varied generalizations of some linear positive operators to the quantum calculus (q-calculus) and their approximation results have been extensively investigated for three decades. Some generalizations of Baskakov operators based on q-integers can be read from [1-3]. Further, quantum calculus is extended to post-quantum calculus, displayed by (p,q)-calculus. The new parameter gives flexibility to the approximation. (p,q)-calculus is used effectively in such areas as neural network, field theory, hypergeometric series, Lie group, and differential equations. In 2015, Mursaleen et al. [4] pioneered (p,q)-calculus in approximation theory. Due to its comprehensive applications, the approximation behaviors of linear positive operators in (p,q)-calculus have been studied actively by different authors [5-10]. Most recently, Aral and Gupta [11] have initiated the (p,q)-analogue of the Baskakov operators.

(p,q)-calculus can be illustrated by some essential notations and definitions. The (p,q)-integer of m is described by

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$$[m]_{p,q} := p^{m-1} + p^{m-2}q + ... + pq^{m-2} + q^{m-1} = \begin{cases} \dfrac{p^m - q^m}{p-q} & , p \neq q \neq 1 \\ mp^{m-1} & , p = q \neq 1 \\ [m]_q & , p = 1 \\ m & , p = q = 1 \end{cases}$$

where p and q are non-negative numbers and $[m]_q$ shows q-integers for $m = 0,1,2,...$ It is seen that $[m]_{p,q} = p^{m-1}[m]_{q/p}$. The (p,q)-factorial is given by

$$[m]_{p,q}! = \prod_{j=1}^{m} [j]_{p,q}, m \geq 1,$$

are the (p,q)-binomial coefficients. The (p,q)-power basis is explained as

$$(x \oplus y)_{p,q}^m = (x+y)(px+qy)(p^2x+q^2y)...(p^{m-1}x+q^{m-1}y).$$

Further details about (p,q)-calculus are given in [12] and [13]. The (p,q)-analogue of Baskakov operators are introduced by Aral and Gupta [11]

$$B_{m,p,q}(f;x) = \sum_{s=0}^{\infty} \begin{bmatrix} m+s-1 \\ s \end{bmatrix}_{p,q} p^{s+m(m-1)/2} q^{s(s-1)/2} \frac{x^s}{(1 \oplus x)_{p,q}^{m+s}} f\left(\frac{p^{m-1}[s]_{p,q}}{q^{s-1}[m]_{p,q}} \right), \quad (1)$$

where $0 < q < p \leq 1$, and $x \in [0, \infty)$.

In the following lemma, we present the moments of (p,q)-Baskakov operators.

Lemma 1. [11] For $x \in [0,\infty)$ and $0 < q < p \leq 1$, the (p,q)-analogue of Baskakov operators $B_{m,p,q}(.;.)$ satisfy the following equalities:

$$B_{m,p,q}(1;x) = 1,$$
$$B_{m,p,q}(t;x) = x,$$
$$B_{m,p,q}(t^2;x) = x^2 + \frac{p^{m-1}x}{[m]_{p,q}}\left(1 + \frac{p}{q}x\right).$$

In section 2, we will extend the operators given by (1) for $0 \leq \alpha \leq \beta$, $x \in [0,\infty)$ and $0 < q < p \leq 1$. Then we will calculate the moments of the newly constructed operators. In addition, we will present the convergence of the operators according to the weighted Korovkin theorem.

2. Construction of the operators

Definition 1. For any $x \in [0,\infty)$, $0 < q < p \leq 1$, $0 \leq \alpha \leq \beta$, we construct the (p,q)-analogue of Stancu type Baskakov operators by

$$B_{m,p,q}^{\alpha,\beta}(f;x) = \sum_{s=0}^{\infty} b_{m,s}^{p,q}(x) f\left(\frac{p^{m-1}q^{1-s}[s]_{p,q} + \alpha}{[m]_{p,q} + \beta} \right), \quad (2)$$

where

$$b_{m,s}^{p,q}(x) = \begin{bmatrix} m+s-1 \\ s \end{bmatrix}_{p,q} p^{s+m(m-1)/2} q^{s(s-1)/2} \frac{x^s}{(1 \oplus x)_{p,q}^{m+s}}. \quad (3)$$

We will give the next lemma to present the moments of the operators (2).

Lemma 2. Let $B_{m,p,q}^{\alpha,\beta}(.;.)$ be defined by (2) and (3). Then we obtain the following equalities

$$B_{m,p,q}^{\alpha,\beta}(1;x)=1, \tag{4}$$

$$B_{m,p,q}^{\alpha,\beta}(t;x)=\frac{[m]_{p,q}}{[m]_{p,q}+\beta}x+\frac{\alpha}{[m]_{p,q}+\beta}, \tag{5}$$

$$\begin{aligned} B_{m,p,q}^{\alpha,\beta}(t^2;x)=&\frac{[m]_{p,q}^2}{\left([m]_{p,q}+\beta\right)^2}\left(1+\frac{p^m}{q[m]_{p,q}}\right)x^2+\frac{[m]_{p,q}}{\left([m]_{p,q}+\beta\right)^2}\left(p^{m-1}+2\alpha\right)x \\ &+\frac{\alpha^2}{\left([m]_{p,q}+\beta\right)^2}. \end{aligned} \tag{6}$$

