SOLVABILITY THE EQUATIONS OF MAGNETO-GAS DYNAMICS IN NON-BOUNDED DOMAIN

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1. STATEMENT OF THE PROBLEM

The system of differential equations describing one-dimensional non-stationary flow of a viscous heat-conducting gas in a magnetic field in a porous medium can be written [1] in terms of Lagrange mass coordinates:

$$\frac{\partial \mathrm{v}}{\partial t}-\frac{\partial u}{\partial x}=0, \qquad \mathrm{v}=\frac{1}{\rho},$$

$$\frac{\partial u}{\partial t}=\mu\frac{\partial}{\partial x}\left(\frac{1}{\mathrm{v}}\frac{\partial u}{\partial x}\right)-\frac{\partial p}{\partial x}-\mu_l H\frac{\partial H}{\partial x}-\beta(x)|u|^a u, \qquad p=k\frac{\theta}{\mathrm{v}},$$

$$\frac{\partial \theta}{\partial t}=\lambda\frac{\partial}{\partial x}\left(\frac{1}{\mathrm{v}}\frac{\partial \theta}{\partial x}\right)-p\frac{\partial u}{\partial x}+\mu\frac{1}{\mathrm{v}}\left(\frac{\partial u}{\partial x}\right)^2+\mu_l\,\mu_H\frac{1}{\mathrm{v}}\left(\frac{\partial H}{\partial x}\right)^2, \tag{1}$$

$$\frac{\partial}{\partial t}(\mathrm{v}H)=\mu_H\frac{\partial}{\partial x}\left(\frac{1}{\mathrm{v}}\frac{\partial H}{\partial x}\right).$$

Here $\rho, \mathrm{v}, u, \theta, H, p$, the density, specific volume, velocity, absolute temperature, magnetic field intensity and pressure, respectively, are the required functions; $\mu, \lambda, k, \mu_l, \mu_H$ are positive physical constants; the variables $x \in R=(-\infty,\infty)$, $t,\ t\in[0,T],\ 0<T<\infty$; $\beta(x)$ is coefficient of penetration – continuous non-negative bounded function and $\int_{-\infty}^{\infty}\beta(x)dx \le C;\quad 0\le\alpha\le 1$.

The functions $\mathrm{v}_0, u_0, \theta_0, H_0$, which have initial values

$$\mathrm{v}\big|_{t=0}=\mathrm{v}_0(x),\ \ u\big|_{t=0}=u_0(x),\ \theta\big|_{t=0}=\theta_0(x),\ H\ \big|_{t=0}=H_0(x),\ \ |x|<\infty, \tag{2}$$

are assumed to be known and continuous, $(\mathrm{v}_0(x),\theta_0(x))$ are strictly positive and bounded: $\quad 0<m_0\le \mathrm{v}_0(x)\le M_0<\infty,\quad m_0\le\theta_0(x)\le M_0,$

and have finite limits at infinity

$$\lim_{x\to-\infty} \mathrm{v}_0(x) = \mathrm{v}_0^1, \quad \lim_{x\to+\infty} \mathrm{v}_0(x) = \mathrm{v}_0^2, \quad \mathrm{v}_0^1 \neq \mathrm{v}_0^2,$$

$$\lim_{x\to-\infty} u_0(x) = u_0^1, \quad \lim_{x\to+\infty} u_0(x) = u_0^2, \quad u_0^1 < u_0^2, \tag{3}$$

$$\lim_{x\to-\infty} \theta_0(x) = \theta_0^1, \quad \lim_{x\to+\infty} \theta_0(x) = \theta_0^2, \quad \theta_0^1 \neq \theta_0^2,$$

$$\lim_{x\to-\infty} H_0(x) = H_0^1, \quad \lim_{x\to+\infty} H_0(x) = H_0^2, \quad H_0^1 \neq H_0^2.$$

It has been proved [2, p.76] that Cauchy problem for system (1) when $H \equiv 0$, $\beta(x) \equiv 0$ and the limits of the initial values at infinity are the same is well posed. In [3] the Cauchy problem for system (1) without a porous medium, viz. $\beta(x) \equiv 0$ and in case when the limits of the initial temperature at infinity is the same is considered. In [4] the Cauchy problem for system (1) without a porous medium is considered. In [5] local solvability of the Cauchy problem for system (1) is proved. In this paper we study whether the problem defined by (1)-(3) is well posed.

