Solving FIDEs by Using Semi-Analytical Techniques

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Abstract

This paper mainly focuses on the recent advances in the semi-analytical approximated methods for solving Fredholm Integro-Differential Equations (FIDEs) of the second kind by using Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM) and Direct Homotopy Analysis Method (DHAM). Convergence analysis of the exact solution of the proposed methods is established. Moreover, we proved the uniqueness of the solution. To illustrate the methods, an example is presented.

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1. Introduction

In this paper, we consider FIDE of the form:

\[ \sum_{j=0}^{k} p_j(x)\Delta^{(j)}(x) = f(x) + \lambda \int_{a}^{b} W(x,t)G(\Delta(t))dt \]  

(1.1)

with the initial conditions

\[ \Delta^{(r)}(a) = b_r, \quad r = 0, 1, 2, \ldots, (k-1), \]  

(1.2)

where \( \Delta^{(j)}(x) \) is the \( j \)th derivative of the unknown function \( \Delta(x) \) that will be determined, \( W(x,t) \) is the kernel of the equation, \( f(x) \) and \( p_j(x) \) are analytic functions, \( G \) is nonlinear function of \( \Delta \) and \( a, b, \lambda, \) and \( b_r \) are real finite constants.

The FIDEs arise in many scientific applications. It was also shown that these equations can be derived from boundary value problems.

The application of homotopy techniques in linear and non-linear problems has been devoted by scientists and engineers, because this method is to continuously deform a simple problem which is easy to solve into the under study problem which is difficult to solve. This method was proposed first by He in 1997 and systematical description in 2000 which is, in fact, a coupling of the traditional perturbation method and homotopy in topology [1]. This method was further developed and improved by He and applied to non-linear oscillators with discontinuities [2]. After that many researchers applied the method to various linear and non-linear problems. For example, it was applied to the quadratic Ricatti differential equation by Abbasbandy [3], to the axisymmetric flow over a stretching sheet by Ariel et al. [4], to the Helmholtz equation and fifth-order KdV equation by Rafei and Ganji [5], for the thin film flow of a fourth grade fluid down a vertical cylinder by Siddiqui et al. [6], to the...
As a result, the Lagrange multipliers can be identified as 
\[ \mu \]
where \[ L \]
is inhomogeneous term. According to variational iteration method [7], the terms of a sequence \( \Delta_n \) are constructed such that this sequence converges to the exact solution. The terms \( \Delta_n \) are calculated by a correction functional as follows:

\[
\Delta_{n+1}(t) = \Delta_n(t) + \int_0^t \mu(\tau)(L\Delta_n(\tau) + N\delta(t) - f(\tau))d\tau. \tag{2.1}
\]

The successive approximation \( \Delta_n(t), n \geq 0 \) of the solution \( \Delta(t) \) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function \( \Delta_0 \). The zeroth approximation \( \Delta_0 \) may be selected using any function that just satisfies at least the initial and boundary conditions. With \( \mu \) determined, several approximations \( \Delta_n(t), n \geq 0 \) follow immediately.

The VIM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions.

To obtain the approximation solution of IVP (1.1) – (1.2), according to the VIM, the iteration formula (2.1) can be written as follows:

\[
\Delta_{n+1}(x) = \Delta_n(x) + L^{-1}\left[ \mu(x) \left[ \sum_{j=0}^{k} p_j(x)\Delta_n^{(j)}(x) - f(x) \right] - \lambda \int_a^b W(x,t)G(\Delta_n(t))dt \right],
\]

where \( L^{-1} \) is the multiple integration operator given as follows:

\[
L^{-1}(\cdot) = \int_a^t \int_a^t \cdots \int_a^t (\cdot)dx_1 \cdots dx_k \quad (k-times).
\]

To find the optimal \( \mu(x) \), we proceed as follows:

\[
\delta \Delta_{n+1}(x) = \delta \Delta_n(x) + \delta L^{-1}\left[ \mu(x) \left[ \sum_{j=0}^{k} p_j(x)\Delta_n^{(j)}(x) - f(x) \right] - \lambda \int_a^b W(x,t)G(\Delta_n(t))dt \right]
\]

\[
= \delta \Delta_n(x) + \mu(x)\delta \Delta_n(x) - L^{-1}\left[ \delta \Delta_n(x)\mu'(x) \right]. \tag{2.2}
\]

From Eq. (2.2), the stationary conditions can be obtained as follows:

\[
\mu'(x) = 0, \quad \text{and} \quad 1 + \mu(x)|_{x=1} = 0.
\]

