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Delay Differential Equations in Sequence Spaces

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Abstract
The standard delay equations are newly studied in the context of classical separable Banach Sequence Spaces. As a classical solution is shown to exist, the associated semigroup and its infinitesimal generator are found, and some important properties of those operators are proven, including some spectral properties. As an application, it is shown how can these results be used to characterize the constrained null-controllability.

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1. Statement of the problem

Consider the l_p spaces for $1 < p < \infty$ consisting of all absolutely p-power summable scalar sequences, with the p-norm. Consider also the subspace of all null scalar sequences c_0 with the supremum norm.

Let X be either $l_p, 1 \leq p < \infty$ or c_0. Let also be $B(X)$ the space of all bounded linear operators on X. Consider the following equation

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{n} A_i x(t-h_i), \quad t \geq 0 \\ x(0) &= r, \\ x(\theta) &= f(\theta), \ -h_n \leq \theta < 0, \end{aligned} \tag{1.1}$$

where $0 < h_1 < \ldots < h_n$ are the delaying points, $x(t) \in X$ for $t > 0$, $A_i \in B(X)$, $i = 0, \ldots, n$ and $f : [-h_n, 0] \to X$ must also satisfy $f(0) = x(0) = r$ (a fixed vector in X), and $f(\theta) \neq 0$ for every θ such that $-h_n \leq \theta < 0$. Here, the convergence is in the norm of X, ie, $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$ if and only if $\|x_n - x\| \to 0$ as $n \to \infty$, where $\|.\|$ stands for the p-norm or the supremum norm, respectively.

The fundamental concepts of derivative and integral for scalar functions of a single variable can be extended to a function $F : [0, \infty) \to X$. We simply express F as a function of its components and do the calculus operations on those components, i.e., if $F(t) = \{f_i(t)\}_{i=1}^{\infty}$, we have $F'(t) = \{f_i'(t)\}_{i=1}^{\infty}$ and $\int_a^b F(t)\,dt = \left\{\int_a^b f_i(t)\,dt\right\}_{i=1}^{\infty}$.

In view of these definitions, it is easily checked that the basic theorems about continuity, differentiability and integrability are also valid in this case. Using standard arguments, it can also be proven that

$$\left|\left|\int_0^t x(s)\,ds\right|\right|_X \leq \int_0^t ||x(s)||_X\,ds.$$

We also have, as usual, $e^{At} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$ for $A \in B(X)$.

Note that the functions $x \in X$ we are working with should satisfy

i) $x(t) \in X$ for every $t \geq 0$.

ii) $x'(t) \in X$ for every $t \geq 0$.

iii) $g(t) = \int_a^t x(s)\,ds \in X$ for each fixed $t \geq 0$ and every $t \geq 0$.

The mapping $x(t) = \left\{ \frac{e^{\lambda t}}{i^q} \right\}_{i=1}^{\infty}$, where $\lambda \in \mathbb{C}$ and $q \geq \frac{1}{p}$ is a example of this. More generally, the same is true for $y(t) = \{g(t)\,a_i\}_{i=1}^{\infty}$, where $\{a_i\}_{i=1}^{\infty} \in X$ and g is differentiable function on $\mathbb{R}$.

In the next pages we will show that (1.1) can be rewritten as an abstract differential equation of the form

$$\begin{aligned} \dot{z}(t) &= A\,z(t) \\ z(0) &= z_0, \end{aligned} \tag{1.2}$$

where A is the infinitesimal generator of a c_0-semigroup $\{T_t\}_{t\geq 0}$ on a suitable Banach space, and we will prove some important properties of $\{T_t\}_{t\geq 0}$ and A (including some spectral properties). Finally, as an application, we will characterize the null-controllability by using some techniques from Functional Analysis and Operator Theory. The control u is constrained to lie in a separable weakly compact subset Ω of an arbitrary Banach space U.

2. Main results

In the following we will prove a standard formula for the solution of (1.1). Then, we will introduce the c_0-semigroup $\{T_t\}_{t\geq 0}$ associated to (1.1), and its infinitesimal generator A.