We introduce four auxiliary functions $\psi(x)$, $f(x)$, $\eta(x)$, $\varphi(x)$, such that:

$$0 < C_1^{-1} < \psi(x) < C_1, \quad \lim_{|x|\to\infty} \mathrm{v}_0(x)\psi(x) = 1, \quad \psi'(x) \in W_2^1(R),$$

$$|f(x)| < C_2 < \infty, \quad \lim_{x\to-\infty} f(x) = u_0^1, \quad \lim_{x\to+\infty} f(x) = u_0^2,$$

$$0 < f'(x) \le C_0, \quad f'(x) \in W_2^1(R), \quad f'(x) \in L_1(R), \tag{4}$$

$$|\eta(x)| < C_3 < \infty, \quad \lim_{x\to-\infty} \eta(x) = H_0^1, \quad \lim_{x\to+\infty} \eta(x) = H_0^2, \quad \eta'(x) \in W_2^1(R),$$

$$0 < C_4^{-1} < \varphi(x) < C_4, \quad \lim_{|x|\to\infty} \theta_0(x)\varphi(x) = 1, \quad \varphi'(x) \in W_2^1(R).$$

$$(\eta'(x))^2 < \delta f'(x), \quad (\varphi'(x))^2 < \delta f'(x), \quad 0 < \delta < 1. \tag{5}$$

It is obvious that such functions exist.

THEOREM. *Let the initial date (2) satisfy conditions (3) and*

$$(u_0 - f, \; H_0 - \eta, \; \theta_0\varphi - 1, \; \mathrm{v}_0\psi - 1) \in W_2^2(R).$$

Then in any finite time interval $[0,T]$, $0 < T < \infty$ *a unique generalized solution of problem (1), (2) exists which satisfies the equations and initial date almost everywhere, and*

$$(u - f, \; \varphi\theta - 1, \; H - \eta) \in L_\infty(0,T; W_2^1(R)) \cap L_\infty(0,T; W_2^2(R)),$$

$$(\mathrm{v}\psi - 1) \in L_\infty(0,T; W_2^1(R)), \quad \left(\frac{\partial \mathrm{v}}{\partial t}, \frac{\partial u}{\partial t}, \frac{\partial H}{\partial t}, \frac{\partial \theta}{\partial t}\right) \in L_2(\Pi),$$

$\mathrm{v}(x,t)$, $\theta(x,t)$ *are strictly positive and bounded functions.*

The proof of the theorems is based on global a priori estimates in which the constants C, C_i, N_i depend only on the problem data and the time T, but not on the interval of existence of the local solution. These estimates permit us to extend the local solution, whose existence follows from [3,5], to the whole time interval $[0,T]$, $0 < T < \infty$.

2. A PRIORI BOUNDS

Without loss of generality we can assume for simplicity that the physical parameters $\mu, \lambda, k, \mu_l, \mu_H$ are equal to unity.

We substitute the independent variable assuming $\frac{\partial \xi}{\partial x} = \frac{1}{\varphi(x)}$. Then system (1) is transformed as

$$\begin{gathered}
\frac{\partial \mathrm{v}}{\partial t} - \frac{1}{\varphi}\frac{\partial u}{\partial \xi} = 0, \quad \mathrm{v} = \frac{1}{\rho}, \\
\frac{\partial u}{\partial t} = \frac{1}{\varphi}\frac{\partial}{\partial \xi}\left(\frac{1}{\varphi \mathrm{v}}\frac{\partial u}{\partial \xi}\right) - \frac{1}{\varphi}\frac{\partial p}{\partial \xi} - \frac{1}{\varphi} H \frac{\partial H}{\partial \xi} - \beta(x)|u|^{\alpha} u, \quad p = \frac{\theta}{\mathrm{v}}, \\
\frac{\partial \theta}{\partial t} = \frac{1}{\varphi}\frac{\partial}{\partial \xi}\left(\frac{1}{\varphi \mathrm{v}}\frac{\partial \theta}{\partial \xi}\right) - \frac{1}{\varphi} p \frac{\partial u}{\partial \xi} + \frac{1}{\varphi^2 \mathrm{v}}\left(\frac{\partial u}{\partial \xi}\right)^2 + \frac{1}{\varphi^2 \mathrm{v}}\left(\frac{\partial H}{\partial \xi}\right)^2, \qquad (6) \\
\mathrm{v}\frac{\partial H}{\partial t} + \frac{1}{\varphi} H \frac{\partial u}{\partial \xi} = \frac{1}{\varphi}\frac{\partial}{\partial \xi}\left(\frac{1}{\varphi \mathrm{v}}\frac{\partial H}{\partial \xi}\right).
\end{gathered}$$

LEMMA 1. If the conditions of theorem are satisfied, the following estimate is true

$$U(t) + \int_0^t W(\tau)\, d\tau \le E = const > 0, \quad t \in [0,T] \qquad (7)$$

where

$$U(t) = \int \left\{ \frac{1}{2}(u-f)^2 + \frac{1}{2}\mathrm{v}(H-\eta)^2 + (\varphi\theta - \ln \varphi\theta - 1) + (\mathrm{v}\psi - \ln \mathrm{v}\psi - 1) \right\} dx,$$

$$W(t) = \int \left\{ \frac{\theta_x^2}{\mathrm{v}\theta^2} + \frac{u_x^2}{\mathrm{v}\theta} + \frac{H_x^2}{\mathrm{v}\theta} + \frac{\theta}{\mathrm{v}} f'(x) + \frac{1}{2} H^2 f'(x) + \beta(x)|u|^{\alpha}(u-f)^2 \right\} dx.$$

The interval of integration with respect to x is from $-\infty$ to ∞.

