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Existence of Solutions for Nonlocal Boundary Value Problem of Hadamard Fractional Differential Equations

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Abstract

We investigate the existence and uniqueness of solutions for Hadamard fractional differential equations with nonlocal integral boundary conditions, by using the Leray-Schauder nonlinear alternative, Leray Schauder degree theorem, Krasnoselskiis fixed point theorem, Schaefers fixed point theorem, Banach fixed point theorem, Nonlinear Contractions. Two examples are also presented to illustrate our results.

Keywords: Hadamard fractional derivative; Hadamard fractional integral; Fractional differential equation; Integral boundary conditions; Existence; Fixed point.
2010 MSC: 26A33, 34A08, 34A12.

1. Introduction

Fractional differential equations have increased extensive consideration from both hypothetical and the applied perspectives as of late years. There are various applications in an assortment of fields, for example, chemical physics, permeable media, signal processing, viscoelasticity, aerodynamics, exhibiting anomalous diffusion, fluid flow, electrical systems, financial aspects, etc., rather than integer-order differential and integral operators, fractional-order differential operators are nonlocal and give the way to investigate the inherited properties of a few materials and procedures. The monographs [14, 15, 16, 18, 20] have ordinarily

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referred to the hypothesis of fractional derivatives and integrals and applications to differential equations of fractional order. For more points of interest and models, see [1, 4, 5, 6, 9, 10, 17, 24, 25, 26, 27, 28, 29, 21, 22, 23, 31], and the references in that.

In any case, it has seen that the greater part of the work on the point is concerned about Riemann-Liouville or Caputo type fractional differential equation. Other than these fractional derivatives, another sort of fractional derivatives established in the literature is the fractional derivative because of Hadamard made known to in 1892 [12], differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of an arbitrary exponent. A point by point depiction of Hadamard fractional derivatives and integral having been discovered in [2, 3, 7, 8, 11, 13, 30, 33]. Recently Qinghua et.al [19] discussed a Lyapunovtype inequality with the Hadamard fractional derivative. Similarly, Wang et.al [32] studied the nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions.

In this paper, we introduce nonlocal boundary value problem on the Hadamard fractional derivatives of the form:

$$\begin{aligned} &{}^{H}\mathcal{D}^{\varrho}y(\tau) = g(\tau, y(\tau)), \quad \tau \in [1, T], \\ &y(1) = 0, \;\; y'(1) = 0, \;\; {}^{H}\mathcal{D}^{\varsigma}y(T) = \omega\, {}^{H}\mathcal{J}^{\gamma}y(\varphi), \; 1 < \varphi < T, \end{aligned} \tag{1}$$

where ${}^{H}\mathcal{D}^{\varrho}$, ${}^{H}\mathcal{D}^{\varsigma}$ denotes the Hadamard fractional derivatives of order $2 < \varrho \leq 3$, $1 < \varsigma < 2$, and g: $[1, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and ω is positive real constant. The rest of the paper has organized as follows: section 2 has devoted to some fundamental concepts of fractional calculus with basic lemmas related to the given problem. The existence and uniqueness results based on, Leray-Schauder nonlinear alternative, Leray-Schauder degree theorem, Krasnoselskii's fixed point theorem, Schaefer's fixed point theorem, Banach fixed point theorem, and Nonlinear contractions have obtained in section 3. The validation of the results has done by providing examples in section 4.

2. Preliminaries

We begin with some basic definitions, properties, and lemmas with results [14, 18].

Definition 2.1. *The Hadamard fractional integral of order $\varrho \in \mathbb{R}^+$ of a function $g \in L^p[b, c]$, $0 \leq b \leq \tau \leq c \leq \infty$, is defined as*

$$({}^{H}\mathcal{J}^{\varrho}g)(\tau) = \frac{1}{\Gamma(\varrho)} \int_b^\tau \left(\log\frac{\tau}{\sigma}\right)^{\varrho-1} g(\sigma)\frac{d\sigma}{\sigma}, \quad \varrho > 0.$$

Definition 2.2. *Let $0 < b < c < \infty$, $\delta = \left(\tau\frac{d}{d\tau}\right)$ and $\mathcal{AC}^n_\delta[b, c] = \{g : [b, c] \to \mathbb{R} : \delta^{n-1}[g(\tau)] \in \mathcal{AC}[b, c]\}$. The Hadamard fractional derivative of order $\varrho > 0$ for a function $g \in \mathcal{AC}^n_\delta[b, c]$ is defined as*

$$({}^{H}\mathcal{D}^{\varrho}g)(\tau) = \frac{1}{\Gamma(n-\varrho)}\left(\tau\frac{d}{d\tau}\right)^n \int_1^\tau \left(\log\frac{\tau}{\sigma}\right)^{n-\varrho-1} g(\sigma)\frac{d\sigma}{\sigma},$$

where $n - 1 < \varrho < n$, $n = [\varrho] + 1$, $[\varrho]$ denotes the integer part of the real number ϱ and $\log(\cdot) = \log_e(\cdot)$.