Theorem 2.1. *Consider the retarded differential equation (1.1). For every $r \in X$ and every $f : [-h_p, 0] \to X$ there exists a unique function x from $(0,\infty)$ to X that is absolutely continuous and satisfies the differential equation (1.1) almost everywhere. This function is called the solution of (1.1) and it satisfies*

$$x(t) = e^{A_0 t} r + \sum_{i=1}^{n} \int_0^t e^{A_0(t-s)} A_i x(s-h_i)\,ds, \text{ for } t \geq 0. \tag{2.1}$$

Proof. Notice first that, for $t \in (0,h_i)$ the term $\sum_{i=1}^{n} A_i x(t-h_i)$ equals the known function $v(t) := \sum_{i=1}^{n} A_i f(t-h_i)$. So we may reformulate the equation (1.1) on $[0,h_i]$ as

$$\begin{aligned} \dot{x}(t) &= A_0\,x(t) + v(t) \\ x(0) &= r. \end{aligned} \tag{2.2}$$

Now, we can proceed coordinatewise an apply finite dimensional theory on each coordinate. We thus find that the solution of (2.2) is given by

$$x(t) = e^{A_0 t} f + \int_0^t e^{A_0(t-s)} v(s)\,ds$$

and this equals (2.1).

Now, we will consider the case $t \geq h_1$. At a given time t, the past is known and so the delayed part $\sum_{i=1}^{n} A_i x(t-h_i)$ is also a known function. Using the same argument as before we conclude that the solution of (1.1) is unique and it satisfies (2.1). $\square$

Lemma 2.2. *If $x(t)$ is the solution of (1.1), then the following inequalities hold*

a) $||x(t)|| \leq C_t[||r|| + ||f(\cdot)||]$,

b) $\int_0^1 ||x(t)||^q dt \leq D_t[||r||^q + ||f(\cdot)||^q]$,

where $1 \leq q < \infty$ and C_t and D_t are constants, depending only on t.

Proof. We know that for some positive constants M_0 and $\omega_0, e^{A_0 t}$ satisfies

$$||e^{A_0 t}|| \leq M_0\, e^{\omega_0 t},\ t \geq 0.$$

Define the positive constant M by $M := \max\left(||A_1||, \ldots, ||A_n|, M_0|\right)$. Then, from (2.1) it follows that

$$\begin{aligned}
||x(t)|| &\leq ||e^{A_0 t} r|| + ||\sum_{i=1}^{n} \int_0^t e^{A_0(t-s)} A_i x(s-h_i)\, ds|| \\
&\leq Me^{\omega_0 t}||r|| + \sum_{i=1}^{n} \int_0^t M\, e^{\omega_0(t-s)} M ||x(s-h_i)||\, ds \\
&= Me^{\omega_0 t}||r|| + \sum_{i=1}^{n} M^2 \int_{-h_i}^{t-h_i} e^{\omega_0(t-\bar{t}-h_i)} ||x(\bar{t})||\, d\bar{t} \\
&= Me^{\omega_0 t}||r|| + M^2 e^{\omega_0 t} \sum_{i=1}^{n} \int_{-h_i}^{t-h_i} e^{-\omega_0(\bar{t}+h_i)} ||x(\bar{t})||\, d\bar{t}.
\end{aligned} \tag{2.3}$$

We now establish the following inequalities for the last term of (2.3)

$$\begin{aligned}
\sum_{i=1}^{n} \int_{-h_i}^{t-h_i} e^{-\omega_0(\bar{t}+h_i)} ||x(\bar{t})||\, d\bar{t} &\leq \sum_{i=1}^{n} \int_{-h_i}^{0} e^{-\omega_0(\bar{t}+h_i)} ||x(\bar{t})||\, d\bar{t} + \sum_{i=1}^{n} \int_0^t e^{-\omega_0(\bar{t}+h_i)} ||x(\bar{t})||\, d\bar{t} \\
&\leq \sum_{i=1}^{n} \int_{-h_i}^{0} ||f(\bar{t})||\, d\bar{t} + \sum_{i=1}^{n} \int_0^t e^{-\omega_0 \bar{t}} ||x(\bar{t})||\, d\bar{t} \text{ since } \omega_0 > 0 \\
&\leq \sum_{i=1}^{n} \int_{-h_r} ||f(\bar{t})||\, d\bar{t} + \sum_{i=1}^{n} \int_0^t e^{-\omega_0 \bar{t}} ||x(\bar{t})||\, d\bar{t}.
\end{aligned}$$

Now, let us fix $\theta \in [-h_n, 0)$ and let $g : [-h_n, 0] \to \mathbb{R}$ be defined by $g(\bar{t}) = \dfrac{||f(\bar{t})||}{||f(\theta)||}$. Since g is a continuous function over the compact set $[-h_n, 0]$, there exists $k > 0$ such that $||f(\bar{t})|| \leq k||f(\theta)||$ for all $\bar{t} \in [-h_n, 0]$.

