Available online: October 2, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 23–36 (2020)
DOI: 10.31801/cfsuasmas.467319
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

APPROXIMATE TEST FOR TESTING A NULL VARIANCE RATIO IN THE UNBALANCED ONE-WAY RANDOM MODEL

SEVGI DEMIRCIOĞLU AND BILGEHAN GÜVEN

Abstract. The approximate test for testing the significance of the random effect is presented in the unbalanced one-way random model in which both random effects and errors are from nonnormal universes. The test is based on the asymptotic distribution of the F-ratio. Under the condition that the number of groups tends to infinity while the average of powers of the group sizes is bounded, the asymptotic distribution of the F statistic is obtained. Robustness of the proposed test is given.

1. Introduction

We derive the approximate test for testing the significance of the random effect in the unbalanced one-way random effects model where both random effects and errors are from nonnormal universes. To derive the approximate test, we first obtain the asymptotic distribution of the F-ratio.

In literature there are two different methods to obtain the asymptotic distribution of the F-ratio. Akritas and Arnold (2000) and Akritas and Papadatos (2004) obtained asymptotic normality of the F-ratio from the difference $MS_\tau - MSE$ and from the fact that MSE converges in probability to constant. Here, MS_τ and MSE are the mean square for the random effects and errors respectively. Westfall (1988) first derived the joint asymptotic distribution of MS_τ and MSE and then used the delta method to obtain asymptotic normality of the F-ratio.

To get the asymptotic distribution of the F-ratio, we use the method of Westfall and establish the following asymptotic condition. The number of groups is large while the average of powers of the group sizes is bounded. This asymptotic condition may be viewed as modification of the asymptotic condition established by Wesfall (1987, 1988). He assumed that the number of groups is large while the group sizes are from a finite set of positive integers.

Received by the editors: October 09, 2018; Accepted: July 16, 2019.

2010 *Mathematics Subject Classification.* 60F05, 62J10.

Key words and phrases. Asymptotic normality; F-test; robustness of the approximate F-test; random effects.

Also it is implicitly shown that the presented approximate test is robust for the size of the test in the balanced model does not depend on the fourth moment of the error term for the balanced case. The size of the test in the non normal case is same as it in the normal case.

This paper differs from the previous studies in three ways. A new asymptotic condition is established by modifying Westfall's asymptotic condition. Robustness of the asymptotic distribution of the F-ratio is analytically shown. Different distributions having positive, null and negative kurtosis are used in simulations.

This paper is organized as follows: Sec. 2 demonstrates the asymptotic condition and its consequences. Sec. 3 gives the asymptotic distribution of the F-ratio under the established asymptotic condition. Sec. 4 proposes the approximate test for testing significance of the random effects. Sec. 5 shows that the approximate test is robust. In Sec. 6 some numerical and simulated results are given to examine the accuracy of the approximate test.

Throughout the paper we shall use the following notations. If d_N is a sequence of N and r is a real number then $d_N = o(N^r)$ if $N^{-r}d_N \to 0$ as $N \to \infty$ and $d_N = O(N^r)$ if $N^{-r}d_N$ has a nonzero finite limit as $N \to \infty$.

2. The Model and Asymptotic

The unbalanced one-way random effects model is:

$$Y_{ij} = \mu + \tau_i + e_{ij} \qquad i = 1, 2, \dots t \qquad j = 1, 2, \dots, n_i \tag{1}$$

where μ is an overall mean, τ_i and e_{ij} are random variables with zero means and variances σ_τ^2 and σ^2 respectively. The model is appropriate for analyzing data involving t random treatments. The number of observation is N where $N = \sum_{i=1}^{t} n_i$.

We shall address the problem of testing $H_0: \quad \rho = 0$ vs. $H_1: \quad \rho > 0$ where the ratio of variances ρ is defined as $\rho = \sigma_\tau^2/\sigma^2$. The statistic for testing H_0 is based on

$$F_N = MS_\tau/MSE \tag{2}$$

where $MS_\tau = (t-1)^{-1}SS_\tau$ and $MSE = (N-t)^{-1}SSE$. SS_τ and SSE are the sum of squares for treatment and for error respectively and they are defined as

$$SS_\tau = \sum_{i=1}^{t} n_i(\overline{Y}_{i.} - \overline{Y}_{..})^2 \qquad \text{and} \qquad SSE = \sum_{i=1}^{t}\sum_{j=1}^{n_i}(Y_{ij} - \overline{Y}_{i.})^2 \tag{3}$$