Lemma 2.3. *Let $b, \varrho, \varsigma > 0$, then*

$$\begin{aligned} ({}^{H}\mathcal{D}^{\varsigma}(\log\frac{\tau}{b})^{\varrho-1})(y) &= \frac{\Gamma(\varrho)}{\Gamma(\varrho-\varsigma)}\left(\log\frac{y}{b}\right)^{\varrho-\varsigma-1}, \\ ({}^{H}\mathcal{J}^{\varsigma}(\log\frac{\tau}{b})^{\varrho-1})(y) &= \frac{\Gamma(\varrho)}{\Gamma(\varrho+\varsigma)}\left(\log\frac{y}{b}\right)^{\varrho+\varsigma-1}. \end{aligned}$$

Lemma 2.4. *Let $\varrho > 0$ and $y \in \mathcal{C}[1,\infty) \cap L^1[1,\infty)$. Then the solution of Hadamard fractional differential equation $({}^{H}\mathcal{D}^{\varrho} y)(\tau) = 0$ is given by*

$$y(\tau) = \sum_{i=1}^{n} a_i (\log \tau)^{\varrho - i}.$$

and the following formula holds:

$${}^{H}\mathcal{J}^{\varrho H}\mathcal{D}^{\varrho} y(\tau) = y(\tau) + \sum_{i=1}^{n} a_i (\log \tau)^{\varrho - i},$$

where $a_i \in \mathbb{R}$, $i = 1, 2, ..., n$ and $n - 1 < \varrho < n$.

We define space $\mathcal{P} = \mathcal{C}([1,T],\mathbb{R})$ the Banach space of all continuous functions from $[1,T] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|y\| = \sup\{|y(\tau)|, \tau \in [1,T]\}$.
Next, we present an auxiliary lemma which plays a key role in the sequel.

Lemma 2.5. *For any $\hat{g} \in \mathcal{C}([1,T],\mathbb{R})$, $y \in \mathcal{C}([1,T],\mathbb{R})$, the function y is the solution of the problem*

$$\begin{aligned} {}^{H}\mathcal{D}^{\varrho} y(\tau) &= \hat{g}(\tau), \quad \tau \in [1,T], \\ y(1) = 0, \quad y'(1) &= 0, \quad {}^{H}\mathcal{D}^{\varsigma} y(T) = \omega {}^{H}\mathcal{J}^{\gamma} y(\varphi), \end{aligned} \tag{2}$$

if and only if

$$y(\tau) = {}^{H}\mathcal{J}^{\varrho} \hat{g}(\tau) + \frac{(\log \tau)^{\varrho-1}}{\Lambda} \left[\omega {}^{H}\mathcal{J}^{\varrho+\gamma} \hat{g}(\varphi) - {}^{H}\mathcal{J}^{\varrho-\varsigma} \hat{g}(T) \right], \tag{3}$$

where

$$\Lambda = \frac{\Gamma(\varrho)}{\Gamma(\varrho - \varsigma)} (\log T)^{\varrho-\varsigma-1} - \omega \frac{\Gamma(\varrho)}{\Gamma(\varrho + \gamma)} (\log \varphi)^{\varrho+\gamma-1}. \tag{4}$$

Proof. Applying the operator ${}^{H}\mathcal{J}^{\varrho}$ on the linear fractional differential equations in (2), we obtain

$$y(\tau) = {}^{H}\mathcal{J}^{\varrho} \hat{g}(\tau) + a_1 (\log \tau)^{\varrho-1} + a_2 (\log \tau)^{\varrho-2} + a_3 (\log \tau)^{\varrho-3}, \tag{5}$$

where a_1,a_2 and $a_3 \in \mathbb{R}$, are arbitrary unknown constants. Using the boundary conditions (2) in (5), we get $a_2, a_3 = 0$ and

$$a_1 = \frac{1}{\Lambda} \left[\omega {}^{H}\mathcal{J}^{\varrho+\gamma} \hat{g}(\varphi) - {}^{H}\mathcal{J}^{\varrho-\varsigma} \hat{g}(T) \right].$$

Substituting the value of a_1, a_2 in (5), we get the solution (3). This completes the proof. □

3. Existence and Uniqueness Results

In view of Lemma 2.5, we interpret an operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ as

$$\begin{aligned} \mathcal{T}(y)(\tau) = {}& {}^{H}\mathcal{J}^{\varrho} g(\sigma, y(\sigma))(\tau) + \frac{(\log \tau)^{\varrho-1}}{\Lambda} \Big[\omega {}^{H}\mathcal{J}^{\varrho+\gamma} g(\sigma, y(\sigma))(\varphi) \\ & - {}^{H}\mathcal{J}^{\varrho-\varsigma} g(\sigma, y(\sigma))(T) \Big]. \end{aligned} \tag{6}$$

In the sequel, we use the following expressions:

$$\begin{aligned}
{}^{H}\mathfrak{J}^{\varrho}g(\sigma,y(\sigma))(\tau) &= \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\left(\log\frac{\tau}{\sigma}\right)^{\varrho-1}g(\sigma,y(\sigma))\frac{d\sigma}{\sigma},\\
\omega^{H}\mathfrak{J}^{\varrho+\gamma}g(\sigma,y(\sigma))(\varphi) &= \omega\frac{1}{\Gamma(\varrho+\gamma)}\int_1^{\varphi}\left(\log\frac{\varphi}{\sigma}\right)^{\varrho+\gamma-1}g(\sigma,y(\sigma))\frac{d\sigma}{\sigma},\\
{}^{H}\mathfrak{J}^{\varrho-\varsigma}g(\sigma,y(\sigma))(T) &= \frac{1}{\Gamma(\varrho-\varsigma)}\int_1^{T}\left(\log\frac{T}{\sigma}\right)^{\varrho-\varsigma-1}g(\sigma,y(\sigma))\frac{d\sigma}{\sigma}.
\end{aligned}$$

Suitable for computation, we represent:

$$\Delta = \frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\frac{(\log T)^{\varrho-1}}{\Lambda}\left(\frac{\omega(\log\varphi)^{\varrho+\gamma}}{\Gamma(\varrho+\gamma+1)}+\frac{(\log T)^{\varrho-\varsigma}}{\Gamma(\varrho-\varsigma+1)}\right). \tag{7}$$

Our first existence result is based on Leray-Schauder nonlinear alternative.