PROOF. We multiply the first equation of system (1) by $\left(\psi - \frac{1}{\mathrm{v}}\right)$ the second by $\varphi(u-f)$ the third by $\left(\varphi - \frac{1}{\theta}\right)$ and the forth by $\varphi(H-\eta)$, add and integrate with respect to R:

$$\frac{d}{dt}\int \left\{ \frac{1}{2}\varphi(u-f)^2 + \frac{1}{2}\varphi \mathrm{v}(H-\eta)^2 + (\varphi\theta - \ln\varphi\theta - 1) + (\mathrm{v}\psi - \ln \mathrm{v}\psi - 1) \right\} d\xi + \qquad (8)$$

$$+\int\left\{\frac{\theta_\xi^2}{\mathrm{v}\theta^2\varphi^2}+\frac{u_\xi^2}{\mathrm{v}\theta\,\varphi^2}+\frac{H_\xi^2}{\mathrm{v}\theta\,\varphi^2}+\frac{\theta}{\mathrm{v}}f'(\xi)+\frac{1}{2}H^2f'(\xi)+\beta(\xi)|u|^\alpha(u-f)^2\varphi\right\}d\xi=$$

$$=\int\frac{\psi}{\varphi}u_\xi d\xi+\int\frac{u_\xi f'}{\mathrm{v}\varphi}d\xi+\int\frac{H_\xi\eta'}{\varphi\mathrm{v}}d\xi+\frac{1}{2}\int\eta^2u_\xi d\xi+\int\frac{\theta_\xi\varphi'}{\mathrm{v}\theta\varphi^3}d\xi+\int\beta(\xi)|u|^\alpha f(u-f)\varphi d\xi$$

Integrating by parts and employing the properties (4), (5) each integral on the right-hand side of (8) can be estimated using the Cauchy inequality, Young's inequality, enclosure inequality. First four integrals can be estimated as in [4]. Consider the last two integrals

$$\int\frac{\theta_\xi\varphi'}{\mathrm{v}\theta\varphi^3}d\xi=\int\frac{\theta_\xi\varphi'\psi^{1/2}}{\mathrm{v}^{1/2}\theta\varphi^3}d\xi-\int\frac{\theta_\xi\varphi'\psi^{1/2}}{\mathrm{v}^{1/2}\theta\varphi^3}\frac{(\mathrm{v}\psi)^{1/2}-1}{(\mathrm{v}\psi)^{1/2}\sqrt{\mathrm{v}\psi-\ln\mathrm{v}\psi-1}}\sqrt{\mathrm{v}\psi-\ln\mathrm{v}\psi-1}\,d\xi$$

Note that

$$\frac{\left|(\mathrm{v}\psi)^{1/2}-1\right|}{(\mathrm{v}\psi)^{1/2}\sqrt{\mathrm{v}\psi-\ln\mathrm{v}\psi-1}}\le C_5, \qquad (9)$$

since

$$\lim_{\mathrm{v}\psi\to\infty}\frac{(\mathrm{v}\psi)^{1/2}-1}{(\mathrm{v}\psi)^{1/2}\sqrt{\mathrm{v}\psi-\ln\mathrm{v}\psi-1}}=0,\quad \lim_{\mathrm{v}\psi\to1}\frac{\left|(\mathrm{v}\psi)^{1/2}-1\right|}{(\mathrm{v}\psi)^{1/2}\sqrt{\mathrm{v}\psi-\ln\mathrm{v}\psi-1}}=\frac{1}{\sqrt{2}},$$

$$\lim_{\mathrm{v}\psi\to0}\frac{\left|(\mathrm{v}\psi)^{1/2}-1\right|}{(\mathrm{v}\psi)^{1/2}\sqrt{\mathrm{v}\psi-\ln\mathrm{v}\psi-1}}=0.$$

