ON THE THEORY OF NEAR SOLITON SOLUTION OF CAUCHY PROBLEM FOR PERTURBED NONLINEAR DIFFERENTIAL EQUATION OF KORTEWEG - DE VRIES TYPE

Prof. Dr. Taalaibek IMANALIEV

Kyrgyz State National University Chair Of Applied Mathematics And İnformation Sciences

Ahmet AYDIN

Kyrgyz State National University Chair Of Applied Mathematics And İnformation Sciences

The near soliton solution of regularized equation Korteweg - de Vries and Boussinesq type for the first time was introduced in works [1-4]. These works shows the existence of near soliton solutions for different classes of perturbed equations. The main feature of these works was implementation of the solution of initial problem of classical equation of Korteweg - de Vries [5].

In this article the problem of existence of near soliton solution of perturbed differential equation in partial derivatives

$$\alpha^3 u_t(t,x) + u(t,x)\big(u_x(t,x) + \alpha u(t,x)\big) + \alpha u_{xxx} + u_{xxxt} = f(t,x,u) \quad (1)$$

type with initial conditions

$$u(0,x) = \varphi(x), \quad (2)$$

where $\alpha = const$, $\alpha > 0$, $\varphi(x)$, $f(t,x,u)$ - are known functions.

In this article the problem of presenting of the solution of Cauchy problem (1)-(2) as

$$u(t,x)=c(t,x)+\int_0^t\int_{-\infty}^{x}e^{-\alpha(x-s+t-v)}Q(v,s)dsdv\ . \tag{3}$$

Here $Q(v,s)$ is the new function to be found,

$c(t,x)$ is soliton solution of the Cauchy problem of wellknown Korteweg - de Vries type

$$\alpha^3 c_t(t,x)+c(t,x)c_x(t,x)+c_{xxx}(t,x)=0 \tag{4}$$

with the initial condition

$$c(0,x)=\varphi(x) \tag{5}$$

From (3) and (5) follows that $u(0,x)=\varphi(x)$.

The main purpose of this work is to effectively define the unknown function $Q(t,x)$. Let propose that $Q(t,x)$, $\| Q(t,x)\| \le R=const$ is determined.

Then from (3) follows the inequity

$$\|u(t,x)-c(t,x)\| \le \int_0^t e^{-\alpha(t-v)}\int_{-\infty}^{x}e^{-\alpha(x-s)}\|Q\|dsdv \le \frac{R}{\alpha^2}=const\ . \tag{6}$$

The solution of the problem (1)-(2) which is presented as (3) is called near soliton solution. To define the unknown function $Q(t,x)$ (3) is substituted to (1). For this purposes, differentiating (3) by *x*,

$$u(t,x)+\alpha u(t,x)=c_x(t,x)+\alpha c(t,x)+\int_0^t e^{-\alpha(t-v)}Q(v,x)dv \tag{7}$$

is received

$$u_{xx}(t,x)+\alpha u_x(t,x)=c_{xx}(t,x)+\alpha c_x(t,x)+\int_0^t e^{-\alpha(t-v)}Q_x(v,x)dv \tag{8}$$

From (7) and (8) the inequity follows

$$u_{xx}(t,x)-\alpha^2 u_x(t,x)=c_{xxx}(t,x)-\alpha^2 c_x(t,x)+\int_0^t e^{-\alpha(t-\nu)}\left[Q_x(\nu,x)-\alpha Q(\nu,x)\right]d\nu$$

(9)

Differentiating both parts of (9) by *x*,

$$u_{xxx}(t,x)-\alpha^2 u_x(t,x)=c_{xxx}(t,x)-\alpha^2 c_x(t,x)+\int_0^t e^{-\alpha(t-\nu)}\left[Q_{xx}(\nu,x)-\alpha Q_x(\nu,x)\right]d\nu$$

(10)

$$u_{xxx}(t,x)+\alpha^3 u_t(t,x)=c_{xxx}(t,x)+\alpha^3 c(t,x)+\int_0^t e^{-\alpha(t-\nu)}\left[Q_{xx}(\nu,x)-\alpha Q_x(\nu,x)+\alpha^2 Q(\nu,x)\right]d\nu$$

(11)

is received.