Theorem 3.1. *Let us speculate that* $g:[1,T]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function and the following conditions hold :*

($\mathcal{E}_1$) *There exists a function* $q\in\mathcal{C}([1,T],\mathbb{R}^+)$*, and* $\psi:\mathbb{R}^+\to\mathbb{R}^+$ *nondecreasing such that* $|g(\tau,y)|\le q(\tau)\psi(\|y\|)$ *for each* $(\tau,y)\in[1,T]\times\mathbb{R}$*;*

($\mathcal{E}_2$) *There exists a number* $\mathcal{L}>0$ *such that*

$$\begin{aligned}
\frac{\mathcal{L}}{\|q\|\psi(\mathcal{L})} &> {}^{H}\mathfrak{J}^{\varrho}q(\sigma)(T)+\mathcal{G}\\
\mathcal{G} &= \frac{(\log T)^{\varrho-1}}{\Lambda}\left[\omega^{H}\mathfrak{J}^{\varrho+\gamma}q(\sigma)(\varphi)+{}^{H}\mathfrak{J}^{\varrho-\varsigma}q(\sigma)(T)\right].
\end{aligned}$$

Then there exists at least one solution for problem (1) on $[1,T]$.

Proof. To begin with, the operator $\mathcal{T}:\mathcal{P}\to\mathcal{P}$ is described by (6). Next, we demonstrate that $\mathcal{T}$ maps bounded sets into bounded sets in $\mathcal{C}([1,T],\mathbb{R})$. For a positive number θ, let $\mathcal{B}_\theta=\{y\in\mathcal{C}([1,T],\mathbb{R}):\|y\|\le\theta\}$ be a bounded set in $\mathcal{C}([1,T],\mathbb{R})$. Then, for each $y\in\mathcal{B}_\theta$, we have

$$\begin{aligned}
&|(\mathcal{T}y)(\tau)|\\
&\le {}^{H}\mathfrak{J}^{\varrho}|g(\sigma,y(\sigma))|(\tau)+\frac{(\log\tau)^{\varrho-1}}{\Lambda}\Big[\omega^{H}\mathfrak{J}^{\varrho+\gamma}|g(\sigma,y(\sigma))|(\varphi)\\
&\quad+{}^{H}\mathfrak{J}^{\varrho-\varsigma}|g(\sigma,y(\sigma))|(T)\Big]\\
&\le \psi(\|y\|)^{H}\mathfrak{J}^{\varrho}q(\sigma)(T)+\frac{\psi(\|y\|)(\log T)^{\varrho-1}}{\Lambda}\Big[\omega^{H}\mathfrak{J}^{\varrho+\gamma}q(\sigma)(\varphi)\\
&\quad+{}^{H}\mathfrak{J}^{\varrho-\varsigma}q(\sigma)(T)\Big],
\end{aligned}$$

and consequently,

$$\begin{aligned}
\|\mathcal{T}y\| \le \psi(\|\theta\|)\Big\{&{}^{H}\mathfrak{J}^{\varrho}q(\sigma)(T)+\frac{(\log T)^{\varrho-1}}{\Lambda}\Big[\omega^{H}\mathfrak{J}^{\varrho+\gamma}q(\sigma)(\varphi)\\
&+{}^{H}\mathfrak{J}^{\varrho-\varsigma}q(\sigma)(T)\Big]\Big\}.
\end{aligned}$$

We shall proceed to prove that the operator $\mathcal{T}$ maps bounded sets into equicontinuous sets of $\mathcal{C}([1,T],\mathbb{R})$. For $\tau_1,\tau_2 \in [1,T]$ with $\tau_1 < \tau_2$, and $y \in \mathcal{B}_\theta$ is a bounded set of $\mathcal{C}([1,T],\mathbb{R})$. Then we have

$$\begin{aligned}
&|(\mathcal{T}y)(\tau_2) - (\mathcal{T}y)(\tau_1)| \\
&\quad \leq \ |{}^H\mathcal{J}^{\varrho}|g(\sigma,y(\sigma))|(\tau_2) - {}^H\mathcal{J}^{\varrho}|g(\sigma,y(\sigma))|(\tau_1)| \\
&\qquad + \frac{|(\log\tau_2)^{\varrho-1} - (\log\tau_1)^{\varrho-1}|}{\Lambda}\Bigg[\omega^H\mathcal{J}^{\varrho+\gamma}|g(\sigma,y(\sigma))|(\varphi) \\
&\qquad + {}^H\mathcal{J}^{\varrho-\varsigma}|g(\sigma,y(\sigma))|(T)\Bigg] \\
&\quad \leq \ \frac{\psi(\|\theta\|)}{\Gamma(\varrho)}\Bigg|\int_0^{\tau_1}\left[\left(\log\frac{\tau_2}{\sigma}\right)^{\varrho-1} - \left(\log\frac{\tau_1}{\sigma}\right)^{\varrho-1}\right]q(\sigma)\frac{d\sigma}{\sigma} \\
&\qquad + \int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{\sigma}\right)^{\varrho-1}q(\sigma)\frac{d\sigma}{\sigma}\Bigg| \\
&\qquad + \frac{|(\log\tau_2)^{\varrho-1} - (\log\tau_1)^{\varrho-1}|}{\Lambda}\Bigg[\omega^H\mathcal{J}^{\varrho+\gamma}q(\sigma)(\varphi) + {}^H\mathcal{J}^{\varrho-\varsigma}q(\sigma)(T)\Bigg].
\end{aligned}$$