From this we deduce that the former equation is lesser or equal than

$$n\, h_n\, Q ||f(\cdot)|| + n \int_0^t e^{-\omega_0 \bar{t}} ||x(\bar{t})||\, d\bar{t}. \tag{2.4}$$

Comparing equations (2.3) and (2.4) gives

$$||x(t)|| \leq e^{\omega_0 t} \left[M||r|| + M^2\, n\, h_n\, Q\, ||f(\cdot)|| + M^2\, n \int_0^t e^{-\omega_0 \bar{t}} ||x(\bar{t})||\, d\bar{t} \right]. \tag{2.5}$$

Setting $\alpha = M||r|| + M^2\, n\, h_n\, Q\, ||f(\cdot)||$, $\beta = n M^2$ and $y(t) = e^{-\omega_0 \bar{t}} ||x(\bar{t})||$, we can reformulate (2.5) as

$$y(t) \leq \alpha + \beta \int_0^t y(\bar{t})\, d\bar{t}.$$

From Gronwall's Lemma we conclude that $y(t) \leq \alpha\, e^{\beta\, t}$. So we have

$$\begin{aligned}
||x(t)|| &\leq \alpha\, e^{(\beta + \omega_0)\, t} \\
&= e^{(\rho\, M^2 + \omega_0)\, t} \left[M||r|| + M^2\, Q\, n\, h_n\, ||f(\cdot)|| \right] \\
&\leq e^{(\rho\, M^2 + \omega_0)\, t} \max\left[M, M^2\, n\, h_n\, Q \right] \left[||r|| + ||f(\cdot)|| \right],
\end{aligned}$$

which proves *a*).

It follows, from the above inequality, that

$$||x(t)||^q \leq e^{q\,(\rho\, M^2 + \omega_0)\, t} \max\left[M, M^2\, Q\, n\, h_n \right]^q \left[||r|| + ||f(\cdot)|| \right]^q.$$

Let us now suppose $r \neq 0$. On the whole compact set $[-h_n, 0]$ we define

$$w(\theta) = \frac{(||r|| + ||f(\theta)||)^q}{||r||^q + ||f(\theta)||^q},$$

w is a continuous function over $[-h_n, 0]$. Then there exists $K > 0$ such that $w(\theta) \le K$. This, in particular, is valid for $\theta \in [-h_n, 0)$ and thus, we have

$$(||r|| + ||f(\theta)||)^q \le K\,(||r||^q + ||f(\theta)||^q)\ \theta \in [-h_n, 0).$$

Consequently

$$||x(t)||^q \le K\, e^{q(\rho M^2 + \omega_0)t} \max\left[M, M^2 Q\, n\, h_n\right]^q [||r|| + ||f(\cdot)||]^q.$$

If $r = 0$, then $(||r|| + ||f(\cdot)||)^q = (||r||^q + ||f(\cdot)||^q)$ and we have the same estimation for $||x(t)||^q$. Integrating this inequality gives b). $\square$

Now we shall introduce the semigroup related to (1.1). Consider the space $X \oplus X$ with the usual norm $||(x_1, x_2)||_{X\oplus X} = ||x_1||_X + ||x_2||_X$.

It should be noted that l_p, $1 \le p < \infty$ and c_0 are **prime** Banach spaces, i.e, every infinite-dimensional complemented subspace of X is isomorphic to X. From this we can deduce that $X \oplus X$ is isomorphic to X and thus, the norm $||(\cdot,\cdot)||_{X\oplus X}$ is, in fact, equivalent to $||\cdot||_X$ (see, for example,[1]).

We define the following family of operators on $X \oplus X$ for $t \ge 0$ by

$$T(t)\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} := \begin{pmatrix} x(t) \\ x(t+\cdot) \end{pmatrix}, \tag{2.6}$$

where $x(\cdot)$ is the solution of (1.1) and $x(-s) = f(-s)$ for $h_p > s > 0$.

Theorem 2.3. *The family $\{T(t)\}_{t\ge 0}$ defined by (2.6) satisfies:*

a) $T(t) \in B(X \oplus X)$ for all $t \ge 0$.

b) $\{T(t)\}_{t\ge 0}$ is a c_0-semigroup on $X \oplus X$

Proof. The linearity of $T(t)$ follows easily from the linearity of (1.1) and the uniqueness of its solution. We will now prove that $T(t)$ is a bounded operator.

