Pseudo-Valuations on UP-Algebras

Daniel A. Romano

International Mathematical Virtual Institute, Banja Luka, Bosnia and Herzegovina

Abstract

Looking at pseudo-valuations on some classes of abstract algebras, such as BCK, BCI, BCC and KU, in this article we introduce the concept of pseudo-valuations on UP-algebras and analyze the relationship of these mappings with UP-substructures.

1. Introduction

The idea that universal algebra should be analyzed by means of pseudo-valuation was first developed by D. Busneag in 1996 [1]. This author has expanded the perception of pseudo-valuation on Hilbert's algebras [2]. Logical algebras and pseudo-valuations on them have become an object of interest for researchers in recent years. For example, Doh and Kang [3, 4] introduced the concept of pseudo-valuation on BCK/BCI-algebras. Ghorbani in 2010 [5] determined a congruence on BCI-algebras based on pseudo-valuation and describe the obtained factorial structure generated by this congruence. Song, Roh and Jun described pseudo-valuation on BCK/BCI-algebras [12] and Song, Bordbar and Jun have described the quotient structure on such algebras generated by pseudo-valuation [13]. Jun, Lee and Song analyzed in article [8] several types of quasi-valuation maps on BCK-algebras and their interactions. Also, Mehrshad and Kouhestani were interested in pseudo-valuations on BCK-algebra [10]. Jun, Ahn and Roh. in [7] described pseudo-valuation on the BCC-algebras. Koam, Haider and Ansari described in 2019 pseudo-valuations on KU algebras [9].

The concept of UP-algebras is introduced and analyzed by Iampan in 2017 [6] as a generalization of the concept of KU-algebras. In this note, we offer one way of determining of pseudo-evaluation on PU-algebras. Apart from showing the features of this pseudo-valuation on UP-algebras, we have demonstrated how to construct a pseudo-metric space by such mapping.

2. Preliminaries

Here we give the definition of UP-algebra and some of its substructures necessary for further work.

Definition 2.1 ([6]). An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra if it satisfies the following axioms:

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, and
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \land y \cdot x = 0) \implies x = y)$.

In $A$ we can define a binary relation $' \leq'$ by

$(\forall x, y \in A)(x \leq y \iff x \cdot y = 0)$.

Definition 2.2 ([6]). A non-empty subset $J$ of a UP-algebra $A$ is called a UP-ideal of $A$ if it satisfies the following conditions:

(1) $0 \in J$, and
(2) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \land y \in J) \implies x \cdot z \in J)$.

Definition 2.3 ([11]). Let $A$ be a UP-algebra. A subset $G$ of $A$ is called a proper UP-filter of $A$ if it satisfies the following properties:

(3) $\neg(0 \in G)$, and
(4) $(\forall x, y, z \in A)((\neg(x \cdot (y \cdot z) \in G) \land x \cdot z \in G) \implies y \in G)$.
3. The concept of pseudo-valuations on UP-algebras

In this section, we introduce the concept of pseudo-valuations on UP-algebras, describe the basics properties of such pseudo-valuation and construct a pseudo-metric space based on this mapping.

**Definition 3.1.** A real-valued function $v$ on a UP-algebra $A$ is called a pseudo-valuation on $A$ if it satisfies the following two conditions:

1. $v(0) = 0$, and
2. $(\forall x, y, z \in A)\left(v(x \cdot z) \leq v(x \cdot (y \cdot z)) + v(y)\right)$.

A pseudo-valuation $v$ on a UP-algebra $A$ satisfying the following condition:

3. $(\forall x \in A)\left(v(x) = 0 \iff x = 0\right)$

is called a valuation on $X$.

**Theorem 3.2.** Let $v$ be a pseudo-valuation on a UP-algebra $A$. Then the following are valid:

4. $(\forall x, y \in A)\left(v(x) \leq v(x \cdot y) + v(y)\right)$.
5. $(\forall x, y \in A)\left(v(x \cdot y) \leq v(y)\right)$.

**Proof.** If we put $x = 0$, $y = x$ and $z = y$ in formula (2), we get

\[ v(y) \leq v(x \cdot y) + v(x). \]

Thus, formula (4) is valid.

If we put $z = y$ in formula (2), we have $v(x \cdot y) \leq v(x \cdot (y \cdot y)) + v(y)$ from which follows $v(x \cdot y) \leq v(y)$ due to the assertion (1) of Proposition 1.7 in [6], (UP-3) and (1). So, (5) is proven.

**Corollary 3.3.** Let $v$ be a pseudo-valuation on a UP-algebra $A$. Then

6. $(\forall x, y \in A)\left(x \leq y \implies v(y) \leq v(x)\right)$.

**Proof.** Let $x$ and $y$ be arbitrary elements of a UP-algebra $A$ such that $x \leq y$. Then $x \cdot y = 0$ and $v(x \cdot y) = 0$ by (1). From here follows $v(y) = v(x \cdot y) + v(x)$ according to (4). Thus $v(y) \leq v(x)$. Thus, any pseudo-valuation on a UP-algebra is an inversely monotone mapping.

**Corollary 3.4.** Let $v$ be a pseudo-valuation on a UP-algebra $A$. Then

7. $(\forall x \in A)(0 \leq v(x))$.

**Proof.** Since $x \cdot 0 = 0$ according to (UP-3), i.e. as always $x \leq 0$ in UP-algebra $A$, we have $0 = v(0) \leq v(x)$ according to Corollary 3.3.