Hence we have that right hand side of the above inequality tends to zero independent of $y \in \mathcal{B}_\theta$ as $\tau_2 - \tau_1 \to 0$. Therefore, the operator $\mathcal{T}(y)$ is equicontinuous and consequently, by Arzela-Ascoli theorem, it is completely continuous. Next, we demonstrate that the boundedness of the set of all solutions to equations $y = \nu\mathcal{T}(y)$, $0 < \nu < 1$. Let y be a solution. Then, for $\tau \in [1,T]$, and using the computations in proving that $\mathcal{T}$ is bounded, we have

$$\begin{aligned}
|(\mathcal{T}y)(\tau)| \ &\leq \ \psi(\|y\|)\Bigg\{{}^H\mathcal{J}^{\varrho}q(\sigma)(T) + \frac{(\log T)^{\varrho-1}}{\Lambda}\Bigg[\omega^H\mathcal{J}^{\varrho+\gamma}q(\sigma)(\varphi) \\
&\qquad + {}^H\mathcal{J}^{\varrho-\varsigma}q(\sigma)(T)\Bigg]\Bigg\} \\
&= \ \psi(\|y\|)^H\mathcal{J}^{\varrho}q(\sigma)(T) + \mathcal{G}.
\end{aligned}$$

In view of $(\mathcal{E}_2)$, there exists $\mathcal{L}$ such that $\|y\| \neq \mathcal{L}$. Let us set

$$\mathcal{M} \ = \ \{y \in \mathcal{C}([1,T],\mathbb{R}) : \|y\| < \mathcal{L}\}.$$

Bearing in mind that the operator $\mathcal{T} : \overline{\mathcal{W}} \to \mathcal{C}([1,T],\mathbb{R})$ is continuous and completely continuous. From the of choice of $\mathcal{W}$, there is no $y \in \partial\mathcal{W}$ such that $y = \nu\mathcal{T}(y)$, $0 < \nu < 1$. Consequently, by the Leray-Schauder nonlinear alternative, we deduce that $\mathcal{T}$ has a fixed point $y \in \overline{\mathcal{W}}$ which is a solution of the problem (1). □

Our second existence result is based on Leray-Schauder degree theorem.

Theorem 3.2. *Let $g : [1,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that*

$(\mathcal{E}_3)$ *There exists constants $0 \leq \phi < \frac{1}{\Delta}$, and $\mathcal{A} > 0$ such that*
$|g(\tau,y)| \leq \phi|y| + \mathcal{A}\ \forall\ (\tau,y) \in [1,T] \times \mathbb{R}$,

where Δ is described by (7). Then the boundary value problem (1) has at least one solution on $[1,T]$.

Proof. We interpret an operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ as in (6). In view of the fixed point problem

$$y \ = \ \mathcal{T}y, \tag{8}$$

we shall to prove the existence of at least one solution $y \in \mathcal{C}([1,T])$ satisfying (8). Set a ball $\mathcal{B}_\mathcal{R} \subset \mathcal{C}([1,T])$, as

$$\mathcal{B}_\mathcal{R} \ = \ \{y \in \mathcal{P} : \max_{\tau\in\mathcal{C}([1,T])}|y(\tau)| < \mathcal{R}\},$$

with a constant radius $\mathcal{R} > 0$. Hence, we shall show that $\mathcal{T} : \overline{\mathcal{B}}_{\mathcal{R}} \to \mathcal{C}([1,T])$ satisfies the following:

$$y \neq \zeta \mathcal{T} y, \quad \forall\, y \in \partial \mathfrak{B}_{\mathcal{R}}, \quad \forall\, \zeta \in [1,T]. \tag{9}$$

We set

$$\mathcal{T}(\zeta, y) = \zeta \mathfrak{T} y, \quad y \in \mathcal{P}, \quad \zeta \in [1,T].$$

As shown in Theorem 3.2 we have that the operator $\mathcal{T}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzela-Ascoli theorem, a continuous map f_ζ described by $f_\zeta(y) = y - \mathcal{T}(\zeta, y) = y - \zeta \mathcal{T} y$ is completely continuous. If (9) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} deg(f_\zeta, \mathcal{B}_{\mathcal{R}}, 0) &= deg(\mathcal{I} - \zeta \mathfrak{T}, \mathcal{B}_{\mathcal{R}}, 0) = deg(f, \mathcal{B}_{\mathcal{R}}, 0) \\ &= deg(f_0, \mathcal{B}_{\mathcal{R}}, 0) = deg(\mathcal{I}, \mathcal{B}_{\mathcal{R}}, 0) = 1 \neq 0, \quad 0 \in \mathcal{B}_{\mathcal{R}}, \end{aligned}$$

where $\mathcal{I}$ denotes the unit operator. By the nonzero property of Leray-Schauder degree, $f_1(y) = y - \mathcal{T} y = 0$ for at least one $y \in \mathcal{B}_{\mathcal{R}}$. Let us assume that $y = \zeta \mathfrak{T} y$ for some $\zeta \in [1,T]$ and for all $\tau \in [1,T]$ so that

$$\begin{aligned} |(\mathcal{T}y)(\tau)| &\leq {}^H\mathcal{J}^{\varrho} |g(\sigma, y(\sigma))|(\tau) + \frac{(\log \tau)^{\varrho-1}}{\Lambda} \Big[\omega\, {}^H\mathcal{J}^{\varrho+\gamma} |g(\sigma, y(\sigma))|(\varphi) \\ &\quad + {}^H\mathcal{J}^{\varrho-\varsigma} |g(\sigma, y(\sigma))|(T) \Big] \\ &\leq \phi|y| + \mathcal{A} \Big[{}^H\mathcal{J}^{\varrho} q(\sigma)(T) + \frac{(\log T)^{\varrho-1}}{\Lambda} \Big[\omega\, {}^H\mathcal{J}^{\varrho+\gamma} q(\sigma)(\varphi) \\ &\quad + {}^H\mathcal{J}^{\varrho-\varsigma} q(\sigma)(T) \Big] \Big] \\ &= (\phi|y| + \mathcal{A}) \Delta \end{aligned}$$

which, on taking norm $\sup_{\tau \in [1,T]} |y(\tau)| = \|y\|$ and solving for $\|y\|$, yields

$$\|y\| \leq \frac{\mathcal{A}\Delta}{1 - \phi\Delta}.$$

If $\mathcal{R} = \frac{\Delta}{1-\phi\Delta} + 1$, inequality (9) holds. This completes the proof. □

Our third existence result is based on Krasnoselskii's fixed point theorem.