Then

$$\left|\int\frac{\theta_\xi\varphi'}{\mathrm{v}\theta\varphi^3}d\xi\right|\le\left(\int\frac{\theta_\xi^2}{\mathrm{v}\theta^2\varphi^2}d\xi\right)^{1/2}\left(\int\frac{\psi}{\varphi^4}\varphi'^2d\xi\right)^{1/2}+$$

$$+C_5\left(\int\frac{\theta_\xi^2}{\mathrm{v}\theta^2\varphi^2}d\xi\right)^{1/2}\left(\int\frac{\psi(\mathrm{v}\psi-\ln\mathrm{v}\psi-1)}{\varphi^4}\varphi'^2d\xi\right)^{1/2}\le$$

$$\le\delta\int\frac{\theta_\xi^2}{\mathrm{v}\theta^2\varphi^2}d\xi+C_6\left(\int(\mathrm{v}\psi-\ln\mathrm{v}\psi-1)d\xi+1\right),$$

$$\int\beta(\xi)|u|^\alpha f(u-f)\varphi d\xi\le\delta\int\beta(\xi)|u|^\alpha(u-f)^2\varphi d\xi+$$

$$+C_7\left[\int\beta(\xi)|u-f|^\alpha f^2\varphi d\xi+\int\beta(\xi)|f|^\alpha f^2\varphi d\xi\right]\le$$

$$\le\delta\int\beta(\xi)|u|^\alpha(u-f)^2\varphi d\xi+C_8\left[\left(\int\varphi(u-f)^2d\xi\right)^{\frac{\alpha}{2}}\left(\int\left(\beta(\xi)f^2\right)^{\frac{2}{2-\alpha}}d\xi\right)^{\frac{2-\alpha}{2}}+1\right]\le$$

$$\leq \delta \int \beta(\xi)|u|^{\alpha}(u-f)^2 \varphi d\xi + C_9\left(\left\|\sqrt{\varphi}\,(u-f)\right\|^2 + 1\right).$$

By integrating the inequality obtained from (8) with respect to time t and using Gronwall's lemma we get (7) after returning to the old independent variable x. This proves the lemma 1.

Let us divide the number axis R and the strip Π into finite segments and rectangles [2]:

$$R = \bigcup_{N=-\infty}^{\infty} \overline{\Omega}_N, \quad \Pi = \bigcup_{N=-\infty}^{\infty} \overline{Q}_N,$$

$$\Omega_N = \{x \mid N < x < N+1\}, \;\; Q_N = \Omega_N \times (0,T), \;\; N = 0, \pm 1, \pm 2, ...$$

As in [2] from (7) it follows that

$$C_{10}^{-1} \leq \int_N^{N+1} \mathrm{v}(x,t)\, dx \leq C_{10}, \quad C_{11}^{-1} \leq \int_N^{N+1} \theta(x,t)\, dx \leq C_{11}, \quad \forall t \in [0,T]. \tag{10}$$

Thus, from the mean value theorem, for any $t \in [0,T]$ in each domain $\overline{\Omega}_N$ points $a(t) = a_N(t) \in [N, N+1]$, $a_1(t) = a_{1N}(t) \in [N, N+1]$ exist such that

$$C_{10}^{-1} \leq \mathrm{v}(a_1(t),t) \leq C_{10}, \quad C_{11}^{-1} \leq \theta(a(t),t) \leq C_{11}. \tag{11}$$

From the first and second equations of system (1), as in [6], we derive an auxiliary relation between the required functions:

$$\mathrm{v}(x,t) = I^{-1}(t) B^{-1}(x,t) K(x,t) \left[\mathrm{v}_0(x) + \int_0^t \left(\theta + \frac{1}{2}\mathrm{v} H^2\right) I(\tau) B(x,\tau) K^{-1}(x,\tau) d\tau \right] \tag{12}$$

Here

$$I(t) = \frac{\mathrm{v}_0(x_0(t))}{\mathrm{v}(x_0(t),t)} \exp\left\{ \int_0^t \left[\frac{\theta(x_0(t),\tau)}{\mathrm{v}(x_0(t),\tau)} + \frac{1}{2} H^2(x_0(t),\tau) \right] d\tau \right\},$$

$$B(x,t) = \exp\left\{ \int_{x_0(t)}^{x} (u_0(\xi) - u(\xi,t))\, d\xi \right\}, \quad K(x,t) = \exp\left\{ \int_0^t \int_{x_0(t)}^{x} \beta(\xi)|u|^{\alpha} u(\xi,\tau) d\xi\, d\tau \right\},$$

where $x_0 = x_0(t)$, x are the arbitrary chosen points in the number axis.