We can suppose that x is not constantly equal to zero (otherwise the result is trivial) and let us choose t_0 such that $x(t_0) \ne 0$. For each t, let $M_t = \sup\limits_{\bar{t}\in[-h_p,0]} ||x(t+\bar{t})||_X$. Then, we have

$$||x(t+\cdot)||_X := \frac{||x(t+\cdot)||_X}{||x(t_0)||_X} ||x(t_0)||_X \le D_t\,[||r||_X + ||f(\cdot)||_X],$$

where $D_t = \dfrac{M_t}{||x(t_0)||} C_{t_0}$ (C_{t_0} as in the previous lemma), and so

$$\left|\left| T(t)\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \right|\right|_{X\oplus X} = ||x(t)||_X + ||x(t+\cdot)||_X \le (C_t + D_t)\,(||r||_X + ||f(\cdot)||_X).$$

The semigroup property can be proven exactly as in Theorem 2.4.4 of [2].

It only remains to prove the strong continuity. For $t < h_1$ we have

$$\left|\left| T(t)\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} - \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \right|\right| = \left|\left| e^{A_0 t} r + \sum_{i=1}^{p} \int_0^t e^{A_0(t-s)} A_i f(s-h_i)\, ds - r \right|\right|_X + ||x(t+\cdot) - f(\cdot)||_X.$$

The first term converges to zero as $t \to 0$ since

$$e^{A_0 t} r + \sum_{i=1}^{p} \int_0^t e^{A_0(t-s)} A_i f(s-h_i)\, ds$$

is continuous. Let us now prove that $||x(t+\cdot)-f(\cdot)||_X$ tends to zero as $t\to 0$. We first suppose $X=l_p$, $1\leq p<\infty$. We thus have

$$\begin{aligned} ||x(t+\cdot)-f(\cdot)||_X^p &\leq (||x(t+\cdot)||_X+||f(\cdot)||_X)^p \\ &\leq \left(\sup_{t\in[0,h_1]}||x(t+\cdot)||+||f(\cdot)||\right)^p \\ &\leq K(\cdot)\left(\sup_{t\in[0,h_1]}||x(t+\cdot)||^p+||f(\cdot)||^p\right) \\ &= \sum_{i=1}^{\infty}\left[\frac{K(\cdot)\,\sup_{t\in[0,h_1]}||x(t+\cdot)||^p}{2^i}+K(\cdot)|f_i(\cdot)|^p\right], \end{aligned}$$

where $K(\cdot)$ is a constant non depending on t. Consequently, the series

$$\sum_{i=1}^{\infty}|x_i(t+\cdot)-f(\cdot)|^p$$

converges uniformly on $[0,h_1]$, according to the classical Weierstrass M Test, and so we have

$$\lim_{t\to 0}\left(\sum_{i=1}^{\infty}|x_i(t+\cdot)-f(\cdot)|^p\right)=\sum_{i=1}^{\infty}\left(\lim_{t\to 0}|x_i(t+\cdot)-f(\cdot)|^p\right)=0.$$

Let us now suppose $X=c_0$, and let $\varepsilon>0$ be given. We use the results for l_p in the particular case $p=1$. Then, there exists $\delta>0$ such that

$$|x_i(t+\cdot)-f(\cdot)|\leq\sum_{i=1}^{\infty}|x_i(t+\cdot)-f(\cdot)|<\varepsilon$$

for $t\in[0,h_1)\cap(-\delta,\delta)$ and every $i\in\mathbb{N}$. Consequently we have,

$$\sup_{i\in\mathbb{N}}|x_i(t+\cdot)-f(\cdot)|<\varepsilon,\ \text{for } t\in[0,h_1)\cap(-\delta,\delta).$$

□

The following two results deal with the infinitesimal generator A. We will give a detailed description of A and prove some important properties of it. Bearing in mind this, and the former comments about $\{T(t)\}_{t\geq 0}$, it can be shown, in a classical manner, that (1.1) can be rewritten as (1.2).

Lemma 2.4. *Consider the c_0-semigroup $T(t)$ defined by (2.6) and let A denote its infinitesimal generator. For sufficiently large $\alpha\in\mathbb{R}$, the resolvent is given by*

$$(\alpha I-A)^{-1}\begin{pmatrix} r \\ f(\cdot)\end{pmatrix}=\begin{pmatrix} g(0) \\ g(\cdot)\end{pmatrix} \tag{2.7}$$

where

$$g(\theta)=e^{\alpha\theta}g(0)-\int_0^{\theta}e^{\alpha(\theta-s)}f(s)\,ds,\ \text{for } \theta\in[-h,0] \tag{2.8}$$

and

$$g(0)=[\Delta(\alpha)]^{-1}\left[r+\sum_{i=1}^{n}\int_{-h_i}^{0}e^{-\alpha(\theta+h_i)}A_i\,f(\theta)\,d\theta\right], \tag{2.9}$$

where

$$\Delta(\lambda)=\left[\lambda I-A_0-\sum_{i=1}^{n}e^{-\lambda h_i}A_i\right],\ \text{for } \lambda\in\mathbb{C}.$$