Theorem 3.3. *Let $g : [1,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the following conditions hold:*

($\mathcal{E}_4$) $|g(\tau, p_1) - g(\tau, p_2)| \leq \mathcal{S}|p_1 - p_2|$, $\forall\, \tau \in [1,T], p_1, p_2 \in \mathbb{R}, \mathcal{S} > 0$.

($\mathcal{E}_5$) $|g(\tau, y(\tau))| \leq \xi(\tau)$ *for* $(\tau, y) \in [1,T] \times \mathbb{R}$, *and* $\xi \in \mathcal{C}([1,T], \mathbb{R}^+)$ *with* $\|\xi\| = \max_{\tau \in [1,T]} |\xi(\tau)|$.

If

$$\left\{ \frac{\mathcal{S}(\log T)^{\varrho-1}}{\Lambda} \left(\frac{\omega(\log \varphi)^{\varrho+\gamma}}{\Gamma(\varrho+\gamma+1)} + \frac{(\log T)^{\varrho-\varsigma}}{\Gamma(\varrho-\varsigma+1)} \right) \right\} < 1. \tag{10}$$

Then, there exists at least one solution for the problem (1) on $[1,T]$.

Proof. Let us interpret $\mathcal{B}_\theta = \{y \in \mathcal{P} : \|y\| \leq \theta\}$, where $\theta \geq \|\xi\|\Delta$. To prove the hypothesis of Krasnoselskii's fixed point result, we split the operator $\mathcal{T}$ given by (6) as $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ on $\mathcal{B}_\theta$, where

$$\begin{aligned} (\mathcal{T}_1 y)(\tau) &= {}^H\mathcal{J}^{\varrho} g(\sigma, y(\sigma))(\tau) \\ (\mathcal{T}_2 y)(\tau) &= \frac{(\log \tau)^{\varrho-1}}{\Lambda} \Big[\omega\, {}^H\mathcal{J}^{\varrho+\gamma} g(\sigma, y(\sigma))(\varphi) - {}^H\mathcal{J}^{\varrho-\varsigma} g(\sigma, y(\sigma))(T) \Big]. \end{aligned}$$

For $\hat{y}_1, \hat{y}_2 \in \mathcal{B}_\theta$,
$|(\mathcal{T}_1\hat{y}_1)(\tau) + (\mathcal{T}_2\hat{y}_2)(\tau)|$

$$\begin{aligned} &\leq \sup_{\tau\in[1,T]} \Bigg\{ {}^H\mathcal{J}^{\varrho}|g(\sigma, y(\sigma))|(\tau) + \frac{(\log\tau)^{\varrho-1}}{\Lambda}\Bigg[\omega\, {}^H\mathcal{J}^{\varrho+\gamma}|g(\sigma, y(\sigma))|(\varphi) \\ &\qquad + {}^H\mathcal{J}^{\varrho-\varsigma}|g(\sigma, y(\sigma))|(T)\Bigg]\Bigg\} \\ &\leq \|\xi\|\Bigg\{\frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \frac{(\log T)^{\varrho-1}}{\Lambda}\Bigg(\frac{\omega(\log\varphi)^{\varrho+\gamma}}{\Gamma(\varrho+\gamma+1)} + \frac{(\log T)^{\varrho-\varsigma}}{\Gamma(\varrho-\varsigma+1)}\Bigg)\Bigg\} \\ &\leq \|\xi\|\Delta \leq \theta, \end{aligned}$$

which implies that $\mathcal{T}_1\hat{y}_1 + \mathcal{T}_2\hat{y}_2 \in \mathcal{B}_\theta$.
Now, we will show that $\mathcal{T}_2$ is a contraction. Let $p_1, p_2 \in \mathbb{R}$, $\tau \in [1, T]$. Then, using the assumption $(\mathcal{E}_4)$ together with (10), we get

$$\|\mathcal{T}_2 p_1 - \mathcal{T}_2 p_2\| \leq \frac{\mathcal{S}(\log T)^{\varrho-1}}{\Lambda}\left(\frac{\omega(\log\varphi)^{\varrho+\gamma}}{\Gamma(\varrho+\gamma+1)} + \frac{(\log T)^{\varrho-\varsigma}}{\Gamma(\varrho-\varsigma+1)}\right)\|p_1 - p_2\|.$$

By the assumption $(\mathcal{E}_4)$, it follows that the operator $\mathcal{T}_2$ is a contraction. Next, we will show that $\mathcal{T}_1$ is compact and continuous. Continuity of g implies that the operator $\mathcal{T}_1$ is continuous. Also, $\mathcal{T}_1$ is uniformly bounded on B_θ as

$$\|\mathcal{T}_1 y\| \leq \frac{\|\xi\|(\log T)^{\varrho}}{\Gamma(\varrho+1)}.$$