LEMMA 2. If the conditions of theorem are satisfied, the following estimate is true

$$N_1^{-1} \leq B(x,t) \leq N_1, \quad N_2^{-1} \leq K(x,t) \leq N_2, \quad N_3^{-1} \leq I(t) \leq N_3, \quad (x,t) \in \overline{Q}_N.$$

PROOF. The bounds for functions $B(x,t)$, $I(t)$ can be derived as in [6]. Let us derive the bounds for function $K(x,t)$. Consider $0 \leq \alpha < 1$. Case $\alpha = 1$ is obvious. Using Gelder's inequality, (4), (7), the properties of the function $\beta(x)$, we have

$$\left| \int_0^t \int_{a(t)}^{x} \beta(\xi)|u|^{\alpha} u(\xi,\tau) d\xi\, d\tau \right| \leq \int_0^t \int_N^{N+1} \beta(x)|u-f|^{\alpha+1} dx d\tau + \int_0^t \int_N^{N+1} \beta(x)|f|^{\alpha+1} dx\, d\tau \leq$$

$$\le \int_0^t \left(\int_N^{N+1} (u-f)^2\, dx \right)^{\frac{1+\alpha}{2}} \left(\int_N^{N+1} \beta^{\frac{2}{1-\alpha}}(x) dx \right)^{\frac{1-\alpha}{2}} + C_{12} \le C_{13}.$$

Hence follow the bounds for function $K(x,t)$. This proves the lemma 2.

Let $h(x,t)$ be a continuous function. We introduce the notation

$$M_h(t) = \max_{x\in R} h(x,t), \quad m_h(t) = \min_{x\in R} h(x,t).$$

LEMMA 3. If the conditions of theorem are satisfied, the following estimate is true

$$m_{\rm v}(t) \ge N_4, \quad m_\theta(t) \ge N_5, \quad \forall\, t \in [0,T].$$

PROOF. First estimate follow from representation (12) and lemma 2. From the heat-conducting equation of system (1) can be derived the second estimate. This proves the lemma 3.

LEMMA 4. If the conditions of theorem are satisfied, the following estimate is true

$$\int_0^t \int \left[\frac{\theta_x^2}{{\rm v}\theta^{3/2}} + \frac{u_x^2}{{\rm v}\theta^{1/2}} + \frac{H_x^2}{{\rm v}\theta^{1/2}} \right] dx d\tau \le N_6, \quad \forall\, t \in [0,T]. \quad (13)$$

PROOF as in [4].

LEMMA 5. If the conditions of theorem are satisfied, the following estimate is true $M_{\rm v}(t) \le N_7, \quad \forall t \in [0,T]$.

PROOF. Using (10), (11) we have

$$\max_{\overline{\Omega}_N} \theta(x,t) \le C_{11} + \int_N^{N+1} |\theta_x|\, dx \le C_{11} + \left(\int_N^{N+1} \frac{\theta_x^2}{{\rm v}\theta^2} dx \right)^{1/2} \left(\int_N^{N+1} {\rm v}\theta^2 dx \right)^{1/2}.$$

Hence

$$M_\theta(t) \le C_{11} + C_{11}^{1/2} A^{1/2}(t) M_\theta^{1/2}(t) M_{\rm v}^{1/2}(t),$$

where $$A(t) = \int \frac{\theta_x^2}{{\rm v}\theta^2} dx.$$

Applying Young's inequality with ε, we deduce

$$M_\theta(t) \le C_\varepsilon A(t) M_{\rm v}(t) + C_{14}. \quad (14)$$

Now estimate $M_H^2(t)$. Consider arbitrary segment $\overline{\Omega}_N = [N, N+1]$. Take points $a(t),\ x \in \overline{\Omega}_N$ and use (10).

$$\theta^{1/4}(x,t) = \theta^{1/4}(a(t),t) + \frac{1}{4} \int_N^{N+1} \frac{\theta_x}{\theta^{3/4}} dx \le$$

$$\le C_{11}^{1/4} + \frac{1}{4} \left(\int_N^{N+1} \frac{\theta_x^2}{{\rm v}\theta^{3/2}} dx \right)^{1/2} \left(\int_N^{N+1} {\rm v} dx \right)^{1/2} \le C_{11}^{1/4} + \frac{1}{4} C_{10}^{1/2} \left(\int \frac{\theta_x^2}{{\rm v}\theta^{3/2}} dx \right)^{1/2}.$$

Hence

$$\max_{\Omega_N}\theta^{1/4}(x,t)\le C_{11}^{1/4}+\frac{1}{4}C_{10}^{1/2}\left(\int\frac{\theta_x^2}{\mathrm{v}\theta^{3/2}}dx\right)^{1/2}.$$

Then

$$\max_{\Omega_N}H^2(x,t)\le C_{15}+\int_N^{N+1}|HH_x|\,dx\le C_{15}+\int_N^{N+1}|(H-\eta)H_x|\,dx+\int_N^{N+1}|\eta H_x|\,dx\le$$

$$\le C_{15}+\left(\int_N^{N+1}\frac{H_x^2}{\mathrm{v}\theta^{1/2}}dx\right)^{1/2}\max_{\Omega_N}\theta^{1/4}(x,t)\left[\left(\int_N^{N+1}\mathrm{v}(H-\eta)^2dx\right)^{1/2}+C_3\left(\int_N^{N+1}\mathrm{v}dx\right)^{1/2}\right],$$