Proof. First of all, by using the definition of the operators (2), we can check the first moment (4) as follows

$$B_{m,p,q}^{\alpha,\beta}(1;x)=\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x)=1.$$

Secondly, we have the following equality for the second moment by the definition of the operators (2)

$$\begin{aligned} B_{m,p,q}^{\alpha,\beta}(t;x)&=\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x)\frac{p^{m-1}q^{1-s}[s]_{p,q}+\alpha}{[m]_{p,q}+\beta} \\ &=\frac{[m]_{p,q}}{[m]_{p,q}+\beta}\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x)\frac{p^{m-1}[s]_{p,q}}{q^{s-1}[m]_{p,q}}+\frac{\alpha}{[m]_{p,q}+\beta}\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x) \\ &=\frac{[m]_{p,q}}{[m]_{p,q}+\beta}B_{m,p,q}(t;x)+\frac{\alpha}{[m]_{p,q}+\beta}B_{m,p,q}(1;x) \\ &=\frac{[m]_{p,q}}{[m]_{p,q}+\beta}x+\frac{\alpha}{[m]_{p,q}+\beta}, \end{aligned}$$

as we see in equality (5).

Lastly, we obtain the third moment $B_{m,p,q}^{\alpha,\beta}(t^2;x)$ as follows

$$\begin{aligned} B_{m,p,q}^{\alpha,\beta}(t^2;x)&=\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x)\left(\frac{p^{m-1}q^{1-s}[s]_{p,q}+\alpha}{[m]_{p,q}+\beta}\right)^2 \\ &=\frac{[m]^2_{p,q}}{\left([m]_{p,q}+\beta\right)^2}\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x)\frac{p^{2m-2}[s]^2_{p,q}}{q^{2s-2}[m]^2_{p,q}}+\frac{2[m]_{p,q}\alpha}{\left([m]_{p,q}+\beta\right)^2}\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x)\frac{p^{m-1}[s]_{p,q}}{q^{s-1}[m]_{p,q}}+\frac{\alpha^2}{\left([m]_{p,q}+\beta\right)^2}\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x) \\ &=\frac{[m]^2_{p,q}}{\left([m]_{p,q}+\beta\right)^2}B_{m,p,q}(t^2;x)+\frac{2[m]_{p,q}\alpha}{\left([m]_{p,q}+\beta\right)^2}B_{m,p,q}(t;x)+\frac{\alpha^2}{\left([m]_{p,q}+\beta\right)^2}B_{m,p,q}(1;x) \end{aligned}$$

$$=\frac{[m]^2_{p,q}}{\left([m]_{p,q}+\beta\right)^2}\left(1+\frac{p^m}{q[m]_{p,q}}\right)x^2+\frac{[m]_{p,q}}{\left([m]_{p,q}+\beta\right)^2}\left(p^{m-1}+2\alpha\right)x$$
$$+\frac{\alpha^2}{\left([m]_{p,q}+\beta\right)^2}. \tag{7}$$

Remark 1. (Central Moments)
We use linearity of the operators $B^{\alpha,\beta}_{m,p,q}$ to get the first central moment $B^{\alpha,\beta}_{m,p,q}(t-x;x)$

$$B^{\alpha,\beta}_{m,p,q}(t-x;x)=\left(\frac{[m]_{p,q}}{\beta+[m]_{p,q}}-1\right)x+\frac{\alpha}{\beta+[m]_{p,q}}. \tag{8}$$

Similarly, for the second central moment $B^{\alpha,\beta}_{m,p,q}\left((t-x)^2;x\right)$, we have

$$B^{\alpha,\beta}_{m,p,q}\left((t-x)^2;x\right)=\mu_1(m)x^2+\mu_2(m)x+\mu_3(m), \tag{9}$$

where we shortly denote

$$\mu_1(m)=\left(\frac{[m]_{p,q}}{\beta+[m]_{p,q}}-1\right)^2+\frac{p^m[m]_{p,q}}{q\left(\beta+[m]_{p,q}\right)^2},$$

$$\mu_2(m)=\frac{[m]_{p,q}\,p^{m-1}-2\alpha\beta}{\left(\beta+[m]_{p,q}\right)^2},$$

$$\mu_3(m)=\frac{\alpha^2}{\left(\beta+[m]_{p,q}\right)^2}.$$

Let us choose $\tilde{\mu}(m):=\max\left\{\mu_1(m),\frac{\mu_2(m)}{2},\mu_3(m)\right\}$. Finally, we can write

$$B^{\alpha,\beta}_{m,p,q}\left((t-x)^2;x\right)\le\tilde{\mu}(m)(1+x)^2. \tag{10}$$

Here the $B^{\alpha,\beta}_{m,p,q}(f;x)$ are linear and positive operators.

