NUMERICAL REGULARIZATION OF CAUCHY PROBLEM FOR SINGULARLY PERTURBED PARABOLIC EQUATION

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In this article we study the following problem:

$$L_\varepsilon u \equiv \partial_t u(x,t,\varepsilon)- \varepsilon^2 a(x)\,\partial_x^2\, u(x,t,\varepsilon) =0, \quad (x,t) \in \Omega \tag{1}$$

$$u(x,t,\varepsilon)\,|_{t=0}=\chi(x), \tag{2}$$

where $\Omega=\{(x,t):\ x\in R=(-\infty,+\infty),\ \ t\in(0,T]\}$, $\chi(x)$-function of Hevisaid's, it's smooth function $a(x)>0\ \forall x\in R$.

As it is known if in degeneration the smoothness or part of the boundary condition are lost, the phenomenon of boundary layer will arise. Our problem concerns with first class problem. Problem (1), (2) was investigated in [1], there asymptotic of this problem solution was constructed and as is shown phenomenon of parabolic boundary layer is observed along the characteristic of limit equation.

Difficulties of solving problems with boundary layers are well known. They arise on applying numerical methods developed for regular problems (for example [2]-[3]). We propose the numerical method, wich lets us construct solution of problems (1), (2) with higher accuracy than in [2], [3]. The idea of our method is the following: applying regularization method of Lomov [4], first, we expand initial problem space with bigger dimension and obtain regular problem by small parameter. Then substituting derivatives with respect to initial variables in regular problem for differential correlation we obtain partially discrete problem. Solution of partially discrete problem is found in special class functions, in addition we get well known difference schemes for definition of this function coefficients. Earlier this method was applied for solving boundary problems for ordinary differential equations [5], [6].

Following regularization method we introduce variables to formulas

$$\tau=\frac{1}{\varepsilon^2}\,t,\ \ \xi=\frac{1}{\varepsilon^2}\,\varphi_j(x) =\frac{1}{\varepsilon^2}\int\limits_0^x \frac{ds}{\sqrt{a(s)}},\ \ \varphi(0) = 0$$

for expended function $\widetilde{u}\ (M,\varepsilon),\ M=(x,t,\xi,\tau)$, so that

$$\widetilde{u}\ (M,\varepsilon)|_{\theta=\theta(x,t,\varepsilon)}\equiv u(x,t,\varepsilon),\ \ \theta=(\xi,\tau),\ \ \theta(x,t,\varepsilon)=\frac{1}{\varepsilon^2}\,(\varphi(x),t),$$

finding derivatives

$$\partial_t u\equiv(\partial_t \widetilde{u} +\frac{1}{\varepsilon^2}\,\partial_\tau \widetilde{u}\,)|_{\theta=\theta(x,t,\varepsilon)}\,,$$

$$\partial_x u \equiv (\partial_x \tilde{u} + \frac{1}{\varepsilon^2}\varphi'(x)\partial_\xi \tilde{u})|_{\theta=\theta(x,t,\varepsilon)}$$

$$\partial_x^2 u \equiv (\partial_x^2 \tilde{u} + (\frac{1}{\varepsilon^2})^2 D_\xi \tilde{u} + \frac{1}{\varepsilon^2} L_\xi \tilde{u})|_{\theta=\theta(x,t,\varepsilon)},$$

$$D_\xi \equiv (\varphi'(x))^2 \partial_\xi^2, \ L_\xi \equiv [2\varphi'(x)\partial_{x,\xi}^2 + \varphi''(x)\partial_\xi]$$

Naturally we set up extended problem

$$\tilde{L}_\varepsilon \tilde{u}(M,\varepsilon) \equiv \frac{1}{\varepsilon^2} T_1 \tilde{u}(M,\varepsilon) + D\tilde{u}(M,\varepsilon) - \varepsilon^2 L_x \tilde{u}(M,\varepsilon) = 0, \ (M)\in\Omega,$$

$$\tilde{u}(M,\varepsilon)|_{t=\tau=0} = \chi(x), \tag{3}$$

where $\Omega=R\times(0,T]\times R\times(0,\infty)$, $T_1 \equiv \partial_\tau - \partial_\xi^2$, $D\equiv\partial_t - a(x)L_\xi$, $L_x \equiv a(x)\partial_x^2$.