Differentiating (11) by *t*,

$$u_{xxx}(t,x)+\alpha^3 u_t(t,x)=c_{xxxt}(t,x)+\alpha^3 c_t(t,x)+Q_{xx}(\nu,x)-\alpha Q_x(\nu,x)+\alpha^2 Q(\nu,x)-$$

$$-\alpha\int_0^t e^{-\alpha(t-\nu)}\left[Q_{xx}(\nu,x)-\alpha Q_x(\nu,x)+\alpha^2 Q(\nu,x)\right]d\nu$$

(12)

is received

From (11) and (12)

$$u_{xxxt}(t,x)+\alpha^3 u_t(t,x)=c_{xxxt}(t,x)+\alpha^3 c_t(t,x)+Q_{xx}(\nu,x)-\alpha Q_x(\nu,x)+\alpha^2 Q(\nu,x)-$$

$$-\left[u_{xxx}(t,x)+\alpha^3 u_t(t,x)-c_{xxx}(t,x)-\alpha^3 c(t,x)\right].$$

is received.

Thus,

$$u_{xxxt}(t,x)+\alpha^3 u_t(t,x)+\alpha u_{xxx}(t,x)+\alpha^4 u(t,x)=c_{xxxt}(t,x)+$$

$$+\alpha^3 c_t(t,x)+\alpha c_{xxx}(t,x)+\alpha^4 c(t,x)+Q_{xx}(\nu,x)-\alpha Q_x(\nu,x)+\alpha^2 Q(\nu,x)$$

(13)

From (3) and (7) the expression follows:

$$u(u_x+\alpha u)=c(t,x)(c_x+\alpha c)+c(t,x)\int_0^t e^{-\alpha(t-\nu)}Q(\nu,x)d\nu+$$

$$\left(c_x+\alpha c(t,x)\right)\int_0^t\int_{-\infty}^x e^{-\alpha(x-s+t-\nu)}Q(\nu,x)\,ds\,d\nu\,*$$

$$*\int_0^t\int_{-\infty}^x e^{-\alpha(x-s+t-\nu)}Q(\nu,x)\,ds\,d\nu$$

(14)

Adding right and left parts (13) and (14),

$$\alpha^3u_t(t,x)+u\left(u_x+\alpha u+\alpha^4\right)+\alpha u_{xxx}(t,x)+u_{xxxt}(t,x)=f(t,x,u)=\alpha^3c_t+cc_x+\alpha c_{xxx}(t,x)+$$

$$+c_{xxxt}(t,x)+\alpha c(t,x)\left(\alpha^3+c(t,x)\right)+c(t,x)\int_0^t e^{-\alpha(t-\nu)}Q(\nu,x)\,d\nu+$$

$$\left(c_x(t,x)+c(t,x)\right)\int_0^t\int_{-\infty}^x e^{-\alpha(x-s+t-\nu)}Q(\nu,s)\,ds\,d\nu+\int_0^t e^{-\alpha(t-\nu)}Q(\nu,s)\,d\nu\int_0^t\int_{-\infty}^x e^{-\alpha(x-s+t-\nu)}Q(\nu,s)\,ds\,d\nu$$

$$+Q_{xx}(t,x)-\alpha Q(t,x)+\alpha^2Q(t,x)$$

(15)

is received.

From (4) and (15), to determine the unknown function *Q (t, x)*, the following nonlinear ordinary integro-differential equation relatively the function *Q (t, x)* in type of

$$Q_{xx}(t,x)-\alpha Q_x(t,x)+\alpha^2Q(t,x)=f\left(t,x,c(t,x)+\int_0^t\int_{-\infty}^x e^{-\alpha(x-s+t-\nu)}Q(\nu,s)\,ds\,d\nu\right)-$$

$$-c(t,x)\int_0^t e^{-\alpha(t-\nu)}Q(\nu,s)\,d\nu-\left(c(t,x)+\alpha c(t,x)\right)\int_0^t\int_{-\infty}^x e^{-\alpha(x-s+t-\nu)}Q(\nu,s)\,ds\,d\nu-$$

$$-\int_0^t e^{-\alpha(t-\nu)}Q(\nu,s)\,d\nu\int_0^t\int_{-\infty}^x e^{-\alpha(x-s+t-\nu)}Q(\nu,s)\,ds\,d\nu-c_{xxxt}(t,x)+\alpha c(t,x)\left(\alpha^3+c(t,x)\right)$$

(16)

is received.