with $\overline{Y}_{i.} = n_i^{-1}\sum_{j=1}^{n_i} Y_{ij}$ and $\overline{Y}_{..} = N^{-1}\sum_{i=1}^{t}\sum_{j=1}^{n_i} Y_{ij}$. Under the normality of the random effects and the error terms, the test rejects H_0 when $F_N > F_{t-1,N-t,\alpha}$ where $F_{\nu_1,\nu_2,\alpha}$ denotes the $1-\alpha$ quantile of the F distribution with degrees of freedom ν_1 and ν_2.

When the random effects and error terms are from nonnormal universes, the approximate distribution of F_N is used for testing problem presented above. With the moment conditions that $E|\tau_i|^{4+\delta} < \infty$ and $E|e_{ij}|^{4+\delta} < \infty$ for some positive δ we establish the following asymptotic condition.

Asymptotic Condition. Consider a sequence of the model (1). The number of groups t tends to infinity in such a way that the average of $n_1^p, n_2^p, \ldots, n_t^p$ is bounded where $p \geq 1$. So there exists a real number $M > 0$ such that

$$t^{-1} \sum_{i=1}^{t} n_i^p < M$$

for all t. It is ensured by finite group sizes.

We are free to put in order the levels of the random effect among the $(t+1)$ levels. The group sizes can be ordered in the ascending order,i.e., $n_i \leq n_{i+1}$. Then

$$\frac{\sum_{i=1}^{t+1} n_i^p}{t+1} - \frac{\sum_{i=1}^{t} n_i^p}{t} = \frac{t n_{t+1}^p - \sum_{i=1}^{t} n_i^p}{t(t+1)}$$

where $t n_{t+1}^p > \sum_{i=1}^{t} n_i^p$. The sequences $t^{-1} \sum_{i=1}^{t} n_i^p$ of t are bounded and monotone and than they have a finite limit as $t \to \infty$. The positive monotone sequence $t^{-1} \sum_{i=1}^{t} (1/n_i^p)$ are bounded from both left by 0 and right by $t^{-1} \sum_{i=1}^{t} n_i^p$ So it has a finite limit as $t \to \infty$.

We have shown that $(1/t) \sum_{i=1}^{t} n_i$ has a finite limit as $t \to \infty$ where $(1/t) \sum_{i=1}^{t} n_i = N/t$. Then $t/N = O(1)$ implying that t and N are of the same order. So t can be replaced by N.

Thus we are ready to define the following limits appearing in calculation of the asymptotic covariance matrix. They are:

$$a = \lim_{N \to \infty} (t/N), \qquad \gamma_p = \lim_{N \to \infty} (1/N) \sum_{i=1}^{t} n_i^p \quad \text{for} \quad p = -1, 2. \tag{4}$$

where $a \in (0,1)$ since $0 < t < N$.

3. Asymptotic Distribution of F_N

In this section we derive the asymptotic distribution of F_N in Eq. (2) where a variance ratio ρ is considered to be positive. The derivation of the asymptotic distribution of F_N is based on obtaining the joint asymptotic distribution of $\sqrt{N}(MS_\tau, MSE)$ and then applying the delta method.

Lemma 3.1. *Suppose the asymptotic condition established in Sec. 3. holds. Then the covariance matrix of* $\sqrt{N}(MS_\tau, MSE)'$ *is:*

$$\boldsymbol{ACOV} = 2\sigma^4 \begin{bmatrix} (\gamma_2 \rho^2 + 2\rho + a)/a^2 & 0 \\ 0 & 1/(1-a) \end{bmatrix} + k_\tau \sigma^4 \begin{bmatrix} \gamma_2 \rho^2/a^2 & 0 \\ 0 & 0 \end{bmatrix}$$
$$+ k_e \sigma^4 \begin{bmatrix} \gamma_{-1}/a^2 & (a - \gamma_{-1})/a(1-a) \\ (a - \gamma_{-1})/a(1-a) & (1 - 2a + \gamma_{-1})/(1-a)^2 \end{bmatrix} \tag{5}$$

as $N \to \infty$ where k_τ and k_e are the kurtosis of the underlying distributions of τ_i and e_{ij} defined as $k_\tau = E|\tau_i|^4/\sigma^4 - 3$ and $k_e = E|e_{ij}|^4/\sigma^4 - 3$, the limits a and γ_p for $p = -1, 2$ are in Eq. (4).