Moreover, with $\sup_{(\tau,y)\in[1,T]\times\mathcal{B}_\theta} |g(\tau, y)| = \hat{g} < \infty$ and $\tau_1 < \tau_2$, $\tau_1, \tau_2 \in [1, T]$, we have

$$\begin{aligned} |(\mathcal{T}_1 y)(\tau_2) - (\mathcal{T}_1 y)(\tau_1)| &= |{}^H\mathcal{J}^{\varrho}|g(\sigma, y(\sigma))|(\tau_2) - {}^H\mathcal{J}^{\varrho}|g(\sigma, y(\sigma))|(\tau_1)| \\ &\leq \frac{\hat{g}}{\Gamma(\varrho)}\Bigg|\int_0^{\tau_1}\left[\left(\log\frac{\tau_2}{\sigma}\right)^{\varrho-1} - \left(\log\frac{\tau_1}{\sigma}\right)^{\varrho-1}\right]\frac{d\sigma}{\sigma} \\ &\quad + \int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{\sigma}\right)^{\varrho-1}\frac{d\sigma}{\sigma}\Bigg|. \end{aligned} \tag{11}$$

Clearly, the right-hand sides of (11) tends to zero independent of y as $\tau_2 - \tau_1 \to 0$. Thus, $\mathcal{T}_1$ is relatively compact on $\mathcal{B}_\theta$. Hence, by the Arzela-Ascoli Theorem, $\mathcal{T}_1$ is compact on $\mathcal{B}_\theta$. Thus, all the assumptions of Krasnoselskii's fixed point theorem are satisfied. Therefore, there exists at least one solution for problem (1) on $[1, T]$. □

Our next existence result is based on Schaefer's fixed point theorem.

Theorem 3.4. *Let $g : [1, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exists a positive constant $\widehat{\mathcal{M}}$ such that $|g(\tau, y)| \leq \widehat{\mathcal{M}}$ for $\tau \in [1, T]$, $y \in \mathbb{R}$. Then there exists at least one solution for problem (1) on $[1, T]$.*

Proof. To begin with, we demonstrate that the operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ is completely continuous. By continuity of the function g, it follows that the operator $\mathcal{T}$ is continuous.

For a positive constant θ, let $\mathcal{B}_\theta = \{y \in \mathcal{P} : \|y\| \leq \theta\}$ be a bounded set in $\mathcal{P}$. Then, for $\tau \in [1, T]$, we obtain

$$\begin{aligned} |(\mathcal{T}y)(\tau)| &\leq {}^H\mathcal{J}^{\varrho}|g(\sigma, y(\sigma))|(\tau) + \frac{(\log \tau)^{\varrho-1}}{\Lambda}\Bigg[\omega\, {}^H\mathcal{J}^{\varrho+\gamma}|g(\sigma, y(\sigma))|(\varphi) \\ &\quad + {}^H\mathcal{J}^{\varrho-\varsigma}|g(\sigma, y(\sigma))|(T)\Bigg] \\ &\leq \widehat{\mathcal{M}}\left\{{}^H\mathcal{J}^{\varrho}(T) + \frac{(\log T)^{\varrho-1}}{\Lambda}\left[\omega\, {}^H\mathcal{J}^{\varrho+\gamma}(\varphi) + {}^H\mathcal{J}^{\varrho-\varsigma}(T)\right]\right\} = \widehat{\mathcal{M}}\Delta. \end{aligned}$$

Hence it follows that $\mathcal{T}$ is uniformly bounded.

We shall proceed to prove that the operator $\mathcal{T}$ is equicontinuous. For $\tau_1, \tau_2 \in [1, T]$ with $\tau_1 < \tau_2$, we have $|(\mathcal{T}y)(\tau_2) - (\mathcal{T}y)(\tau_1)|$

$$\begin{aligned} &\leq |{}^H\mathcal{J}^{\varrho}|g(\sigma, y(\sigma))|(\tau_2) - {}^H\mathcal{J}^{\varrho}|g(\sigma, y(\sigma))|(\tau_1)| \\ &\quad + \frac{|(\log \tau_2)^{\varrho-1} - (\log \tau_1)^{\varrho-1}|}{\Lambda}\Bigg[\omega\, {}^H\mathcal{J}^{\varrho+\gamma}|g(\sigma, y(\sigma))|(\varphi) \\ &\quad + {}^H\mathcal{J}^{\varrho-\varsigma}|g(\sigma, y(\sigma))|(T)\Bigg] \\ &\leq \frac{\widehat{\mathcal{M}}}{\Gamma(\varrho)}\left|\int_0^{\tau_1}\left[\left(\log\frac{\tau_2}{\sigma}\right)^{\varrho-1} - \left(\log\frac{\tau_1}{\sigma}\right)^{\varrho-1}\right]\frac{d\sigma}{\sigma} + \int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{\sigma}\right)^{\varrho-1}\frac{d\sigma}{\sigma}\right| \\ &\quad + \frac{|(\log \tau_2)^{\varrho-1} - (\log \tau_1)^{\varrho-1}|}{\Lambda}\left[\omega\, {}^H\mathcal{J}^{\varrho+\gamma}(\varphi) + {}^H\mathcal{J}^{\varrho-\varsigma}(T)\right]. \end{aligned}$$

Hence we have that right hand side of the above inequality tends to zero independent of $y \in \mathcal{B}_\theta$ as $\tau_2 - \tau_1 \to 0$. Therefore, the operator $\mathcal{T}(y)$ is equicontinuous and consequently, by Arzela-Ascoli theorem, it is completely continuous. Next, we consider the set $\mathcal{H} = \{y \in \mathcal{P} : y = \upsilon\mathcal{T}(y),\ 0 < \upsilon < 1\}$. Then, we have to show that $\mathcal{H}$ is bounded, let $y \in \mathcal{H}$ and $\tau \in [1, T]$. Then

$$\|y\| \leq \widehat{\mathcal{M}}\left\{\frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \frac{(\log T)^{\varrho-1}}{\Lambda}\left(\frac{\omega(\log \varphi)^{\varrho+\gamma}}{\Gamma(\varrho+\gamma+1)} + \frac{(\log T)^{\varrho-\varsigma}}{\Gamma(\varrho-\varsigma+1)}\right)\right\} = \widehat{\Theta}.$$

Thus, $\mathcal{H}$ is bounded. Hence it follows by Schaefer's fixed point result that the equation (1) has at least one solution on $[1, T]$. $\square$

Next, we establish the uniqueness of solution using Banach fixed point theorem for the problem (1).