e function *Q (t, x)* is searched in class of functions {*Q (t, x)*}, satisfying the conditions $Q(t,+\infty)=0$, $Q_x(t,\infty)=0$. Let's show that every nonlinear and bounded solution of nonlinear integral equation

$$z(t,x)\equiv f\left(t,x,c+\int_0^t\int_{-\infty}^x e^{-\alpha(x-s+t-\nu)}Q\,ds\,d\nu\right)-c(t,x)\int_0^t e^{-\alpha(t-\nu)}Q\,d\nu-$$

$$-c(t,\nu)\int_0^t e^{-\alpha(t-\nu)}Q(\nu,\gamma)\,d\nu-c_x(t,\gamma)\int_0^t\int_{-\infty}^{\gamma} e^{-\alpha(\gamma-\rho+t-\nu)}Q(\nu,\rho)\,d\rho\,d\nu-$$

$$-\int_0^t e^{-\alpha(t-\nu)}Q(\nu,\gamma)\,d\nu\int_0^t\int_{-\infty}^{x} e^{-\alpha(\gamma-\rho+t-\nu)}Q(\nu,\rho)\,d\rho\, d\nu+F(t,\gamma),$$

(17)

where $F(t,x)\equiv -c_{xxx}(t,x)-\alpha c(t,x)\left(\alpha^3+c(t,x)\right),$

is the same solution of nonlinear integro-differential equation (16) and vice versa. Indeed, the right part of (16) is named as *z (t, x)* to be short. Then, the equation (16) will accept the following type

$$Q_{xx}(t,x)-\alpha Q_x(t,x)+\alpha^2 Q(t,x)\equiv z(t,x),$$

and the equation (17) will be written as

$$Q(t,x)\equiv\int_{\infty}^{x} e^{\frac{\alpha\left(1+i\sqrt{3}\right)}{2}(x-s)}\int_{\infty}^{s} e^{\frac{\alpha\left(1-i\sqrt{3}\right)}{2}(s-\gamma)} z(t,\gamma)\,d\gamma\, ds\,.$$

There is

$$Q_x(t,x)\equiv\frac{\alpha\left(1+i\sqrt{3}\right)}{2}Q(t,x)+\int_{-\infty}^{x} e^{\frac{\alpha\left(1-i\sqrt{3}\right)}{2}(x-\gamma)} z(t,\gamma)\,d\gamma\, ds\,,$$

(18)

$$Q_{xx}(t,x)\equiv\frac{\alpha\left(1+i\sqrt{3}\right)}{2}Q(t,x)+z(t,x)+\frac{\alpha\left(1-i\sqrt{3}\right)}{2}\int_{\infty}^{x} e^{\frac{\alpha\left(1-i\sqrt{3}\right)}{2}(x-\gamma)} z(t,\gamma)\,d\gamma\,.$$

Then

$$Q_{xx}(t,x)-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}Q_x-\frac{\alpha\left(1-i\sqrt{3}\right)}{2}Q_x-\frac{\alpha\left(1-i\sqrt{3}\right)\left(1+i\sqrt{3}\right)\alpha^2}{4}Q(t,x)\equiv z(t,x)\,.$$

That is why the left and right part of the last identity will be written as

$$Q_{xx}(t,x)-\alpha Q_x(t,x)-\alpha^2 Q(t,x)\equiv z(t,x)$$

(19)

Further lets consider that the condition (A) is fulfilled, if: 1) known function

$c_x(t,x) \in \overset{-}{c}^{(1,2)}(D)$, $c_{xxxt}(t,x) \in \overset{-}{c}^{(2,4)}(D)$, $D = \{(t,x) \mid t \geq 0, x \in R\}$ satisfied inequity $\| c_t \|, \| c(t,x) \|, \| c_x(t,x) \|, \| c_{xxxt}(t,x) \| \leq c_0 = const$;

2) in area $D = \{(t,x,u) \mid t \geq 0, x,u \in R\}$ $f(t,x,u) \in \overline{c}(D) \cup Lip(N|_u)$,

3) $\frac{4}{\alpha^5}\left[N\alpha + c_0(1+\alpha c_0)\alpha + 2R\right] \leq \frac{1}{2}$, where

$$R = \frac{8}{\alpha^2}\left[M + c_0\left(1+\alpha\left(\alpha^3 + c_0\right)\right)\right],$$

$$\frac{4}{\alpha^5}\left\{N\alpha + c_0(1+\alpha c_0)\alpha + \frac{16}{\alpha^2}\left[M + c_0\left(1+\alpha\left(\alpha^3+c_0\right)\right)\right]\right\} < 1.$$

Implementing of principle of compressed mapping could show the existence and uniqueness of the solution of nonlinear integral equation (17). Indeed, the right part of (17) lets consider as operator $H[Q]$ acting on function $\{Q(t,x)\} : H[Q] : \{Q(t,x)\} \to \{Q(t,x)\}$.