Using (9), (10), (13) we find

$$M_H^2(t)\le C_{16}\left[\int\frac{\theta_x^2}{\mathrm{v}\theta^{3/2}}dx+\int\frac{H_x^2}{\mathrm{v}\theta^{1/2}}dx+1\right]. \tag{15}$$

From (12), (16) and lemma 2 follow representation

$$M_{\mathrm{v}}(t)\le C_{17}\left[1+\int_0^t\left(A(\tau)+M_H^2(\tau)\right)M_{\mathrm{v}}(\tau)d\tau\right].$$

Using Gronwall's lemma and (7), (13), (15) we derive the required estimate. This proves the lemma 5.

From (7), (13), (14), (15), lemma 5 it follows that

$$\int_0^T\left(M_\theta(t)+M_H^2(t)\right)dt\le N_8. \tag{16}$$

LEMMA 6. If the conditions of theorem are satisfied, the following estimate is true

$$\int_0^t\left(\|u_x(t)\|^2+\|H_x(t)\|^2\right)dt\le N_9,\quad \forall t\in[0,T].$$

PROOF. We multiply the momentum equation of system (1) by $(u-f)$, the magnetic field equation by $(H-\eta)$, add and integrate with respect to R.

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|u-f\|^2+\int\mathrm{v}(H-\eta)^2dx\right)+\int\frac{1}{\mathrm{v}}\left(u_x^2+H_x^2\right)dx+\\&+\int\left(\frac{\theta}{\mathrm{v}}+\frac{1}{2}H^2\right)f_xdx+\int\beta(x)|u|^\alpha(u-f)^2dx=\\&=\int\left(\frac{1}{\mathrm{v}}u_xf_x+\frac{1}{\mathrm{v}}H_x\eta_x+\beta(x)|u|^\alpha f(u-f)+\frac{1}{2}\eta^2u_x+\frac{\theta}{\mathrm{v}}u_x\right)dx=\sum_{i=1}^5B_i.\end{aligned} \tag{17}$$

Let us estimate the integrals $B_i\ (i=\overline{1,5})$ on the right-hand side of (17) using integrating by parts, Young's inequality with ε, the properties (4) and the bounds (7)

$$\sum_{i=1}^4B_i\le\varepsilon_1\left(\int\frac{1}{\mathrm{v}}u_x^2dx+\int\frac{1}{\mathrm{v}}H_x^2dx+\int\beta(x)|u|^\alpha(u-f)^2dx\right)+C_{18},\qquad 0<\varepsilon_1<\frac{1}{2},$$

$$B_5 = \int \frac{\theta}{\mathrm{v}} u_x dx = \int \frac{\varphi\theta - 1}{\varphi \mathrm{v}} u_x dx + \int \frac{1}{\varphi \mathrm{v}} u_x dx = J_1 + J_2 .$$

In order to estimate J_1 we partition the number axis R into following domains:

$$\Omega_1(t) = \{ x \in R : \varphi(x)\theta(x,t) > N_{10} \}, \quad N_0 = const > 1,$$

$$\Omega_2(t) = \{ x \in R : \varphi(x)\theta(x,t) \le N_{10}, \quad \varphi(x)\theta(x,t) \ne 1 \}, \quad \Omega_3(t) = \{ x \in R : \varphi(x)\theta(x,t) = 1 \}$$

It is easy to verify that in $\Omega_1(t)$ we have

$$\frac{\varphi\theta - 1}{\varphi\theta - \ln\varphi\theta - 1} < C_{19} \quad \text{since} \quad \lim_{\varphi\theta \to \infty} \frac{\varphi\theta - 1}{\varphi\theta - \ln\varphi\theta - 1} = 1 .$$

In $\Omega_2(t)$ we have

$$\frac{|\varphi\theta - 1|}{\sqrt{\varphi\theta - \ln\varphi\theta - 1}} < C_{20} \quad \text{since} \quad \lim_{\varphi\theta \to 0} \frac{|\varphi\theta - 1|}{\sqrt{\varphi\theta - \ln\varphi\theta - 1}} = 0, \quad \lim_{\varphi\theta \to 1} \frac{|\varphi\theta - 1|}{\sqrt{\varphi\theta - \ln\varphi\theta - 1}} = \sqrt{2} .$$