**Corollary 3.5.** Let $v$ be a pseudo-valuation on a UP-algebra $A$. Then

8. $(\forall x, y \in A)\left(v(x \cdot y) \leq v(x) + v(y)\right)$.

**Proof.** Let $x$ and $y$ be arbitrary elements of $A$. Thus $v(x \cdot y) \leq y(y)$ by (5). Thus $v(x \cdot y) \leq v(x) + v(y)$ by Corollary 3.4.

**Theorem 3.6.** Let $v$ be a pseudo-valuation on a UP-algebra $A$. Then the set $J_v = \{x \in A : v(x) = 0\}$ is an UP-ideal of $A$ and the set $G = \{x \in A : 0 < v(x)\}$ is a proper UP-filter of $A$.

**Proof.** Since $v(0) = 0$, follows $0 \in J_v$.

Let $x, y$ and $z$ be arbitrary elements of $A$ such that $x \cdot (y \cdot z) \in J_v$ and $y \in J_v$. Then $v(x \cdot (y \cdot z)) = 0$ and $v(y) = 0$. By (2) we have

\[ v(x \cdot z) \leq v(x \cdot (y \cdot z)) + v(y) = 0 + 0 = 0. \]

Thus $v(x \cdot z) = 0$ according to Corollary 3.4. Hence $x \cdot z \in J_v$. So, the set $J_v$ is a UP-ideal of UP-algebra $A$.

The set $G$ is a proper UP-filter of $A$ by Theorem 3.7 in [11].

**Corollary 3.7.** Let $v$ be a pseudo-valuation on a UP-algebra $A$. Then $v$ is a valuation on $A$ if and only if $J_v = \{0\}$.

**Proof.** The claim follows from the definition of the concept of valuations on a UP-algebra $A$.

**Remark 3.8.** The previous corollary suggested that a valuation on an UP-algebra $A$ can be defined if $\{0\}$ is a UP-ideal at $A$.

**Example 3.9.** For any ideal $J$ of a UP-algebra $A$, define a map $v_J : A \rightarrow \mathbb{R}$ by $(\forall x \in J)(v_J(x) = 0)$ and $(\forall x \in A \setminus J)(v_J(x) \in \mathbb{R}^+)$. Then, $v_J$ is a pseudo-valuation of $A$.

**Example 3.10.** Let $A = \{0, 1, 2, 3, 4\}$ be given and an operations on it as in Example 2.2 in [6]. Then $(A, \cdot, 0)$ is a UP-algebra. It is easy to directly verify that $v : A \rightarrow \mathbb{R}$, given with $v(0) = v(1) = v(2) = 0$, $v(3) = v(4) = 3$, is a pseudo-valuation on $A$.

**Theorem 3.11.** Let $f : (A, \cdot, 0_A) \rightarrow (B, \ast, 0_B)$ be a homomorphism of UP-algebras. If $v$ is a pseudo-valuation on $B$, then the composition $\circ f$ is a pseudo-valuation on $A$.

**Proof.** First, we have $(v \circ f)(0_A) = v(f(0_B)) = v(0_B) = 0$.

For any $x, y, z \in A$, we get $(v \circ f)(x \cdot z) = v(f(x \cdot z)) = v(f(x) \ast f(z)) \leq v(f(x) + v(f(x)) + v(f(z))) = (v \circ f)(x \cdot (y \cdot z)) + (v \circ f)(y)$.

Hence, $v \circ f$ is a pseudo-valuation on $A$.

**Lemma 3.12.** Suppose that $A$ is a UP-algebra. Then every pseudo-valuation $v$ on $A$ satisfies the following inequality:

9. $(\forall x, y, z \in A)(v(x \cdot z) \leq v(x \cdot y) + v(y \cdot z))$. 
Proof. From (UP-1) follows $y \cdot z \leq (x \cdot y) \cdot (x \cdot z)$. Thus $v(y \cdot z) \geq v((x \cdot y) \cdot (x \cdot z))$ by (6) and $v(y \cdot z) \geq v(x \cdot z) - v(x \cdot y)$ by (4). Therefore, $v(x \cdot z) \leq v(x \cdot y) + v(y \cdot z)$. □

Now, we define pseudo-metric on UP-algebras and show related results.

**Theorem 3.13.** Let $A$ be a UP-algebra and $v$ be a pseudo-valuation on $A$. Then the mapping $d_v : A \times A \ni (x, y) \mapsto v(x \cdot y) + v(y \cdot x) \in \mathbb{R}$ is a pseudo-metric on $A$.

Proof. Clearly, $0 \leq d_v(x, y); d_v(x, x) = 0$ and $d_v(x, y) = d_v(y, x)$ for any $x, y \in A$. For any $x, y, z \in A$ from Lemma 3.12, we get that

\[
d_v(x, y) + d_v(y, z) = (v(x \cdot y) + v(y \cdot x)) + (v(y \cdot z) + v(z \cdot y)) \geq (v(x \cdot y) + v(y \cdot z)) + (v(z \cdot y) + v(y \cdot x)) = v(x \cdot z) + v(z \cdot x) = d_v(x \cdot z).
\]

Hence $(A, d_v)$ is a pseudo-metric space. □

4. Conclusion

The aim of this paper was to study the concept of pseudo-valuation and their induced pseudo-metrics on UP-algebras. This work can be the basis for further and deeper research of the properties of UP-algebras.

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References