As a result, the Lagrange multipliers can be identified as \( \mu(x) = -1 \) and by substituting in Eq. (2.2), the following iteration formula is obtained:

\[
\Delta_0(x) = L^{-1}\left[ \frac{f(x)}{p_k(x)} \right] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r, \tag{2.3}
\]

\[
\Delta_{n+1}(x) = \Delta_n(x) - L^{-1}\left[ \sum_{j=0}^{k} p_j(x)\Delta_n^{(j)}(x) - f(x) \right] - \lambda \int_a^b W(x,t)G(\Delta_n(t))dt, \quad n \geq 0.
\]

The term \( \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r \) is obtained from the initial conditions, \( p_k(x) \neq 0 \). Relation (2.3) will enable us to determine the components \( \Delta_n(x) \) recursively for \( n \geq 0 \). Consequently, the approximation solution may be obtained by using

\[
\Delta(x) = \lim_{n \to \infty} \Delta_n(x).
\]
3. Homotopy perturbation method (HPM)

The homotopy perturbation method first proposed by He [1, 2]. To illustrate the basic idea of this method, we consider the following nonlinear differential equation

\[ A(\Delta) - f(r) = 0, \quad r \in \Omega, \]  

(3.1)

under the boundary conditions

\[ B \left( \Delta, \frac{\partial \Delta}{\partial n} \right) = 0, \quad r \in \Gamma, \]

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(r) \) is a known analytic function, \( \Gamma \) is the boundary of the domain \( \Omega \).

In general, the operator \( A \) can be divided into two parts \( L \) and \( N \), where \( L \) is linear, while \( N \) is nonlinear. Eq. (3.1) therefore can be rewritten as follows [19]:

\[ L(\Delta) + N(\Delta) - f(r) = 0. \]

By the homotopy technique, we will construct a homotopy \( G(r,p) : \Omega \times [0,1] \rightarrow \mathbb{R} \) which satisfies

\[ H(v, p) = (1 - p)[L(v) - L(\Delta_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1]. \]

(3.2)

or

\[ H(v, p) = L(v) - L(\Delta_0) + pL(\Delta_0)] + p[N(v) - f(r)] = 0, \]

(3.3)

where \( p \in [0,1] \) is an embedding parameter, \( \Delta_0 \) is an initial approximation of Eq.(3.1) which satisfies the boundary conditions. From Eqs.(3.2), (3.3) we have

\[ H(v,0) = L(v) - L(\Delta_0) = 0, \]

\[ H(v,1) = A(v) - f(r) = 0. \]

The changing in the process of \( p \) from zero to unity is just that of \( v(r,p) \) from \( \Delta_0(r) \) to \( \Delta(r) \). In topology this is called deformation, the \( L(v) - L(\Delta_0) \), and \( A(v) - f(r) \) are called homotopic. Now, assume that the solution of Eqs. (3.2) and (3.3) can be expressed as

\[ v = v_0 + pv_1 + p^2v_2 + \cdots. \]

The approximate solution of Eq.(3.1) can be obtained by setting \( p = 1 \).

\[ \Delta = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \cdots. \]

Then equating the terms with identical power of \( P \), we obtain the following series of linear equations:

\[ P^0 : \Delta_0(x) = \sum_{r=0}^{k-1} \frac{1}{r!} (x-a)^r b_r, \]

\[ P^1 : \Delta_1(x) = L^{-1} \left( \frac{f(x)}{p_k(x)} \right) + \lambda L^{-1} \left( \int_a^b \frac{W(x,t)}{p_k(x)} G(\Delta_0(t)) dt \right) - \sum_{j=0}^{k-1} L^{-1} \left( \frac{p_j(x)}{p_k(x)} \Delta_0^{(j)}(x) \right), \]

\[ P^2 : \Delta_2(x) = \lambda L^{-1} \left( \int_a^b \frac{W(x,t)}{p_k(x)} G(\Delta_1(t)) dt \right) - \sum_{j=0}^{k-1} L^{-1} \left( \frac{p_j(x)}{p_k(x)} \Delta_1^{(j)}(x) \right), \]

\[ P^3 : \Delta_3(x) = \lambda L^{-1} \left( \int_a^b \frac{W(x,t)}{p_k(x)} G(\Delta_2(t)) dt \right) - \sum_{j=0}^{k-1} L^{-1} \left( \frac{p_j(x)}{p_k(x)} \Delta_2^{(j)}(x) \right), \]

\[ \vdots \]
4. Direct homotopy analysis method (DHAM)

Consider FIDE (1.1) and substitute the kernel $W(x,t) = g(x)h(t)$ we obtain

$$
\sum_{j=0}^{k} p_j(x)\Delta^{(j)}(x) = f(x) + \lambda g(x) \int_{a}^{b} h(t) G(\Delta(t)) dt.
$$