Furthemore, g satisfies the following relation:

$$\alpha g(0)=r+A_0\,g(0)+\sum_{i=1}^{n}A_i\,g(-h_i).$$

Proof. The proof is essentially the same as in Lemma 2.4.5 of [2]. One should only note the following: c_0 and l_p, $1 \le p < \infty$ have Schauder bases. For X being either l_p, $1 \le p < \infty$, or c_0, we have that every bounded linear operator $T : X \to X$ can be written as an infinite matrix M in the usual way. If $\{e_i\}_{i=1}^{\infty}$ is the standard unit vector basis of X, and $T(e_1) = \sum_{k=1}^{\infty} \alpha_{k_1} e_k$, $T(e_2) = \sum_{k=1}^{\infty} \alpha_{k_2} e_k$, etc., then

$$M = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdot & \cdot & \cdot \\ \alpha_{21} & \alpha_{22} & \cdot & \cdot & \cdot \\ \alpha_{31} & \alpha_{32} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

For more details, see [3]. It can also be proven, using standard arguments, that the equalities

$$\begin{aligned} \int_0^{\infty} e^{-\alpha t} A_0 x(t) dt &= A_0 \int_0^{\infty} e^{-\alpha t} x(t) dt, \\ \sum_{i=1}^{n} \int_0^{\infty} e^{-\alpha t} A_i x(t-h_i) dt &= \sum_{i=1}^{n} A_i g(-h_i), \\ \sum_{i=1}^{n} \int_{h_i}^{\infty} e^{-\alpha t} A_i x(t-h_i) dt &= \sum_{i=1}^{n} e^{-\alpha h_i} A_i g(0) \end{aligned}$$

from Lemma 2.4.5 of [2] remain valid for the present case. $\square$

Theorem 2.5. *Consider the c_0-semigroup defined by (2.6). Its infinitesimal generator is given by*

$$A \begin{pmatrix} r \\ \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} A_0 r + \sum_{i=1}^{p} A_i f(-h_i) \\ \\ \frac{df}{d\theta}(\cdot) \end{pmatrix} \tag{2.10}$$

with domain

$$D(A) = \left\{ \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in X \oplus X : f \text{ is absolutely continuous}, \frac{df}{d\theta}(\cdot) \in X,\, f(0) = r \right\}.$$

Furthermore, the spectrum of A is discreet and is given by

$$\sigma(A) = \sigma_p(A) = \left\{ \lambda \in \mathbb{C} : \Delta(\lambda)^{-1} \text{ does not exist} \right\},$$

where $\Delta(\lambda)$ is defined in the former Lemma.

If $\lambda \in \sigma_p(A)$, then $\begin{pmatrix} r \\ e^{\lambda \cdot} r \end{pmatrix}$, where $r \neq 0$ satisfies $\Delta(\lambda) r = 0$, is an eigenvector of A, with eigenvalue λ. On the other hand, if v is an eigenvector of A with eigenvalue λ, then $v = \begin{pmatrix} r \\ e^{\lambda \cdot} r \end{pmatrix}$ with $\Delta(\lambda) r = 0$.

Proof. Denote by $\tilde{A}$ the operator

$$\tilde{A} \begin{pmatrix} r \\ \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} A_0 r + \sum_{i=1}^{p} A_i f(-h_i) \\ \\ \frac{df}{d\theta}(\cdot) \end{pmatrix}$$

with domain

$$D(\tilde{A}) = \left\{ \begin{pmatrix} \sigma \\ f(\cdot) \end{pmatrix} \in X \oplus X : f \text{ is absolutely continuos}, \frac{df}{d\theta}(\cdot) \in X,\, f(0) = r \right\}.$$

We have to show that the infinitesimal generator A equals $\tilde{A}$. Let α_0 be a sufficiently larger real number such that the results of the former Lemma hold. If we can show that the inverse of $(\alpha_0 I - \tilde{A})$ equals $(\alpha_0 I - A)^{-1}$, then $A = \tilde{A}$. To this end, we calculate

$$\begin{aligned}
(\alpha_0 I - \tilde{A})\,(\alpha_0 I - A)^{-1} \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} &= (\alpha_0 I - \tilde{A}) \begin{pmatrix} g(0) \\ g(\cdot) \end{pmatrix} \text{ with } g \text{ as in the former Lemma} \\
&= \begin{pmatrix} \alpha_0 g(0) - A_0\, g(0) + \sum_{i=1}^{p} A_i\, g(-h_i) \\ \\ \alpha_0 g(\cdot) - \dfrac{dg}{\partial\theta}(\cdot) \end{pmatrix} \\
&= \begin{pmatrix} r \\ \alpha_0 g(\cdot) - \dfrac{dg}{\partial\theta}(\cdot) \end{pmatrix} \text{ from (2.3) of the former Lemma} \\
&= \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \text{ by differentiating (1.2).}
\end{aligned}$$