Remark 2. For $0<q<p\le 1$, $\lim_{m\to\infty}[m]_{p,q}=\frac{1}{p-q}$. To reach the convergence results of the operators $B^{\alpha,\beta}_{m,p,q}(f;x)$, we choose the sequences $0<q_m<p_m\le 1$ such that $\lim_{m\to\infty}p_m=1$, $\lim_{m\to\infty}q_m=1$, $\lim_{m\to\infty}p_m^m=1$ and $\lim_{m\to\infty}q_m^m=1$. Thus, we have $\lim_{m\to\infty}[m]_{p_m,q_m}=\infty$. Additionally, here we assume that $\mu_1(m)\to 0$, $\mu_2(m)\to 0$, $\mu_3(m)\to 0$ as $m\to\infty$, hence $\tilde{\mu}(m)\to 0$ as $m\to\infty$.

Just now, we primarily recall some important definitions of the weighted spaces:
$C[0,\infty)$ denotes the set of all continuous functions defined on the semi-axis $[0,\infty)$. $B_2[0,\infty)$ is the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)|\le M\left(1+x^2\right)$, where M is a positive constant. Further, $B_2[0,\infty)$ denotes a linear normed space with $\|f\|_2=\sup_{x\ge 0}\frac{|f(x)|}{1+x^2}$.

$C_B\left[0,\infty\right)$ is the class of all real valued continuous and bounded functions f on $\left[0,\infty\right)$. The norm is given by $\|f\|_{C_B}=\sup_{x\in[0,\infty)}\left|f\left(x\right)\right|$. Also $C_B^2\left[0,\infty\right)$ denotes the space of f, for which f'', f' and f are continuous on $\left[0,\infty\right)$. Then $C_2\left[0,\infty\right)$ signifies the subspace of all continuous functions in $B_2\left[0,\infty\right)$. Moreover, $C_2^*\left[0,\infty\right)$ indicates the subspace of all continuous functions in $B_2\left[0,\infty\right)$ for which $\lim_{x\to\infty}\frac{f\left(x\right)}{1+x^2}$ is finite.

Theorem 1. Let $B_{m,p,q}^{\alpha,\beta}\left(f;x\right)$ satisfy the conditions, given in Remark 2 for $0\le\alpha\le\beta$ and $0<q_m<p_m\le1$. And then for each $f\in C_2^*\left[0,\infty\right)$, $B_{m,p_m,q_m}^{\alpha,\beta}\left(f;x\right)$ converge uniformly to f on $\left[0,\infty\right)$.

Proof. To give the proof, it is satisfactory by the weighted Korovkin theorem [14] to see that

$$\lim_{m\to\infty}\left\|B_{m,p_m,q_m}^{\alpha,\beta}e_k-e_k\right\|_2=0,$$

where $e_k\left(x\right)=x^k$, $k=0,1,2$.

(i) Using equality (4), it is clear that

$$\lim_{m\to\infty}\left\|B_{m,p_m,q_m}^{\alpha,\beta}e_0-e_0\right\|_2=\lim_{m\to\infty}\sup_{x\ge0}\frac{\left|B_{m,p_m,q_m}^{\alpha,\beta}\left(1;x\right)-1\right|}{1+x^2}=0.$$

(ii) Using equality (5), we write

$$\begin{aligned}\lim_{m\to\infty}\left\|B_{m,p_m,q_m}^{\alpha,\beta}e_1-e_1\right\|_2&=\lim_{m\to\infty}\sup_{x\ge0}\frac{\left|B_{m,p_m,q_m}^{\alpha,\beta}\left(t;x\right)-x\right|}{1+x^2}\\&=\lim_{m\to\infty}\sup_{x\ge0}\frac{\left|\left(\frac{[m]_{p_m,q_m}}{\beta+[m]_{p_m,q_m}}-1\right)x+\frac{\alpha}{\beta+[m]_{p_m,q_m}}\right|}{1+x^2}\\&\le\lim_{m\to\infty}\left(1-\frac{[m]_{p_m,q_m}}{\beta+[m]_{p_m,q_m}}\right)\sup_{x\ge0}\frac{x}{1+x^2}+\lim_{m\to\infty}\frac{\alpha}{\beta+[m]_{p_m,q_m}}\sup_{x\ge0}\frac{1}{1+x^2}\\&\le\lim_{m\to\infty}\frac{1}{2}\left(1-\frac{[m]_{p_m,q_m}}{\beta+[m]_{p_m,q_m}}\right)+\lim_{m\to\infty}\frac{\alpha}{\beta+[m]_{p_m,q_m}}\\&=0.\end{aligned}$$

(iii) By using equality (6), we have

$$\lim_{m\to\infty}\left\|B^{\alpha,\beta}_{m,p_m,q_m}e_2-e_2\right\|_2=\lim_{m\to\infty}\sup_{x\geq 0}\frac{\left|B^{\alpha,\beta}_{m,p_m,q_m}\left(t^2;x\right)-x^2\right|}{1+x^2}$$

$$=\lim_{m\to\infty}\sup_{x\geq 0}$$

$$\times\frac{\left|\left(\frac{[m]^2_{p_m,q_m}}{\left(\beta+[m]_{p_m,q_m}\right)^2}\left(1+\frac{p_m^m}{q_m[m]_{p_m,q_m}}\right)-1\right)x^2+\frac{[m]_{p_m,q_m}}{\left(\beta+[m]_{p_m,q_m}\right)^2}\left(p_m^{m-1}+2\alpha\right)x+\frac{\alpha^2}{\left(\beta+[m]_{p_m,q_m}\right)^2}\right|}{1+x^2}$$