Proof. We first derive the asymptotic covariance matrix of $N^{-1/2}(SS_\tau, SSE)^{'}$. Let $\boldsymbol{Y}_i = (Y_{i1}, Y_{i2}, \ldots, Y_{in_i})^{'}$, $\boldsymbol{Y} = (\boldsymbol{Y}_1^{'}, \boldsymbol{Y}_2^{'}, \ldots, \boldsymbol{Y}_t^{'})^{'}$. We follow Searle's notation (see Searle 1987, p 212-213). SS_τ and SSE in Eq. (3) can be expressed in a matrix notation as $SS_\tau = \boldsymbol{Y}^{'}\boldsymbol{Q}_1\boldsymbol{Y}$ and $SSE = \boldsymbol{Y}^{'}\boldsymbol{Q}_2\boldsymbol{Y}$ where symmetric idempotent matrices $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are:

$$\boldsymbol{Q}_1 = \{_d(1/n_i)\boldsymbol{J}_{n_i}\}_{i=1}^{t} - (1/N)\boldsymbol{J}_N \quad \text{and} \quad \boldsymbol{Q}_2 = \boldsymbol{I}_N - \{_d(1/n_i)\boldsymbol{J}_{n_i}\}_{i=1}^{t}. \tag{6}$$

Here $\boldsymbol{I}_m$ and $\boldsymbol{J}_m$ are matrices of identity and ones of the order $m \times m$ respectively.

The model (1) is in a matrix notation as $\boldsymbol{Y} = \boldsymbol{1}_N\mu + \boldsymbol{U}\boldsymbol{\tau} + \boldsymbol{e}$ where $\boldsymbol{1}_m$ denotes a vector of ones of the order $m \times 1$, $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots \tau_t)^{'}$ and $\boldsymbol{e}$ is defined similarly to $\boldsymbol{Y}$. The matrix $\boldsymbol{U}$ of the order $N \times t$ is defined as

$$\boldsymbol{U} = \{_d\boldsymbol{1}_{n_i}\}_{i=1}^{i=t}. \tag{7}$$

It follows that SS_τ and SSE are rewritten as

$$SS_\tau = (\boldsymbol{\tau}^{'}, \boldsymbol{e}^{'}) \begin{bmatrix} \boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U} & \boldsymbol{U}^{'}\boldsymbol{Q}_1 \\ \boldsymbol{Q}_1\boldsymbol{U} & \boldsymbol{Q}_1 \end{bmatrix} \begin{pmatrix} \boldsymbol{\tau} \\ \boldsymbol{e} \end{pmatrix}, \quad SSE = (\boldsymbol{\tau}^{'}, \boldsymbol{e}^{'}) \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}_2 \end{bmatrix} \begin{pmatrix} \boldsymbol{\tau} \\ \boldsymbol{e} \end{pmatrix}. \tag{8}$$

From Eqs. (6) and (7), the matrix $\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U}$ of the order $t \times t$ is of the form

$$\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U} = \begin{cases} n_i - (1/N)n_i^2, & \text{if} \quad i = j \\ -(1/N)n_in_j & \text{if} \quad i \neq j \end{cases} \tag{9}$$

and the matrix $\boldsymbol{U}^{'}\boldsymbol{Q}_1$ of the order $t \times N$ is equal to $\{\boldsymbol{B}_{ij}\}_{i=1,j=1}^{i=t,j=t}$ where the matrix $\boldsymbol{B}_{ij}$ of the order $1 \times n_j$ is of the form

$$\boldsymbol{B}_{ij} = \begin{cases} (1 - \frac{1}{N}n_i)\boldsymbol{1}_{n_i}^{'} & \text{if} \quad i = j \\ -\frac{1}{N}n_i\boldsymbol{1}_{n_j}^{'} & \text{if} \quad i \neq j \end{cases} \tag{10}$$