Theorem 3.5. *Let $g : [1, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumptions $(\mathcal{E}_4)$. In addition, it is assumed that $\mathcal{S}\Delta < 1$, where Δ is described by (7). Then there exists at most one solution for problem (1) on $[1, T]$.*

Proof. Let us interpret $\sup_{\tau\in[1,T]} |g(\tau, 0)| = \mathcal{Q} < \infty$. Nominating $\theta \geq \frac{\mathcal{Q}\Delta}{1 - \mathcal{S}\Delta}$, we demonstrate that $\mathcal{T}\mathcal{B}_\theta \subset \mathcal{B}_\theta$, where $\mathcal{B}_\theta = \{y \in \mathcal{P} : \|y\| \le \theta\}$. For $y \in \mathcal{B}_\theta$, we have

$$\begin{aligned}
&|(\mathcal{T}y)(\tau)| \\
&\le \sup_{\tau\in[1,T]} \Bigg\{ {}^H\mathcal{J}^{\varrho}|g(\sigma, y(\sigma))|(\tau) + \frac{(\log\tau)^{\varrho-1}}{\Lambda}\Bigg[\omega\, {}^H\mathcal{J}^{\varrho+\gamma}|g(\sigma, y(\sigma))|(\varphi) \\
&\quad + {}^H\mathcal{J}^{\varrho-\varsigma}|g(\sigma, y(\sigma))|(T)\Bigg]\Bigg\} \\
&\le (\mathcal{S}\theta + \mathcal{Q}) \sup_{\tau\in[1,T]} \Bigg\{ {}^H\mathcal{J}^{\varrho}(T) + \frac{(\log T)^{\varrho-1}}{\Lambda}\Bigg[\omega\, {}^H\mathcal{J}^{\varrho+\gamma}(\varphi) + {}^H\mathcal{J}^{\varrho-\varsigma}(T)\Bigg]\Bigg\} \\
&\le (\mathcal{S}\theta + \mathcal{Q})\Delta. \qquad (12)
\end{aligned}$$

Thus, it follows from (12) that $\|(\mathcal{T}y)\| \le \theta$.

Now, for $y, \hat{y} \in \mathcal{P}$, we obtain

$$\begin{aligned}
|\mathcal{T}y(\tau) - \mathcal{T}\hat{y}(\tau)| &\le \sup_{\tau\in[1,T]} \Bigg\{ {}^H\mathcal{J}^{\varrho}|g(\sigma, y(\sigma)) - g(\sigma, \hat{y}(\sigma))|(\tau) \\
&\quad + \frac{(\log\tau)^{\varrho-1}}{\Lambda}\Bigg[\omega\, {}^H\mathcal{J}^{\varrho+\gamma}|g(\sigma, y(\sigma)) - g(\sigma, \hat{y}(\sigma))|(\varphi) \\
&\quad + {}^H\mathcal{J}^{\varrho-\varsigma}|g(\sigma, y(\sigma)) - g(\sigma, \hat{y}(\sigma))|(T)\Bigg]\Bigg\} \\
&\le \Bigg[\mathcal{G}\|y - \hat{y}\|\, {}^H\mathcal{J}^{\varrho}(T) + \frac{\mathcal{G}\|y - \hat{y}\|(\log T)^{\varrho-1}}{\Lambda}\Bigg(\omega\, {}^H\mathcal{J}^{\varrho+\gamma}(\varphi) \\
&\quad + {}^H\mathcal{J}^{\varrho-\varsigma}(T)\Bigg)\Bigg] \\
&= \mathcal{G}\Delta\|y - \hat{y}\|.
\end{aligned}$$

Thus,

$$\|\mathcal{T}y - \mathcal{T}\hat{y}\| \le \mathcal{S}\Delta\|y - \hat{y}\|.$$

Since $\mathcal{S}\Delta < 1$ by the given assumption, therefore $\mathcal{T}$ is a contraction. Hence it follows by Banach fixed point theorem that the equation (1) has at most one solution on $[1, T]$. □

Finally, we establish the uniqueness of solution using nonlinear contractions for the problem (1).

Theorem 3.6. *Let $g : [1, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

($\mathcal{E}_6$) $|g(\tau, u) - g(\tau, v)| \le f(\tau)\frac{|u-v|}{\vartheta+|u-v|}$, $\forall\ \tau \in [1, T], u, v \ge 0$, *where $f : [1, T] \to \mathbb{R}^+$ is continuous and ϑ the constant described by*

$$\vartheta = {}^H\mathcal{J}^{\varrho} f(T) + \frac{(\log T)^{\varrho-1}}{\Lambda}\Bigg[\omega\, {}^H\mathcal{J}^{\varrho+\gamma} f(\varphi) + {}^H\mathcal{J}^{\varrho-\varsigma} f(T)\Bigg].$$

Then there exists at most one solution for problem (1) on $[1, T]$.