$$\leq\lim_{m\to\infty}\left|\frac{[m]^2_{p_m,q_m}}{\left(\beta+[m]_{p_m,q_m}\right)^2}\left(1+\frac{p_m^m}{q_m[m]_{p_m,q_m}}\right)-1\right|\sup_{x\geq 0}\frac{x^2}{1+x^2}$$

$$+\lim_{m\to\infty}\frac{[m]_{p_m,q_m}}{\left(\beta+[m]_{p_m,q_m}\right)^2}\left(p_m^{m-1}+2\alpha\right)\sup_{x\geq 0}\frac{x}{1+x^2}+\lim_{m\to\infty}\frac{\alpha^2}{\left(\beta+[m]_{p_m,q_m}\right)^2}\sup_{x\geq 0}\frac{1}{1+x^2}$$

$$\leq\lim_{m\to\infty}\left(\left|\frac{[m]^2_{p_m,q_m}}{\left(\beta+[m]_{p_m,q_m}\right)^2}\left(1+\frac{p_m^m}{q_m[m]_{p_m,q_m}}\right)-1\right|+\frac{1}{2}\frac{[m]_{p_m,q_m}}{\left(\beta+[m]_{p_m,q_m}\right)^2}\left(p_m^{m-1}+2\alpha\right)+\frac{\alpha^2}{\left(\beta+[m]_{p_m,q_m}\right)^2}\right)$$

$$=0.$$

In section 3, we will give an auxiliary lemma to verify the main results and then treat the local approximation properties by the help of Peetre's K-functionals and modulus of continuities.

3. Local approximation properties

First of all, we remember the properties of Peetre's K-functionals. The norm in $C_B^2[0,\infty)$ is defined as

$$\|f\|_{C_B^2[0,\infty)}=\|f\|_{C_B[0,\infty)}+\|f'\|_{C_B[0,\infty)}+\|f''\|_{C_B[0,\infty)}.$$

Peetre's K-functionals are defined as follows:

$$K_2(f,\delta):=\inf_{g\in C_B^2[0,\infty)}\left\{\|f-g\|_{C_B[0,\infty)}+\delta\|g\|_{C_B^2[0,\infty)}\right\}.$$

The modulus of continuity of the function $f\in C_B[0,\infty)$ is given by

$$\omega(f,\delta):=\sup_{0<h<\delta}\ \sup_{x,\ x+h\in[0,\infty)}\left|f(x+h)-f(x)\right|.$$

It is obvious that $\lim_{\delta\to 0^+}w(f,\delta)=0$ for $f\in C_B[0,\infty)$; and for each $x,t\in[0,1]$ and any $\delta>0,$ we have

$$\left|f(t)-f(x)\right|\leq w(f,\delta)\left(\frac{|t-x|}{\delta}+1\right). \qquad (11)$$

For $\delta>0,$ second order modulus of smoothness of the function f is identified by

$$\omega_2(f,\delta):=\sup_{0<h<\sqrt{\delta}}\ \sup_{x,x+h\in[0,\infty)}\left|f(x+2h)-2f(x+h)+f(x)\right|.$$

By Devore [15], for $M>0,$

$$K_2(f,\delta)\leq M\omega_2\left(f,\sqrt{\delta}\right).$$

Lemma 3. For $f\in C_B[0,\infty)$, we have

$$\left|B_{m,p,q}^{\alpha,\beta}(f;x)\right|\leq\|f\|_{C_B}. \tag{12}$$

Proof.

$$\begin{aligned}\left|B_{m,p,q}^{\alpha,\beta}(f;x)\right| &= \left|\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x)\,f\left(\frac{p^{m-1}q^{1-s}[s]_{p,q}+\alpha}{[m]_{p,q}+\beta}\right)\right| \\ &\leq\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x)\left|f\left(\frac{p^{m-1}q^{1-s}[s]_{p,q}+\alpha}{[m]_{p,q}+\beta}\right)\right| \\ &\leq\|f\|_{C_B}\,B_{m,p,q}^{\alpha,\beta}(1;x) \\ &=\|f\|_{C_B}.\end{aligned}$$

Lemma 4. Let $f\in C_B[0,\infty)$ and $g\in C_B^2[0,\infty)$. We describe the auxiliary operators $B_{m,p,q}^{*}$ *as*

$$B_{m,p,q}^{*}(g;x)=B_{m,p,q}^{\alpha,\beta}(g;x)+g(x)-g\left(\frac{\alpha}{\beta+[m]_{p,q}}+\frac{[m]_{p,q}}{\beta+[m]_{p,q}}x\right). \tag{13}$$