To obtain the approximate solution, we integrating $(k)$-times in the interval $[a,x]$ with respect to $x$ we obtain,

$$
\Delta(x) = L^{-1} \left( \frac{f(x)}{p_k(x)} \right) + \sum_{r=0}^{k-1} \frac{1}{r!} (x-a)^{r} b_r + \lambda L^{-1} \left( \frac{g(x)}{p_k(x)} \int_{a}^{b} h(t) G(\Delta(t)) dt \right) - \sum_{j=0}^{k-1} L^{-1} \left( \frac{p_j(x)}{p_k(x)} \Delta^{(j)}(x) \right),
$$

Setting

$$
Q = \int_{a}^{b} h(t) G(\Delta(t)) dt,
$$

$$
F = L^{-1} \left( \frac{f(x)}{p_k(x)} \right) + \sum_{r=0}^{k-1} \frac{1}{r!} (x-a)^{r} b_r - \sum_{j=0}^{k-1} L^{-1} \left( \frac{p_j(x)}{p_k(x)} \Delta^{(j)}(x) \right),
$$

Therefore, we can rewrite Eq. (4.1) as

$$
\Delta(x) = F(x) + \lambda L^{-1} \left( \frac{g(x)}{p_k(x)} Q \right),
$$

we define the nonlinear homotopy operator as:

$$
N[\Delta(x)] = \Delta(x) - F(x) - \lambda L^{-1} \left( \frac{g(x)}{p_k(x)} Q \right),
$$

The corresponding $m$th-order deformation equation is as follows

$$
L[\Delta_m(x) - \chi_m \Delta_{m-1}(x)] = BH(x) R_m(\Delta_{m-1}(x))
$$

where

$$
R_m(\Delta_{m-1}(x)) = \Delta_{m-1}(x) - F(x)(1 - \chi_m) - \lambda L^{-1} \left( \frac{g(x)}{p_k(x)} Q \right),
$$

and

$$
\chi_m = \begin{cases} 
1, & m > 1, \\
0, & m \leq 1.
\end{cases}
$$

choosing the auxiliary linear operator $L[\Delta] = \Delta$, we obtain

$$
\Delta_0(x) = \text{Choosing initial guess}
$$

$$
\Delta_1(x) = BH(x) \left[ \Delta_0(x) - L^{-1} \left( \frac{f(x)}{p_k(x)} \right) - \sum_{r=0}^{k-1} \frac{1}{r!} (x-a)^{r} b_r - \lambda L^{-1} \left( \frac{g(x)}{p_k(x)} \int_{a}^{b} h(t) G(\Delta_0(t)) dt \right) \right]
$$

$$
+ \sum_{j=0}^{k-1} L^{-1} \left( \frac{p_j(x)}{p_k(x)} \Delta^{(j)}(x) \right),
$$

$$
\Delta_m(x) = \chi_m \Delta_{m-1}(x) + BH(x) \left[ \Delta_{m-1}(x) - \lambda L^{-1} \left( \frac{g(x)}{p_k(x)} \int_{a}^{b} h(t) G(\Delta_{m-1}(t)) dt \right) \right]
$$

$$
+ \sum_{j=0}^{k-1} L^{-1} \left( \frac{p_j(x)}{p_k(x)} \Delta^{(j)}(m-1)(x) \right), m > 1.
$$

with auxiliary function $H(x)$ and auxiliary parameter $B$.

Then, $\Delta(x) = \sum_{i=0}^{m} \Delta_i$ as the approximate solution.
5. Uniqueness results

In this section, we shall give an uniqueness results of Eq. (1.1), with the initial condition (1.2) and prove it [22, 23].

We can be written equation (1.1) in the form of:

\[ \Delta(x) = L^{-1} \left[ \frac{f(x)}{p_k(x)} + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r \right] + \lambda_1 L^{-1} \left[ \int_a^b \frac{1}{p_k(x)} W(x,t) G(\Delta_n(t)) dt \right] - L^{-1} \left[ \sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} \Delta^{(j)}(x) \right]. \]

we can write

\[ L^{-1} \left[ \int_a^b \frac{1}{p_k(x)} W(x,t) G(\Delta_n(t)) dt \right] = \int_a^b \frac{(x-t)^k}{k! p_k(x)} W(x,t) G(\Delta_n(t)) dt. \]

\[ \sum_{j=0}^{k-1} L^{-1} \left[ \frac{p_j(x)}{p_k(x)} \right] \Delta^{(j)}(x) = \sum_{j=0}^{k-1} \int_a^b \frac{(x-t)^{k-1} p_j(t)}{k! p_k(t)} \Delta^{(j)}(t) dt. \]

We set,

\[ \Psi(x) = L^{-1} \left[ \frac{f(x)}{p_k(x)} \right] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r. \]

Before starting and proving the main results, we introduce the following hypotheses:

(H1) There exist two constants \( \alpha \) and \( \gamma_j > 0, j = 0, 1, \cdots, k \) such that, for any \( \Delta_1, \Delta_2 \in C(J, \mathbb{R}) \)

\[ |G(\Delta_1) - G(\Delta_2)| \leq \alpha |\Delta_1 - \Delta_2| \]

and

\[ |D^j(\Delta_1) - D^j(\Delta_2)| \leq \gamma_j |\Delta_1 - \Delta_2|, \]

we suppose that the nonlinear terms \( G(\Delta(x)) \) and \( D^j(\Delta) = \left( \frac{d^j}{dt^j} \right) \Delta(x) = \sum_{i=0}^{m} \gamma_j, \) (\( D^j \) is a derivative operator), \( j = 0, 1, \cdots, k \), are Lipschitz continuous.

(H2) We suppose that for all \( a \leq t \leq x \leq b \), and \( j = 0, 1, \cdots, k \):

\[ \left| \frac{\lambda(x-t)^k W(x,t)}{k! p_k(x)} \right| \leq \theta_1, \quad \left| \frac{\lambda(x-t)^k W(x,t)}{k!} \right| \leq \theta_2, \]

and

\[ \left| \frac{(x-t)^{k-1} p_j(t)}{(k-1)! p_k(t)} \right| \leq \theta_3, \quad \left| \frac{(x-t)^{k-1} p_j(t)}{(k-1)!} \right| \leq \theta_4, \]

(H3) There exist three functions \( \theta^*_{\alpha}, \theta^*_{\gamma} \), and \( \gamma^* \in C(D, \mathbb{R}^+) \), the set of all positive function continuous on \( D = \{(x,t) \in \mathbb{R} \times \mathbb{R} : 0 \leq t \leq x \leq 1 \} \) such that:

\[ \theta^*_{\alpha} = \max_{\theta_1}, \quad \theta^*_{\gamma} = \max_{\theta_4}, \quad \gamma^* = \max |\gamma_j|. \]

(H4) \( \Psi(x) \) is bounded function for all \( x \) in \( J = [a,b] \).

Theorem 5.1. Assume that (H1)–(H4) hold. If

\[ 0 < \psi = (\alpha \theta_1 + k \gamma^* \theta^*_{\gamma})(b-a) < 1, \]

then there exists a unique solution \( \Delta(x) \in C(J) \) to IVP (1.1) – (1.2).
Proof. Let $\Delta_1$ and $\Delta_2$ be two different solutions of IVP (1.1) – (1.2), then
\[
\begin{align*}
|\Delta_1 - \Delta_2| &= \left| \int_a^b \frac{\lambda(x-t)^k W(x,t)}{p_k(x) k!} (G(\Delta_1) - G(\Delta_2)) dt \right. \\
&\quad - \sum_{j=0}^{k-1} \left. \int_a^b \frac{(x-t)^{k-1-p_j(t)}}{p_k(t) (k-1)!} (D^j(\Delta_1) - D^j(\Delta_2)) dt \right| \\
&\leq \int_a^b \left| \frac{\lambda(x-t)^k W(x,t)}{p_k(x) k!} (G(\Delta_1) - G(\Delta_2)) dt \right| \\
&\quad - \sum_{j=0}^{k-1} \int_a^b \left| \frac{(x-t)^{k-1-p_j(t)}}{p_k(t) (k-1)!} (D^j(\Delta_1) - D^j(\Delta_2)) dt \right| \\
&\leq (\alpha \theta_1 + k^\gamma \theta_1^*(b-a)) |\Delta_1 - \Delta_2|,
\end{align*}
\]
we get $(1-\psi)|\Delta_1 - \Delta_2| \leq 0$. Since $0 < \psi < 1$, so $|\Delta_1 - \Delta_2| = 0$. Therefore, $\Delta_1 = \Delta_2$ and the proof is completed.

6. Example

In this section, we present the semi-analytical techniques based on VIM, HPM and DHAM to solve FIDEs. To show the efficiency of the present methods for our problem in comparison with the exact solutions.

Example 6.1. Consider the following FIDE:
\[
\Delta'(x) = e^x (1+x) - x + \int_0^1 x \Delta(t) dt,
\]
with the initial condition
\[
\Delta(0) = 0,
\]
and the the exact solution is $\Delta(x) = xe^x$.

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7. Discussion and conclusion

We discussed the VIM, HPM and DHAM for solving FIDEs of the second kind. To assess the accuracy of each method, the test example with known exact solution is used. In this work, the above methods have been successfully employed to obtain the approximate solution of a FIDE. The results show that these methods are very efficient, convenient and can be adapted to fit a larger class of problems. The comparison reveals that although the numerical results of these methods are similar approximately, Table 1 shows that the numerical results obtained with DHAM agree with the exact solutions.
References