So for $\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in X \oplus X$ we have shown that

$$(\alpha_0 I - \tilde{A})\,(\alpha_0 I - A)^{-1} \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} r \\ f(\cdot) \end{pmatrix}. \tag{2.11}$$

It remains to show that

$$(\alpha_0 I - A)^{-1}\,(\alpha_0 I - \tilde{A}) \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} r \\ f(\cdot) \end{pmatrix},$$

for $\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in D(A)$.

For $\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in D(A)$ we define

$$\begin{pmatrix} r_1 \\ f_1(\cdot) \end{pmatrix} := (\alpha_0 I - A)^{-1}\,(\alpha_0 I - \tilde{A}) \begin{pmatrix} r \\ f(\cdot) \end{pmatrix}.$$

Then, from (2.11) we have that

$$(\alpha_0 I - \tilde{A}) \begin{pmatrix} r_1 \\ f_1(\cdot) \end{pmatrix} = (\alpha_0 I - \tilde{A}) \begin{pmatrix} r \\ f(\cdot) \end{pmatrix}.$$

So $\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} r_1 \\ f_1(\cdot) \end{pmatrix}$ if and only if $(\alpha_0 I - \tilde{A})$ is injective. Suppose, on the contrary, that there exists a $\begin{pmatrix} r_0 \\ f_0(\cdot) \end{pmatrix} \in D(A)$ such that

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= (\alpha_0 I - \tilde{A}) \begin{pmatrix} r_0 \\ f_0(\cdot) \end{pmatrix} = (\alpha_0 I - \tilde{A}) \begin{pmatrix} f_0(0) \\ f_0(\cdot) \end{pmatrix} \\
&= \begin{pmatrix} \alpha_0 f_0(0) - A_0\, f_0(0) - \sum_{i=1}^{n} A_i\, f_0(-h_i) \\ \\ \alpha_0 f_0(\cdot) - \dfrac{df_0}{d\theta}(\cdot) \end{pmatrix},
\end{aligned}$$

where we have used the definitions of $\tilde{A}$ and $D(\tilde{A})$ in the last two steps. Then, working coordinatewise as it has been established, we have

$$\begin{aligned}
f_0(\theta) &= f_0(0)\, e^{\alpha_0 \theta} \\
\alpha_0 f_0(0) - A_0\, f_0(0) - \sum_{i=1}^{n} A_i\, f_0(0)\, e^{-\alpha_0 h_i} &= 0.
\end{aligned}$$

Since $\left(\alpha_0 I - A_0 - \sum_{i=1}^n A_i e^{-\alpha_0 h_i}\right)$ is invertible, this implies that $f_0(0) = 0$ and thus $f_0(\cdot) = f_0(0)\, e^{\alpha_0 \cdot} = 0$. This is a contradiction and thus $(\alpha_0 I - \tilde{A})$ is injective. This proves the assertion that A equals $\tilde{A}$.

It remains to calculate the spectrum of A. In the previous Lemma we obtained the explicit expression (2.7) for the resolvent operator for sufficiently large $\alpha \in \mathbb{R}$ in terms of g given by (2.8) and (2.9). Denote by Q_λ the extension of (2.7) to $\mathbb{C}$

$$Q_\lambda \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} := \begin{pmatrix} g(0) \\ g(\cdot) \end{pmatrix}.$$

A simple calculation shows that if $\lambda \in \mathbb{C}$ satisfies $(\lambda I - A_0 - \sum_{i=1}^n A_i e^{-\lambda h_i})$ is invertible then Q_λ is a bounded linear operator from $X \oplus X$ to $X \oplus X$ (working coordinatewise, as ever, we have that each component is continuous). Furthemore, for these λ we have $(\lambda I - A)\, Q_\lambda = I$ and $(\lambda I - A)$ is injective. So, as in the first part of the proof, we conclude that $Q_\lambda = (\lambda I - A)^{-1}$, the resolvent operator of A. We have that

$$\{\lambda \in \mathbb{C} : (\lambda I - A_0 - \sum_{i=1}^n A_i e^{-\lambda h_i}) \text{is invertible}\} \subseteq \rho(A).$$