So, for all $g\in C_B^2[0,\infty)$, we get

$$\left|B_{m,p,q}^{*}(g;x)-g(x)\right|\leq\|g''\|_{C_B}\left(\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x)\right),$$

where

$$\eta_m(\alpha,\beta,x)=\frac{\alpha-\beta x}{\beta+[m]_{p,q}}.$$

Proof. From the auxiliary operators $B_{m,p,q}^{*}$, we have

$$B_{m,p,q}^{*}(g;x)=B_{m,p,q}^{\alpha,\beta}(g;x)+g(x)-g\left(\frac{\alpha}{\beta+[m]_{p,q}}+\frac{[m]_{p,q}}{\beta+[m]_{p,q}}x\right). \tag{14}$$

It is obvious from Lemma 2 that

$$B_{m,p,q}^{*}(1;x)=1,$$

$$\begin{aligned}B_{m,p,q}^{*}(t-x;x) &= B_{m,p,q}^{\alpha,\beta}((t-x);x)+(x-x)-\left(\frac{\alpha}{\beta+[m]_{p,q}}+\frac{[m]_{p,q}}{\beta+[m]_{p,q}}x-x\right) \\ &=\left(\frac{[m]_{p,q}}{\beta+[m]_{p,q}}-1\right)x+\frac{\alpha}{\beta+[m]_{p,q}}-\left(\frac{[m]_{p,q}}{\beta+[m]_{p,q}}x+\frac{\alpha}{\beta+[m]_{p,q}}-x\right) \\ &=0,\end{aligned} \tag{15}$$

which shows that $B_{m,p,q}^{*}(f;x)$ operators are linear. We have the Taylor expansion for $g\in C_B^2[0,\infty)$ as

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t (t-v)g''(v)\,dv\ ,\ t\in[0,\infty)\,. \tag{16}$$

When we apply $B^*_{m,p,q}$ operators to the equation (16), we obtain

$$B^*_{m,p,q}(g;x)=B^*_{m,p,q}\left(g(x)+(t-x)g'(x)+\int_x^t (t-v)g''(v)\,dv;x\right)$$

$$=g(x)+B^*_{m,p,q}\left((t-x)g'(x);x\right)+B^*_{m,p,q}\left(\int_x^t (t-v)g''(v)\,dv;x\right).$$

Then

$$B^*_{m,p,q}(g;x)-g(x)=g'(x)B^*_{m,p,q}\left((t-x);x\right)+B^*_{m,p,q}\left(\int_x^t (t-v)g''(v)\,dv;x\right).$$

From (14) and (15), we get

$$\begin{aligned} B^*_{m,p,q}(g;x)-g(x) &= B^*_{m,p,q}\left(\int_x^t (t-v)g''(v)\,dv;x\right) \\ &= B^{\alpha,\beta}_{m,p,q}\left(\int_x^t (t-v)g''(v)\,dv;x\right)+\int_x^x (t-v)g''(v)\,dv \\ &\quad -\int_x^{\frac{[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}}\left(\frac{[m]_{p,q}x+\alpha}{\beta+[m]_{p,q}}-v\right)g''(v)\,dv. \end{aligned} \tag{17}$$

Moreover,

$$\left|\int_x^t (t-v)g''(v)\,dv\right|\le\int_x^t |t-v||g''(v)|\,dv\le\|g''\|_{C_B}\int_x^t |t-v|\,dv\le(t-x)^2\|g''\|_{C_B} \tag{18}$$

and

$$\begin{aligned} \left|\int_x^{\frac{[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}}\left(\frac{[m]_{p,q}x+\alpha}{\beta+[m]_{p,q}}-v\right)g''(v)\,dv\right| &\le \left|\|g''\|_{C_B}\int_x^{\frac{[m]_{p,q}x+\alpha}{\beta+[m]_{p,q}}}\left(\frac{[m]_{p,q}x+\alpha}{\beta+[m]_{p,q}}-v\right)dv\right| \\ &\le \|g''\|_{C_B}\left(\frac{[m]_{p,q}x+\alpha}{\beta+[m]_{p,q}}-x\right)^2 \\ &= \|g''\|_{C_B}\left(\frac{\alpha-\beta x}{\beta+[m]_{p,q}}\right)^2. \end{aligned} \tag{19}$$

Let rewrite (18) and (19) in the absolute value of (17). So, we get

$$\left|B^{*}_{m,p,q}(g;x)-g(x)\right|\leq\|g''\|_{C_B}\left(\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x)\right). \tag{20}$$

Now, we will mention the rate of convergence of the constructed operators by using Peetre's K-functionals.

Theorem 2. Let $f\in C_B[0,\infty)$, $0<q<p\leq 1$ and $0\leq\alpha\leq\beta.$ Then there is a positive constant M such that

$$\left|B^{\alpha,\beta}_{m,p,q}(f;x)-f(x)\right|\leq M\omega_2\left(f,\sqrt{\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x)}\right)+\omega\left(f,\eta_m(\alpha,\beta,x)\right), \tag{21}$$

where $x\in[0,\infty)$.