In the range R×[0,T] where variables x and t change we introduce networks $\omega_h=\{x_i = ih,\ i=0,\pm1,\ \pm2,\dots\}$, $\omega_l=\{t_j=jl,\ j=0,1,2,\dots,l_0\}$, $\omega_{h,l}=\omega_h\times\omega_l=\{(ih,jl):\ i=0,\pm1,\ \pm2,\dots,\ j=0,1,2,\dots,l_0\}$, $h = x_i-x_{i-1}$, $l=T/l_0$ and draw straight lines $x= x_i$, $t=t_j$. Let us suppose that $\tilde{u}(M,\varepsilon)$ has necessary derivatives with respect to x and t. Let is assume in this equation that $x= x_k$, $t=t_j$, and replace derivatives with respect to x and t difference relations

$$\partial_x \tilde{u}(M,\varepsilon) = \frac{1}{h}[\tilde{u}|_{x=x_{k+1}} - \tilde{u}|_{x=x_k}] + O(h),$$

$$\partial_x^2 \tilde{u}(x,t,\varepsilon)|_{x=ih} = \frac{1}{h^2}[\tilde{u}|_{x=x_{k-1}} - 2\tilde{u}|_{x=x_k} + \tilde{u}|_{x=x_{k+1}}] + O(h^2)$$

$$\partial_t \tilde{u}(M,\varepsilon) = \frac{1}{l}[\tilde{u}|_{t=t_{k+1}} - \tilde{u}|_{t=t_k}] + O(l).$$

Neglecting members $O(\varepsilon^2 h)$, $O(\varepsilon^2 l)$ and higher ones, then introducing notation $y_k^j(\xi,\tau,\varepsilon)=\tilde{u}(x_k,t_j,\xi,\tau,\varepsilon)$ we obtain the system

$$L_{\varepsilon,k} y_k^j(\xi,\tau) \equiv T_1 y_k^j(\xi,\tau,\varepsilon) + \varepsilon^2 D_j y_k^j(\xi,\tau,\varepsilon) - \varepsilon^2 L_{\xi,k} y_k^j(\xi,\tau,\varepsilon) - \varepsilon^4 \Lambda y_k^j(\xi,\tau,\varepsilon) = 0, \tag{4}$$

$$(\xi,\tau)\in(-\infty,+\infty)\times(0,\infty),\ k=0,\pm1,\ \pm2,\dots,j=1,2,\dots,l_0,$$

initial condition goes into condition

$$y_k^0(\xi,\tau,\varepsilon)|_{\tau=0} = \chi_k. \tag{4$_1$}$$

Here following notations was introduced

$$L_{\xi,k} y_k^j(\xi,\tau) \equiv a(x_k)\,\partial_\xi\,[q_k^1 \frac{y_{k+1}^j(\tau,\xi) - y_k^j(\tau,\xi)}{h} + q_k^2\, y_k^j(\xi,\tau)],$$

$$\Lambda y_k^j(\xi,\tau,\varepsilon) \equiv a(x_k)\ \frac{y_{k+1}^j(\tau,\xi) - 2y_k^j(\tau,\xi) + y_{k-1}^j}{h^2}$$

$$D_j \equiv \frac{1}{l}\ [y_k^{j+1}(\xi,\tau) - y_k^j(\xi,\tau)],\quad q_k^1 = 2\varphi'(x_k),\ q_k^2 = \varphi''(x_k).$$

Remembering exact formula of difference relations we will rewrite equation (4) to obtain

$$L_{\varepsilon,k} y_k^j(\xi,\tau) = T_1 y_k^j(\xi,\tau,\varepsilon) + \varepsilon^2 D_j y_k^j(\xi,\tau,\varepsilon) - \varepsilon^2 L_{\xi,k} y_k^j(\xi,\tau,\varepsilon) -$$

$$-\varepsilon^4 \Lambda y_k^j(\xi,\tau,\varepsilon) = O(\varepsilon^2 (h+l)). \tag{5}$$

By analogy with asymptotic theory of problem (4), (4_1) solution we will determine it in the form

$$y_k^j(\xi,\tau,\varepsilon) = c_k^j\ erfc(\frac{\xi}{2\sqrt{\tau}}) + v_k^j,\ k=0,\pm1,\pm2,\ldots,\ j=0,1,\ldots,j_0, \tag{6}$$