If, on the other hand, $\left(\lambda I - A_0 - \sum_{i=1}^n A_i e^{-\lambda h_i}\right)$ is not invertible, there exists $\xi \in X, \xi \neq 0$, such that the following element of $X \oplus X$: $z_0 = \begin{pmatrix} \xi \\ e^{\lambda \cdot}\xi \end{pmatrix}$ is in $D(A)$ and

$$(\lambda_0 I - \tilde{A})\, z_0 = \begin{pmatrix} \lambda\, \xi - A_0\, \xi - \sum_{i=1}^n A_i\, e^{-h_i \lambda\, \xi} \\ \\ \lambda\, e^{\lambda_0} \xi - \dfrac{d}{d\theta} e^{\lambda \theta}\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So

$$\sigma_p(A) \supset \{\lambda \in \mathbb{C} : (\lambda I - A_0 - \sum_{i=1}^n A_i\, e^{-\lambda h_i}) \text{is not invertible}\}.$$

Let $v = \begin{pmatrix} r \\ f(\cdot) \end{pmatrix}$ be an eigenvector of A with eigenvalue λ. From (2.10) we obtain that for $\theta \in [-h_p, 0)$

$$\frac{df}{d\theta}(\theta) = \lambda\, f(\theta),$$

which gives $f(\theta) = e^{\lambda\, \theta\, f(0)}$. Since $v \in D(A)$ we have $f(0) = r$. Using the first equation of (2.10) gives

$$A_0\, r + \sum_{i=1}^n A_i\, e^{-\lambda h_i} r = \lambda\, r.$$

This shows that $\Delta(\lambda)\, r = 0$. □

3. An application: Constrained null-controllability

We will now consider a system like the following

$$\begin{aligned} \dot{\omega}(t) &= A\, \omega(t) + \overline{B} u(t), t > 0 \\ \omega(0) &= \omega_0 = (f(0), f(\cdot)), \end{aligned} \tag{3.1}$$

where A is the infinitesimal generator of the semigroup $\{T(t)\}_{t \geq 0}$, X is as before, U is a Banach space, $B : U \to X$ is a bounded linear operator, $u : [0, \infty) \to U$ is a strongly measurable, essentially bounded function and $\overline{B}u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}$. Note that the homogeneous part of (3.1) is exactly (1.2). On the other hand, the mild solution of (3.1) is given by

$$\omega(t) = T(t)\, \omega_0 + \int_0^t T(t-s)\, \overline{B} u(s)\, ds.$$

Let Ω be a non-empty separable weakly compact subset of U, and let $\overline{\Omega}_r$ be defined as follows:

$$\overline{\Omega}_r = \{u \in L_U^\infty[0,r] : u \in \Omega\, a.e\}.$$

$\overline{\Omega}_r$ is called the set of *admissible controls* of (3.1), while the set

$$A_r(\omega_0) = \left\{ T(t)\,\omega_0 + \int_0^r T(r-s)\,\overline{B}\,u(s)\,ds : u \in \overline{\Omega}_r \right\}$$

is the set of *accesible points* of (3.1). The system (3.1) is controllable if $0 \in A_r(\omega_0)$.

The *controllability map* on $[0,r]$ for some $r \geq 0$ is the linear map

$$B^r : L_\infty([0,r];U) \to X \quad \text{defined by} \quad B^r u = \int_0^r T(r-s)\,\overline{B}\,u(s)\,ds.$$

Now, one says that the system is exactly controllable on $[0,r]$ if every point in X can be reached from the origin at r, i.e., if $ran(B^r) = X$.

If $ran(B^r) = X$ then $0 \in A_r(0)$. On the other hand, one can prove, using the Open Mapping Theorem, the following: if $0 \in \text{interior}(A_r(0))$, then $ran(B^r) = X$, see [4].

Next we recall a result that we will use to characterize the null controllability, see [5].

Theorem 3.1. Bárcenas-Diestel *Let X and U be Banach spaces, let $B : U \to X$ be a bounded linear operator and $A : X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on X whose dual semigroup is strongly continuous on $(0,\infty)$. Suppose Ω is a non-empty separable weakly compact convex subset of U containing 0. Then for each $T > 0$, $0 \in A_T(x_0)$ if and only if for each $x^* \in X^*$,*

$$\langle x^*, S(T)x_0\rangle + \int_0^T \max_{v\in\Omega} \langle x^*, S(T)\,Bv\rangle\,dt \geq 0.$$

The Bárcenas-Diestel theorem is an important and recent achievement on exact controllability. Using techniques from Banach space theory and the theory of vector measures, the authors show how to translate the question of accesibility of controls to a problem in semigroups of operators, namely, given a c_0-semigroup $\{S(t)\}_{t\geq 0}$ of operators on a Banach space X, under what conditions is the dual semigroup $\{S^*(t)\}_{t\geq 0}$ strongly continuous on $(0,\infty)$? This is the question we will try to answer in the following.