Proof. From (13), for every $g\in C_B^2[0,\infty)$

$$\left|B^{\alpha,\beta}_{m,p,q}(f;x)-f(x)\right|=\Bigg|B^{*}_{m,p,q}(f;x)-f(x)+f\left(\frac{\alpha}{\beta+[m]_{p,q}}+\frac{[m]_{p,q}}{\beta+[m]_{p,q}}x\right)-f(x)$$

$$+B^{*}_{m,p,q}(g;x)-B^{*}_{m,p,q}(g;x)+g(x)-g(x)\Bigg|$$

$$\leq\left|B^{*}_{m,p,q}(f-g;x)-(f-g)(x)\right|+\left|f\left(\frac{\alpha}{\beta+[m]_{p,q}}+\frac{[m]_{p,q}}{\beta+[m]_{p,q}}x\right)-f(x)\right|$$

$$+\left|B^{*}_{m,p,q}(g;x)-g(x)\right|.$$

Using Lemma 3 and Lemma 4 we obtain

$$\left|B^{\alpha,\beta}_{m,p,q}(f;x)-f(x)\right|\leq 4\|f-g\|_{C_B}+\left|f\left(\frac{\alpha}{\beta+[m]_{p,q}}+\frac{[m]_{p,q}}{\beta+[m]_{p,q}}x\right)-f(x)\right|$$

$$+\|g''\|_{C_B}\left(\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x)\right) \tag{22}$$

and then we take the infimum on the right-hand side. Consequently, by using the property of Peetre's K-functionals, we have

$$\left|B^{\alpha,\beta}_{m,p,q}(f;x)-f(x)\right|\leq 4K_2\left(f,\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x)\right)+\omega\left(f,\eta_m(\alpha,\beta,x)\right)$$

$$\leq M\omega_2\left(f,\sqrt{\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x)}\right)+\omega\left(f,\eta_m(\alpha,\beta,x)\right).$$

Just now, we calculate the rate of convergence of our $B^{\alpha,\beta}_{m,p,q}(f;x)$ operators by means of the modulus of continuity on the finite interval.

Theorem 3. Let $f\in C_2[0,\infty)$, $0<q<p\leq 1$, $0\leq\alpha\leq\beta$ and $\omega_{c+1}(f,\delta)$ represent the modulus of continuity on the finite $[0,c+1]\subset[0,\infty)$, where $c>0$. Then we have the following inequality for all $x\in[0,\infty)$,

$$\left|B_{m,p,q}^{\alpha,\beta}(f;x)-f(x)\right|\leq 4M_f\left(1+c^2\right)\tilde{\mu}(m)(1+x)^2+2\omega_{c+1}\left(f,(1+x)\sqrt{\tilde{\mu}(m)}\right).$$

There is a positive constant M_f, which is independent of m and $\tilde{\mu}(m)$.

Proof. We already know the following property of $\omega_{c+1}(.,\delta)$

$$\left|f(t)-f(x)\right|\leq 4M_f\left(1+c^2\right)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{c+1}(f,\delta),\ \delta>0. \tag{23}$$

By choosing $\delta=(1+x)\sqrt{\tilde{\mu}(m)}$ and applying Cauchy-Schwarz inequality, we obtain

$$\left|B_{m,p,q}^{\alpha,\beta}(f;x)-f(x)\right|\leq 4M_f\left(1+c^2\right)B_{m,p,q}^{\alpha,\beta}\left((t-x)^2;x\right)+\left(1+\frac{\sqrt{B_{m,p,q}^{\alpha,\beta}\left((t-x)^2;x\right)}}{\delta}\right)\omega_{c+1}(f,\delta)$$

$$\leq 4M_f\left(1+c^2\right)\tilde{\mu}(m)(1+x)^2+2\omega_{c+1}\left(f,(1+x)\sqrt{\tilde{\mu}(m)}\right).$$

Section 4 gives the rate of convergence locally by using functions, which belong to the Lipschitz class.

4. Pointwise estimates

Definition 2. Let $0<c\leq 1$ and $E\subset[0,\infty)$. Then if $f\in C_B[0,\infty)$ is locally in $Lip(c)$,

$$\left|f(y)-f(x)\right|\leq M|y-x|^c,\ y\in E,\ x\in[0,\infty) \tag{24}$$

holds.

Theorem 4. For each $x\in[0,\infty)$, $f\in Lip(c)$ and $0\leq\alpha\leq\beta$, we obtain

$$\left|B_{m,p,q}^{\alpha,\beta}(f;x)-f(x)\right|\leq M\left(\tilde{\mu}(m)^{c/2}(1+x)^c+2\left(d(x,E)\right)^c\right), \tag{25}$$

where the constant M depends on c and f. Here, $d(x,E)=\inf\{|t-x|:t\in E\}$ defines the distance between x and E.

Proof. Let x_0 be in the closure of E such that $|x-x_0|=d(x,E)$. By using the triangle inequality, we write

$$\left|f(t)-f(x)\right|\leq\left|f(t)-f(x_0)\right|+\left|f(x)-f(x_0)\right|. \tag{26}$$

From the inequality (24), we obtain

$$\left|B_{m,p,q}^{\alpha,\beta}(f;x)-f(x)\right|\leq B_{m,p,q}^{\alpha,\beta}\left(\left|f(t)-f(x_0)\right|;x\right)+B_{m,p,q}^{\alpha,\beta}\left(\left|f(x)-f(x_0)\right|;x\right)$$

$$\leq M\left\{B_{m,p,q}^{\alpha,\beta}\left(|t-x_0|^c;x\right)+|x-x_0|^c\right\}$$

$$\leq M\left\{B_{m,p,q}^{\alpha,\beta}\left(|t-x|^c+|x-x_0|^c;x\right)+|x-x_0|^c\right\}$$

$$=M\left\{B_{m,p,q}^{\alpha,\beta}\left(|t-x|^{c};x\right)+2|x-x_0|^{c}\right\}.$$