We have

$$\|H[Q]\| \leq \int_{-\infty}^{x} |e^{\frac{\alpha(1+i\sqrt{3})}{2}(x-s)}| \int_{\infty}^{s} |e^{\frac{\alpha(1-i\sqrt{3})}{2}(s-\gamma)}| \{\| f\left(t,\gamma,c+\int_0^t\int_{-\infty}^{\gamma} e^{-\alpha(\gamma-\rho+t-\nu)} Q(\nu,\rho)\,d\rho\,ds\right) - f(t,\gamma,c) \| +$$

$$+\| f(t,\gamma,c) \| + \| c(t,\gamma) \| \int_0^t e^{-\alpha(t-\nu)} \| Q \| d\nu + \| c_x(t,\gamma) \| \int_0^t \int_{-\infty}^{x} e^{-\alpha(\gamma-\rho+t-\nu)} \| Q \| d\rho\, d\nu +$$

$$+\int_0^t e^{-\alpha(t-\nu)} \| Q \| d\nu \int_0^t \int_{-\infty}^{\infty} e^{-\alpha(\gamma-\rho+t-\nu)} \| Q \| d\rho\, d\nu + \| c_{xxxt}(t,\gamma) \| + \alpha \| c(t,\gamma) \| \left(\alpha^3 + \| c(t,\gamma) \|\right)\} d\gamma\, ds \leq$$

$$\int_{\infty}^{x} e^{\frac{\alpha}{2}(x-s)} \int_{\infty}^{x} e^{\frac{\alpha}{2}(s-\gamma)} \left\{ N\int_0^t \int_{-\infty}^{\gamma} e^{-\alpha(\gamma-\rho+t-\nu)} \| Q \| d\rho\, d\nu + M + \frac{c_0 R}{\alpha} + \frac{c_0 R}{\alpha^2} + \frac{c_0 R}{\alpha^3} + c_0(1+\alpha)\left(\alpha^3 + c_0\right) \right\} d\gamma\, ds \leq$$

$$\leq \frac{4}{\alpha^2}\left[\frac{NR}{\alpha^2} + \frac{c_0(\alpha+1)R}{\alpha^2} + \frac{R^2}{\alpha^3} + M_0\right] \leq \frac{4}{\alpha^5}\left[N + c_0(\alpha+1)\alpha + 2R\right] + \frac{4M_0}{\alpha^2} \leq \frac{1}{2}R + \frac{4M_0}{\alpha^2} = R;$$

$$R = \frac{8M_0}{\alpha^2}; M_0 = M + c_0(1+\alpha)\left(\alpha^3 + c_0\right).$$

Let us show that $H[Q]:\|Q\|\to\|Q\|$. Indeed,

$$\|H[Q]-H[G]\|\le\int_{-\infty}^{x}e^{\frac{\alpha}{2}(x-s)}\int_{-\infty}^{s}e^{\frac{\alpha}{2}(s-\gamma)}\left\{\left[\frac{N}{\alpha^2}+\frac{c_0(\alpha-1)}{\alpha^2}\right]\|Q-G\|+\right.$$

$$+\int_0^t e^{-\alpha(t-v)}\|Q-G\|\,dv\int_0^t\int_{-\infty}^{x}e^{-\alpha(\gamma-\rho+t-v)}\|Q\|\,d\rho dv+$$

$$\left.+\int_0^t e^{-\alpha(t-v)}\|G\|\,dv\int_0^t\int_{-\infty}^{\gamma}e^{-\alpha(\gamma-\rho+t-v)}\|Q-G\|\,d\rho dv\right\}d\gamma dv\le\frac{4}{\alpha^2}\left[\frac{N}{\alpha^2}+\frac{c_0(\alpha+1)}{\alpha^2}+\frac{2R}{\alpha^3}\right]\|Q-G\|\le$$

$$\le\frac{4}{\alpha^5}\left[N\alpha+c_0(\alpha+1)\alpha+2R\right]\|Q-G\|\le\|Q-G\|.$$

Thus, we have shown that nonlinear integral equation (17) has the only one continuous and bounded solution $Q(t,x)\in\tilde{C}(D)$, therefore, the Cauchy problem (1)-(2) has a solution which could be written as (3), which satisfied inequity (6).

Let us search the differential properties of the solution of problem (1)-(2).

From (7) in D the inequity

$$\|u(t,x)\|\le\alpha\|u(t,x)\|+\|c_x(t,x)\|+\alpha\|c(t,x)\|+\int_0^t e^{-\alpha(t-v)}\|Q\|\,dv\le$$

$$\le\alpha\left(c_0+\frac{R}{\alpha^2}\right)+c_0(1+\alpha)+\frac{R}{2}=R_0=const;$$

follows.