Proof. Let us interpret the operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ as in (6) and the continuous nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Phi(\varpi) = \frac{\vartheta\varpi}{\vartheta + \varpi}, \qquad \varpi > 0.$$

Now, for $y, \hat{y} \in \mathcal{P}$ and for each $\tau \in [1, T]$, we obtain

$$\begin{aligned}
|\mathcal{T}y(\tau) - \mathcal{T}\hat{y}(\tau)| &\leq \sup_{\tau\in[1,T]} \Bigg\{ {}^{H}\mathcal{J}^{\varrho}|g(\sigma, y(\sigma)) - g(\sigma, \hat{y}(\sigma))|(\tau) \\
&\quad + \frac{(\log\tau)^{\varrho-1}}{\Lambda}\Bigg[\omega\,{}^{H}\mathcal{J}^{\varrho+\gamma}|g(\sigma, y(\sigma)) - g(\sigma, \hat{y}(\sigma))|(\varphi) \\
&\quad + {}^{H}\mathcal{J}^{\varrho-\varsigma}|g(\sigma, y(\sigma)) - g(\sigma, \hat{y}(\sigma))|(T)\Bigg]\Bigg\} \\
&\leq \Bigg[{}^{H}\mathcal{J}^{\varrho}\Big(f(\sigma)\frac{|y-\hat{y}|}{\Phi + |y-\hat{y}|}\Big)(T) \\
&\quad + \frac{(\log T)^{\varrho-1}}{\Lambda}\Bigg(\omega\,{}^{H}\mathcal{J}^{\varrho+\gamma}\Big(f(\sigma)\frac{|y-\hat{y}|}{\Phi + |y-\hat{y}|}\Big)(\varphi) \\
&\quad + {}^{H}\mathcal{J}^{\varrho-\varsigma}\Big(f(\sigma)\frac{|y-\hat{y}|}{\Phi + |y-\hat{y}|}\Big)(T)\Bigg)\Bigg] \\
&\leq \frac{\Phi\|y-\hat{y}\|}{\vartheta}\Bigg[{}^{H}\mathcal{J}^{\varrho}f(T) + \frac{1}{\Lambda}\Bigg(\omega\,{}^{H}\mathcal{J}^{\varrho}f(\varphi) + {}^{H}\mathcal{J}^{\varrho-\varsigma}f(T)\Bigg)\Bigg] \\
&= \Phi(\|y-\hat{y}\|).
\end{aligned}$$

This implies that, $\|\mathcal{T}y - \mathcal{T}\hat{y}\| \leq \Phi(\|y-\hat{y}\|)$. Therefore $\mathcal{T}$ is a nonlinear contraction. Hence it follows by nonlinear contractions that the equation (1) has at most one solution on $[1, T]$. □

4. Examples

Example 4.1. *Consider the following fractional-order boundary value problem*

$$ {}^{H}\mathcal{D}^{\frac{9}{4}}y(\tau) = \frac{|y(\tau)|}{1+2|y(\tau)|}\cdot\frac{e^{(\log\tau)^{\varrho-1}}}{(1+\tau)^2} + \frac{1}{1+(\log\tau)^{\varrho-1}}, \quad \tau \in [1, T], \tag{13}$$

subject to the integral boundary conditions

$$ y(1) = 0, \quad y^{'}(1) = 0, \quad {}^{H}\mathcal{D}^{\frac{4}{3}}y(T) = \omega\,{}^{H}\mathcal{J}^{\frac{3}{2}}y(\varphi). \tag{14}$$

Here, $\varrho = \frac{9}{4}$, $\varsigma = \frac{4}{3}$, $\gamma = \frac{3}{2}$, $\omega = \frac{1}{40}$, $\varphi = \frac{5}{4}$, $T = 2$.
In addition, we find that

$$\begin{aligned}
|g(\tau, y(\tau))| &= \frac{1}{1+(\log\tau)^{\varrho-1}} + \frac{|y|}{1+2|y|}\frac{1}{(1+\tau)^2} \qquad \textit{as} \\
|g(\tau, p_1(\tau)) - g(\tau, p_2(\tau))| &\leq \frac{1}{4}\|p_1 - p_2\|.
\end{aligned}$$

With the above specifics, we find that $\Lambda \cong 0.5930036058127669$, $\Delta \cong 1.1969781392352639$. *Thus,the presumptions of Theorem 3.3 are satisfied. Hence, by Theorem 3.3, the boundary value problem (13)-(14) has at least one solution on* $[1, T]$.

Example 4.2. *Consider the following fractional-order boundary value problem*

$$ {}^{H}\mathcal{D}^{\frac{8}{3}}y(\tau) = \frac{\tau}{1+\tau} + \frac{|y(\tau)|}{1+|y(\tau)|}\cdot\frac{1}{(2+\tau)^2}+, \quad \tau \in [1, T], \tag{15}$$

subject to the integral boundary conditions

$$y(1) = 0, \quad y'(1) = 0, \quad {}^H\mathcal{D}^{\frac{5}{3}}y(T) = \omega {}^H\mathcal{J}^{\frac{6}{5}}y(\varphi). \tag{16}$$

Here, $\varrho = \frac{8}{3}$, $\varsigma = \frac{5}{3}$, $\gamma = \frac{6}{5}$, $\omega = \frac{1}{50}$, $\varphi = \frac{6}{5}$, $T = 2$.
In addition, we find that

$$\begin{aligned} |g(\tau, y(\tau))| &= \frac{\tau}{1+\tau} + \frac{|y|}{1+|y|}\frac{1}{(2+\tau)^2} \qquad as \\ |g(\tau, p_1(\tau)) - g(\tau, p_2(\tau))| &\leq \frac{1}{9}\|p_1 - p_2\|. \end{aligned}$$

With the above specifics, we find that $\Lambda = 1.3274143884549592$ *and* $\Delta = 0.4217023815634139$. *Thus,* $\mathcal{S}\Delta \cong 0.046855820173712655 < 1$, *the presumptions of Theorem 3.5 are satisfied. Hence, by Theorem 3.5, the boundary value problem (15)-(16) has at most one solution on* $[1, T]$.

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