Then by taking $p=2/c$ and $q=2/(2-c)$ in the Hölder inequality, we get

$$\left|B_{m,p,q}^{\alpha,\beta}(f;x)-f(x)\right|\le M\left\{\left[B_{m,p,q}^{\alpha,\beta}\left(|t-x|^{cp};x\right)\right]^{1/p}+2\left(d(x,E)\right)^{c}\right\}$$

$$=M\left\{\left[B_{m,p,q}^{\alpha,\beta}\left(|t-x|^{2};x\right)\right]^{c/2}+2\left(d(x,E)\right)^{c}\right\}$$

$$\le M\left\{\left(\tilde{\mu}(m)(1+x)^{2}\right)^{c/2}+2\left(d(x,E)\right)^{c}\right\}$$

$$=M\left\{\left(\tilde{\mu}(m)^{c/2}(1+x)^{c}\right)+2\left(d(x,E)\right)^{c}\right\}.$$

In the following theorem, we will get a local direct estimation of the $B_{m,p,q}^{\alpha,\beta}$ operators by the Lipschitz-type maximal function of order c. Lenze [16] defined Lipschitz-type maximal function $\tilde{\omega}_c$ as follows

$$\tilde{\omega}_c(f,x)=\sup_{t\neq x,\, t\in[0,\infty)}\frac{|f(t)-f(x)|}{|t-x|^{c}},\ x\in[0,\infty) \text{ and } c\in(0,1]. \tag{27}$$

Theorem 5. Let $0<c\le 1$ and $f\in C_B[0,\infty)$. For all $x\in[0,\infty)$,

$$\left|B_{m,p,q}^{\alpha,\beta}(f;x)-f(x)\right|\le\tilde{\omega}_c(f,x)\,\tilde{\mu}(m)^{c/2}(1+x)^{c}. \tag{28}$$

Proof. By using inequality (24) and the definition of the maximal function (27), we have

$$\left|B_{m,p,q}^{\alpha,\beta}(f;x)-f(x)\right|\le\tilde{\omega}_c(f,x)\,B_{m,p,q}^{\alpha,\beta}\left(|t-x|^{c};x\right).$$

Then by taking $p=2/c$ and $q=2/(2-c)$ in the Hölder inequality, we get

$$\left|B_{m,p,q}^{\alpha,\beta}(f;x)-f(x)\right|\le\tilde{\omega}_c(f,x)\left[B_{m,p,q}^{\alpha,\beta}\left(|t-x|^{2};x\right)\right]^{c/2}$$

$$\le\tilde{\omega}_c(f,x)\,\tilde{\mu}(m)^{c/2}(1+x)^{c}.$$

5. Graphical analysis

This section presents illustrative graphics and comparisons to show the convergence of (p,q)-Baskakov-Stancu operators to the selected functions.

First of all, Matlab algorithms of the constructed operators are illustrated. The first algorithm is the definition of (p,q)-integers.

Algorithm 1

```
function y=pqinteger(n,p,q)
    y=(p^(n)-q^(n))/(p-q);
end
```

With the help of the second algorithm, we exemplify (p,q)-Baskakov-Stancu operators. In addition, we check the convergence of the operators with the selected function for various values of p and q. Let the function f be chosen as $f(x)=2x^{2}-3x+4$, $x\in[0,100]$, $\alpha=0.2$ and $\beta=0.5$.

Algorithm 2

```
clear all close all clc
n=[20,30,40];
syms t
inf=100;
alpha=0.2;
beta=0.5;
b=100;
for j=1:3
    m=n(j);
p1=[0.97, 0.98, 0.99];
q1=[0.15, 0.2, 0.25];
 subplot(1,3,j)
for i=1:3
    p=p1(i);
    q=q1(i);
    u=1;
    for x=0:1:b
        ts=0;
 for s=0:inf
            z=1;
            for j=1:m+s
                z=z*((p^(j-1))+(q^(j-1)).*x);
            end
  h1=1;
  for a1=0:m+s-2
  h1=h1*pqinteger(m+s-1-a1,p,q);
  end
  if (m==1)
      h2=1;
  end
  if (m~=1)
       h2=1;
       for a2=0:m-2
       h2=h2*pqinteger(m-1-a2,p,q);
       end
  end
  h3=1;
  for a3=0:s-1
  h3=h3*pqinteger(s-a3,p,q);
  end
  fact=h1/(h2*h3);
            z;
            x1=(p^(m-1))*(q^(1-s)+alpha)/(pqinteger(m,p,q)+beta);
           f1=2*x1^2-3*x1+4;
            B=fact*(p^(s+m*(m-1)/2))*(q^(s*(s-1)/2))*(x^s)*f1;
            ts=ts+B/z;
        end
        a(u)=ts;
        u=u+1;
    end
     x=0:1:b;
    if (i==1)
        c=plot(x,a,'g');
         hold on
   elseif (i==2)
        c=plot(x,a,'r');
         hold on
    else (i==3)
        c=plot(x,a,'m');
end
end
 x=0:1:b;
 y=2*x.^2-3.*x+4;
plot(x,y,'--p')
     legend('p=0.97, q=0.15','p=0.98, q=0.2','p=0.99, q=0.25','function')
end
```

In Figure 1 we have plotted the $B_{m,p,q}^{\alpha,\beta}(f;x)$ operators and $f(x)=2x^2-3x+4$ for different values of parameters p, q and m.