Using (4), (7), lemma 3, we find

$$J_1 = \int \frac{\varphi\theta - 1}{\varphi \mathrm{v}} u_x dx = \int_{\Omega_1(t)} (\varphi\theta - \ln\varphi\theta - 1) \frac{\varphi\theta - 1}{\varphi\theta - \ln\varphi\theta - 1} \frac{u_x}{\varphi \mathrm{v}} dx +$$

$$+ \int_{\Omega_2(t)} \sqrt{\varphi\theta - \ln\varphi\theta - 1} \frac{\varphi\theta - 1}{\sqrt{\varphi\theta - \ln\varphi\theta - 1}} \frac{u_x}{\varphi \mathrm{v}} dx \le$$

$$\le C_{21} \left(\int (\varphi\theta - \ln\varphi\theta - 1) dx \right)^{1/2} \left(\int \frac{1}{\mathrm{v}} u_x^2 dx \right)^{1/2} \left(M_\theta^{1/2} + 1 \right) \le \varepsilon_2 \int \frac{1}{\mathrm{v}} u_x^2 \, dx + C_{\varepsilon_2} (M_\theta + 1),$$

where $0 < \varepsilon_2 < \frac{1}{2}$. Transform and estimate the second addend in B_5 using Cauchy inequality and bounds (7).

$$J_2 = \int \frac{u_x}{\varphi \mathrm{v}} dx = \int \frac{1}{\varphi} \frac{\partial \ln \mathrm{v}\psi}{\partial t} dx = -\frac{d}{dt} \int \frac{1}{\varphi} (\mathrm{v}\psi - \ln \mathrm{v}\psi - 1) dx + \int \frac{1}{\varphi} \frac{\partial \, \mathrm{v}\psi}{\partial t} dx ,$$

$$\int \frac{1}{\varphi} \frac{\partial \, \mathrm{v}\psi}{\partial t} dx = \int \frac{\psi}{\varphi} \frac{\partial (u - f)}{\partial x} dx + \int \frac{\psi}{\varphi} f' \, dx =$$

$$= \int \frac{\psi}{\varphi^2} \varphi' (u - f) dx - \int \frac{\psi'}{\varphi} (u - f) dx + \int \frac{\psi}{\varphi} f' \, dx \le C_{22} .$$

Hence

$$B_5 \le -\frac{d}{dt} \int \frac{1}{\varphi} (\mathrm{v}\psi - \ln \mathrm{v}\psi - 1) dx + \varepsilon_2 \int \frac{1}{\mathrm{v}} u_x^2 \, dx + C_{23} (M_\theta(t) + 1).$$

Substitute the obtained relations for $B_i \left(i = \overline{1,5} \right)$ to (17). After integrating the inequality obtained from (17) with respect to t using (7), (16) we derive the required estimate. This proves the lemma 6.

LEMMA 7. If the conditions of theorem are satisfied, the following estimate is true

$$\| \mathrm{v}_x(t) \|^2 \le N_{11}, \quad \forall t \in [0, T] .$$

PROOF. We multiply the second and forth equations of system (1)

$$\frac{\partial}{\partial t}\left(\frac{\partial \ln \mathrm{v}\psi}{\partial x}\right)=\frac{\partial (u-f)}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\theta}{\mathrm{v}}\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}H^2\right)+\beta(x)|u|^{\alpha}u\,,$$
$$\frac{\partial}{\partial t}\mathrm{v}\,(H-\eta)=\frac{\partial}{\partial x}\left(\frac{1}{\mathrm{v}}\frac{\partial H}{\partial x}\right)-\eta\frac{\partial u}{\partial x}$$

by $(\ln \mathrm{v}\psi)_x$ and $\mathrm{v}(H-\eta)$ respectively, integrate with respect to R and add.

$$\frac{1}{2}\frac{d}{dt}\int\left[(\ln \mathrm{v}\psi)_x^2+(\mathrm{v}(H-\eta))^2\right]dx+\int\left[H_x^2+\frac{\theta}{\mathrm{v}}(\ln \mathrm{v}\psi)_x^2\right]dx=\frac{d}{dt}\int(u-f)\frac{\partial \ln \mathrm{v}\psi}{\partial x}\,dx+$$
$$+\int\frac{1}{\mathrm{v}}u_x^2\,dx+\int\frac{1}{\mathrm{v}}\frac{\partial \theta}{\partial x}\frac{\partial \ln \mathrm{v}\psi}{\partial x}\,dx\;-\int\frac{1}{\mathrm{v}}\frac{\partial u}{\partial x}f'\,dx+\int\frac{\theta}{\mathrm{v}}\frac{\partial \ln \mathrm{v}\psi}{\partial x}(\ln\psi)'\,dx+$$
$$+\int\frac{\partial H}{\partial x}\eta'\,dx+\int H\frac{\partial H}{\partial x}(\ln\psi)'\,dx+\int\eta\frac{\partial H}{\partial x}\frac{\partial \ln \mathrm{v}\psi}{\partial x}\,dx-\int\eta\frac{\partial H}{\partial x}(\ln\psi)'\,dx-$$
$$-\int\eta\frac{\partial u}{\partial x}\mathrm{v}(H-\eta)\,dx+\int\beta(x)|u|^{\alpha}u\frac{\partial \ln \mathrm{v}\psi}{\partial x}\,dx\,. \qquad (18)$$