From (8) in D

$$\|u_{xx}(t,x)\|\le\alpha\|u_x(t,x)\|+\|c_{xx}(t,x)\|+\alpha\|c_x(t,x)\|+\int_0^t e^{-\alpha(t-v)}\|Q_x(v,x)\|\,dv. \quad (20)$$

is obtained.

From equity

$$z(t,x)\equiv f\left(t,x,c+\int_0^t\int_{-\infty}^{x}e^{-\alpha(x-s+t-v)}Q\,ds dv\right)-c(t,x)\int_0^t e^{-\alpha(t-v)}Q\,dv-$$

$$-\left[c_x(t,x)+\alpha c(t,x)\right]\int_0^t\int_{-\infty}^x e^{-\alpha(x-s+t-\nu)}Q\,ds\,d\nu-\int_0^t e^{-\alpha(t-\nu)}Q\,d\nu\int_0^t\int_{-\infty}^x e^{-\alpha(x-s+t-\nu)}Q(\nu,s)\,ds\,d\nu+F(t,x).$$

In D inequity

$$z(t,x)\equiv\| f(t,x,u)+\|c\|\frac{1}{\alpha}R+\left(\|c_x\|+\alpha\|c\|\right)\frac{R}{\alpha^2}+\frac{R^2}{\alpha^2}+\|F(t,x)\|\leq$$

$$\leq M+\frac{c_0\alpha(1+\alpha)R}{\alpha^2}+\frac{R^2}{\alpha^3}+c_0(1+\alpha)\left(\alpha^3+c_0\right)\leq z_0=const. \tag{21}$$

follows.

From identity (18) in D

$$\|Q_x(\nu,x)\|\leq\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\|Q\|+\int_{\infty}^x e^{\frac{\alpha}{2}(x-\gamma)}\|Q(\nu,\gamma)\|d\gamma\leq R\left(\frac{\alpha}{2}+\frac{2}{\alpha}\right)=z_{00}=const, \tag{22}$$

is obtained.

From (19) the inequity follows:

$$\|Q_{xx}(t,x)\|\leq\alpha\|Q_x(t,x)\|+\alpha^2\|Q(t,x)\|+\|z(t,x)\|\leq\alpha z_{00}+\alpha^2R+z_0=z_{00}=const.$$

Therefore, in D from (20),

$$\|u_{xx}(t,x)\|\leq\alpha R_0+c_0(1+\alpha)+z_{00}\frac{1}{\alpha}=\beta=const.$$

is obtained.

From (10) in D

$$\|u_{xxx}(t,x)\|\leq\alpha^2\|u_x(t,x)\|+\|c_{xxx}(t,x)\|+\alpha\|c(t,x)\|+\int_0^t e^{-\alpha(t-\nu)}\left[\|Q_{xx}\|+\alpha\|Q\|\right]d\nu\leq$$

$$\alpha^2R_0+c_0\left(1+\alpha^2\right)+\left(z_0+\alpha R\right)\frac{1}{\alpha}=\beta_0=const \tag{23}$$

is obtained.

Let us evaluate the $u_{xxxt}(t,x)$ in *D*. From the equity (13) the inequity

$$\|u_{xxxt}(t,x)\| \le \alpha^3\|u_t(t,x)\| + \alpha\|u_{xxx}(t,x)\| + \alpha^4\|u(t,x)\| + \|c_{xxxt}(t,x)\| + \alpha^3\|c_t(t,x)\| +$$

$$+\|c_{xxx}(t,x)\| - \alpha^4\|c(t,x)\| + \|Q_{xx}\| + \alpha\|Q_x\| + \alpha^2\|Q\| \le \alpha^3\Big[\|c_t\| + \int_{-\infty}^{x} e^{-\alpha(x-s)}\|Q\|ds +$$

$$+\alpha\int_{-\infty}^{t}\int_{-\infty}^{x} e^{-\alpha(x-s+t-v)}\|Q\|dsdv + \alpha\beta_0 + \alpha^4\left(c_0 + \frac{R}{\alpha^2}\right) + c_0\left(2+\alpha^3+\alpha^4\right) + z_{00} + \alpha z_{00} + \alpha^2 R =$$

$$= z_{0000} = const \qquad (24)$$

follows.

From evaluations (20)-(24) consequent that the derivative solutions of (1)-(2) in D are uniformly bounded.

Theorem. Let the conditions (A) are fulfilled. Then the problem (1)-(2) has unique solution $u(t,x) \in C(D)$, and the derivative solutions are uniformly bounded and the inequity (6) is fulfilled.

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