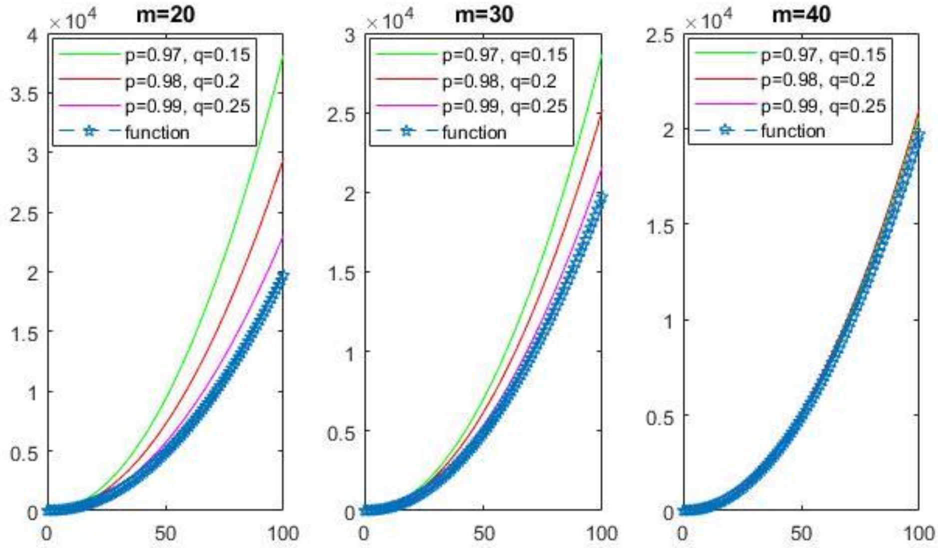

Figure 1. Convergence of $B_{m,p,q}^{\alpha,\beta}(f;x):(p,q)$-analogue of Baskakov-Stancu operators for different values of m, p and q.

As a second example, we take $f(x)=16x^3-12x^2+15$, $x\in[0,100]$, $\alpha=0.1$ and $\beta=1.1$. The convergence of the constructed operators with $f(x)=16x^3-12x^2+15$ is illustrated in Figure 2 by using various values of parameters p, q and m.

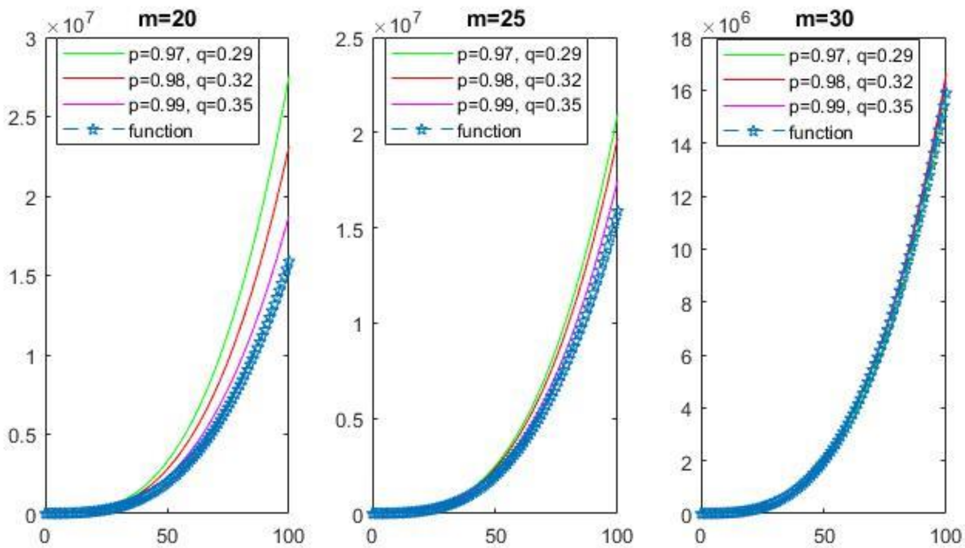

Figure 2. Convergence of $B_{m,p,q}^{\alpha,\beta}(f;x):(p,q)$-analogue of Baskakov-Stancu operators for different values of m, p and q.

5. Conclusion

In this study, we have constructed Stancu type (p,q)-Baskakov operators and investigated the approximation properties of the new operators. The rate of convergence of these operators is examined by using Peetre's K-functionals, modulus of continuities and for the functions belong to a Lipschitz class. Finally, we have presented some figures to show the convergence of the $B_{m,p,q}^{\alpha,\beta}$ operators with some chosen functions by using MATLAB.

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