Let us estimate the integrals on the right-hand side of (18) using the Gelder, Young and Cauchy inequalities, the conditions of theorem, the properties (4) and the known estimates. After some reductions we have

$$\frac{d}{dt}\left(\left\|(\ln \mathrm{v}\psi)_x\right\|^2+\|H-\eta\|^2\right)+\|H_x\|^2+\int\frac{\theta}{\mathrm{v}}(\ln \mathrm{v}\psi)_x^2dx\le\frac{d}{dt}\int(u-f)\frac{\partial \ln \mathrm{v}\psi}{\partial x}\,dx+$$
$$+C_{24}\left[\int\frac{1}{\mathrm{v}}u_x^2dx+\left(\int\frac{\theta_x^2}{\mathrm{v}\theta^{3/2}}dx+1\right)\left(\left\|(\ln \mathrm{v}\psi)_x\right\|^2+1\right)+M_\theta(t)+M_H^2(t)\right].$$

Here

$$\left|\int\beta(x)|u|^{\alpha}u\frac{\partial \ln \mathrm{v}\psi}{\partial x}\,dx\right|\le\left(\int(\ln \mathrm{v}\psi)_x^2dx\right)^{1/2}\left(\int\beta^2(x)|u|^{2\alpha}u^2dx\right)^{1/2}\le$$
$$\le\frac{1}{2}\left\|(\ln \mathrm{v}\psi)_x\right\|^2+C_{25}\left(\max_{x\in R}|u-f|^2+1\right)\int\beta(x)|u|^{2\alpha}dx\,,$$
$$\max_{x\in R}|u-f|^2\le 2\int|(u-f)(u-f)_x|dx\le C\left[\left(\int\frac{1}{\mathrm{v}}u_x^2dx\right)^{1/2}+\left(\int(f')^2dx\right)^{1/2}\right]\le C_{27}\left[\int\frac{1}{\mathrm{v}}u_x^2dx+1\right],$$
$$\int\beta(x)|u|^{2\alpha}dx\le\int\beta\left(|u-f|^{2\alpha}+|f|^{2\alpha}\right)dx\le\left(\int(u-f)^2dx\right)^{\alpha}\left(\int\beta^{1/(1-\alpha)}dx\right)^{1-\alpha}+C\le C_{27}\,,$$
$$\int\frac{\partial \ln \mathrm{v}\psi}{\partial x}(u-f)\,dx\le C_\gamma+\gamma\left\|(\ln \mathrm{v}\psi)_x\right\|^2,\quad 0<\gamma<1\,.$$

By integrating the inequality obtained from (18) with respect to t and using Gronwall's lemma and (13), (16), lemma 6 we deduce estimate

$$\max_{0\le t\le T}\left\|(\ln \mathrm{v}\psi)_x\right\|^2\le C_{28}\,.$$

Using the properties (4) we have affirmation of lemma. This proves the lemma 7.

We multiply the forth equation of system (1) by H_{xx} and integrate with respect to R and to t. After some reductions [6] we conclude that

$$\max_{0\le t\le T}\| H_x(t)\|^2 + \int_0^T \| H_{xx}(t)\|^2 dt \le N_{12}\,.$$

Hence we have $M_H^2(t)\le N_{13}, \qquad \forall t\in[0,T]$.

We multiply the second and third equations of system (1) by u_{xx} and $(\varphi\theta-1)$ respectively, integrate with respect to R and add. Reasoning as in [6] we derive

$$\max_{0\le t\le T}\left(\| \varphi\theta(t)-1\|^2 + \| u_x(t)\|^2\right) + \int_0^T\left(\| \theta_x(t)\|^2 + \| u_{xx}(t)\|^2\right) dt \le N_{14}\,.$$

After multiplying the heat-conducting equation of system (1) by θ_{xx} and some reductions [6] we have

$$\max_{0\le t\le T}\| \theta_x(t)\|^2 + \int_0^T \| \theta_{xx}(t)\|^2\, dt \le N_{15}\,.$$

From system (1) it follows that

$$\max_{0\le t\le T}\| \mathrm{v}_t(t)\|^2 \le N_{16}, \qquad \int_0^T\left(\| u_t(t)\|^2 + \| H_t(t)\|^2 + \| \theta_t(t)\|^2 + \| \mathrm{v}_{xt}(t)\|^2\right) dt \le N_{17}\,.$$

Thus, all the a priori estimates need to prove the existence of a generalized solution have been obtained. The uniqueness of the solution can be derived in the usual way, viz., by constructing a homogeneous equation for the difference between the two possible solutions.

The theorem is completely proved.

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