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-s^2)\,ds$$

Let's substitute the function into (4), (4_1). Taking into account that

$$T_1 y_k^j(\xi,\tau,\varepsilon) \equiv 0,$$

$$L_{\xi,k} y_k^j(\xi,\tau,\varepsilon) \equiv a(x_k)\ [q_k^1 \frac{c_{k+1}^j - c_k^j}{h} + q_k^2\ c_k^j]\ \partial_\xi\ [erfc(\frac{\xi}{2\sqrt{\tau}})\ ],$$

$$D_j y_k^j(\xi,\tau,\varepsilon) \equiv \frac{1}{l}\ [\ c_k^{j+1}(\xi,\tau) - c_k^j\ ] erfc(\frac{\xi}{2\sqrt{\tau}}) + \frac{1}{l}[v_k^{j+1} - v_k^j\ ]$$

$$\Lambda y_k^j(\xi,\tau,\varepsilon) \equiv a(x_k) h^{-2}\{[c_{k-1}^j - 2c_k^j + c_{k+1}^j] erfc(\frac{\xi}{2\sqrt{\tau}}) + v_{k-1}^j - 2v_k^j + v_{k+1}^j\}$$

and on the basis of (4), (4_1), we choose arbitrary functions v_k^j, c_k^j, as a solution of following network equations

$$q_k^1 \frac{c_{k+1}^j - c_k^j}{h} + q_k^2\ c_k^j = 0, \tag{7_1}$$

$$\frac{1}{l}\ [c_k^{j+1}(\xi,\tau) - c_k^j\ ] - a(x_k) h^{-2}\{[c_{k-1}^j - 2c_k^j + c_{k+1}^j] = 0 \tag{7_2}$$

$$\frac{1}{l}[v_k^{j+1} - v_k^j] - a(x_k)h^{-2}[v_{k-1}^j - 2v_k^j + v_{k+1}^j] = 0 \qquad (7_3)$$

$$k=0,\pm 1, \pm 2,\dots, j=0,1,\dots,j_0 .$$

Initial conditions are satisfied by $v_k^0 = \chi_k$ and the other components of (6) transform to zero because of $erfc(\frac{\xi}{2\sqrt{\tau}})\,|_{\tau=0}=0$. In virtue of this we can choose initial conditions for equations for equation (7_1)-(7_2) arbitrary.

On the basis of

$$|erfc(\frac{\xi}{2\sqrt{\tau}})\,| < c\, exp(-\frac{\xi^2}{8\tau})$$

it's easy to establish estimate

$$\max_{j,k} |\tilde{u}(x_k, t_j, \xi, \tau, \varepsilon) - y_k^j(\xi, \tau)| \le c\, \varepsilon^2(h+l)|,\ k=0,\pm 1, \pm 2,\dots, j=0,1,\dots,j_0. \qquad (8)$$

To obtain solution of initial problem we hase to narrow down function (6) by regulating function i/e/ assuming in relation (6) $\xi=\frac{1}{\varepsilon^2}\varphi(x_k)$, $\tau=t_j/\varepsilon^2$, we will obtain

$$u(x_k,t_j,\varepsilon)= c_k^j\, erfc(\frac{\varphi(x_k)}{2\varepsilon\sqrt{t_j}}) + v_k^j,\ k=0,\pm 1, \pm 2,\dots, j=0,1,\dots,j_0.$$

Making narrowing in inequality (8) we get estimate of solution error for initial problem. For practical realization of given algorithm equation (3) can be made discreet by Cranck-Nikolson's method. Then instead of estimating (8) we can get estimate

$$||z||_U \le c\varepsilon^2(h^2+l^2).$$

Summing up we can formulate result as a following theorem.

Theorem. *In the above mentioned condition for function a(x) is true, then with the* $\xi_k = \frac{1}{\varepsilon^2}\varphi(x_k)$, $\tau_j = \frac{1}{\varepsilon^2} t_j$ *the solution semidiscrete problem (4),(4_1) will result in the approximation of the initial problem's (1),(2) solution and the uniform estimation will takes place*

$$||u(x_k,t_j,\varepsilon) - y_k^j(\frac{1}{\varepsilon^2}\varphi(x_k), \frac{1}{\varepsilon^2}t_j||_U < c\varepsilon^2(h+l).$$

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