We recall that a Banach space is a *Grothendieck space* if every weakly* convergent sequence in X^* is also weakly convergent. Equivalently, X is a Grothendieck space if every linear bounded from X to any separable Banach Space is a weakly compact. Among Grothendieck spaces we will list all reflexive Banach spaces and $L^\infty(\Omega,\Sigma,\mu)$, where (Ω,Σ,μ) is a positive measure space (see for example [6]). We also recall that a bounded linear operator $T : X \to Y$, (where X and Y are Banach spaces) *factors through a Banach space* Z, if there are bounded linear operators $u : X \to Z$ and $v : Z \to Y$ such that $T = vu$.

It is proven in [7] that if X is a Banach space and $\{T(t)\}_{t\geq 0}$ a c_0-semigroup defined on X such that for every $a > 0$ there exists a Grothendieck space Y_a such that $T(a)$ factors through Y_a, then $\{T^*(t)\}_{t\geq 0}$ is strongly continuous on $(0,\infty)$. Among semigroups satisfying those assumptions (and hence having adjoints which are strongly continuous on $(0,\infty)$) we mention weakly compact semigroups, i.e, semigroups such that $T(t)$ is weakly compact for each t (see [7] for more detrails). There are many examples of weakly compact semigroups, a category that includes all compact semigroups. Moreover, in $X = l_1$, the terms "weakly compact" and "compact" are equivalent, due to the classical Schur Theorem. This will prove useful to establish our result for the non reflexive cases.

We are, in our case, working with X being either c_0 or l_p, $1 \leq p < \infty$, and we have a c_0-semigroup $\{T(t)\}_{t\geq 0}$ (and its infinitesimal generator A) defined on $X \oplus X$, which, as we have indicated before, is isomorphic to X. If $p \in (1,\infty)$, we have a reflexive Banach space, hence a Grothendieck space. Then, for every $a > 0$ there exists, in an obvious manner, a Grothendieck space Y_a (X itself) such that $T(a)$ factors through Y_a. $\{T^*(t)\}_{t\geq 0}$ is thus strongly continuous on $(0,\infty)$, and we are under the hypotheses of the Bárcenas-Diestel Theorem.

For the cases $X = l_1$ or $X = c_0$, we can additionally suppose that $\{T(t)\}_{t\geq 0}$ factors through a Grothendieck space as before (this happens, for example, if $\{T(t)\}_{t\geq 0}$ is weakly compact, as we have previously indicated). Then we can apply the Bárcenas-Diestel Theorem again.

All this can be sumarized in the following:

Theorem 3.2. *For each $r > 0, 0 \in A_r(\omega_0)$ (i.e., system (3.1) is controllable) if and only if, for each $x^* \in l_q$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$*

$$\langle x^*, T(r)\omega_0\rangle + \int_0^r \max_{v\in\Omega} \left\langle x^*, T(t)\,\overline{B}v(t)\right\rangle\,dt \geq 0.$$

If, additionally, we suppose that the associated semigroup satisfies that, for every $a > 0$ there exists a Grothendieck space Y_a such that $T(a)$ factors through Y_a (in particular, if it is is weakly compact) then we have similar results for $X = l_1$ and $X = c_0$.

4. Final remarks

Problems of this kind are usually set in the context of Hilbert function spaces (see, for example, [2]). But according to the Riesz-Fischer Theorem (see [8]) every separable infinite-dimensional Hilbert space H is isomorphic to l_2. For an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of H and $x \in H$, the map $Tx = \{(x, e_i)\}_{i=1}^{\infty}$ is an isometry. We can thus identify any Hilbert function space with a specific sequence space, namely l_2. Hence, by studying and solving this type of problems in l_2 (as we have, in particular, done) we are, in a certain important sense, studying and solving problems set in any Hilbert function space.

On the other hand, since the function f is allowed, in the present work, to belong to l_p, $1 \leq p < \infty$ or c_0, we are able to study these classical problems (and also, in particular, their null controllability) in a considerably more general context.

In the same line of research to the ones presented here but considering other topological spaces that are not Banach spaces, we refer the reader to [9, 10, 11, 12].

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