Using Lemma 1 of Westfall (1987) that simplifies calculation of covariance between quadratic forms in a vector of mean zero random variables, we get

$$\begin{aligned} Var(SS_\tau) = \sigma^4[&2\rho^2 tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U})^2 + 4\rho tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U}) + 2tr(\boldsymbol{Q}_1)^2 \\ &+ \rho^2 k_\tau tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U} diag(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U})) + k_e tr(\boldsymbol{Q}_1 diag(\boldsymbol{Q}_1))], \end{aligned} \tag{11}$$

$$Var(SSE) = \sigma^4[2tr(\boldsymbol{Q}_2)^2 + k_e tr(\boldsymbol{Q}_2 diag(\boldsymbol{Q}_2))], \tag{12}$$

and

$$Cov(SS_\tau, SSE) = \sigma^4 k_e tr(\boldsymbol{Q}_1 diag(\boldsymbol{Q}_2)). \tag{13}$$

Using Eqs. (6) and (9), we get the following traces

$$tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U})^2 = \sum_{i=1}^{t} n_i^2 + b_N, \quad tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U}) = N + c_N, \tag{14}$$

$$tr(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U}diag(\boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U})) = \sum_{i=1}^{t} n_i^2 + d_N,\ tr(\boldsymbol{Q}_1)^2 = t-1,\ tr(\boldsymbol{Q}_2)^2 = N-t, \quad (15)$$

$$tr(\boldsymbol{Q}_1 diag(\boldsymbol{Q}_1)) = \sum_{i=1}^{t}(1/n_i) + e_N,\ tr(\boldsymbol{Q}_2 diag(\boldsymbol{Q}_2)) = N-2t+\sum_{i=1}^{t}(1/n_i), \quad (16)$$

$$tr(\boldsymbol{Q}_1 diag(\boldsymbol{Q}_2)) = t - \sum_{i=1}^{t}(1/n_i) + f_N \quad (17)$$

where

$$b_N = -(2/N)\sum_{i=1}^{t} n_i^3 + (1/N^2)\sum_{i=1}^{t} n_i^4 + (1/N^2)\sum_{i=1}^{t}\sum_{j=1}^{t} n_i^2 n_j^2,$$

$$c_N = -(1/N)\sum_{i=1}^{t} n_i^2,\ d_N = -(2/N)\sum_{i=1}^{t} n_i^3,$$

$$e_N = (-2t+1)/N, \quad f_N = -(1/N)\sum_{i=1}^{t} n_i + (t/N).$$

Then the sequences b_N, c_N, d_N, e_N and f_N are all $o(N)$.

From the asymptotic condition given in Sec.2. and Eqs. (14)-(17), we get

$$\lim_{N\to\infty}(1/N)Var(SS_\tau) = \sigma^4[2\rho^2\gamma_2 + 4\rho + 2a + k_\tau\rho^2\gamma_2 + k_e\gamma_{-1}], \quad (18)$$

$$\lim_{N\to\infty}(1/N)Var(SSE) = \sigma^4[2(1-a) + k_e(1-2a+\gamma_{-1})] \quad (19)$$

and

$$\lim_{N\to\infty}(1/N)Cov(SS_\tau, SSE) = \sigma^4(a-\gamma_{-1}) \quad (20)$$

where $Var(SS_\tau)$, $Var(SSE)$ and $Cov(SS_\tau, SSE)$ are given in Eqs. (11), (12) and (13) respectively and the limits a and γ_p for $lp-1, 2$ are defined by Eq. (4). From these, the covariance matrix of $N^{-1/2}(SS_\tau, SSE)^{'}$ is:

$$\boldsymbol{\Delta} = \sigma^4\begin{bmatrix} 2\rho^2\gamma_2+4\rho+2a & 0 \\ 0 & 2(1-a)\end{bmatrix} + k_\tau\sigma^4\begin{bmatrix}\rho^2\gamma_2 & 0 \\ 0 & 0\end{bmatrix}$$
$$+ k_e\sigma^4\begin{bmatrix}\gamma_{-1} & a-\gamma_{-1} \\ a-\gamma_{-1} & 1-2a+\gamma_{-1}\end{bmatrix} \quad (21)$$

as $N \to \infty$. We have the equality $\sqrt{N}(MS_\tau, MSE)^{'} = \boldsymbol{\Lambda}_N N^{-1/2}(SS_\tau, SSE)^{'}$ where $\boldsymbol{\Lambda}_N = diag(N/(t-1), N/(N-t))$. $\boldsymbol{\Lambda}_N$ converges to $\boldsymbol{\Gamma}$ as $N\to\infty$ where $\boldsymbol{\Gamma} = diag(1/a, 1/(1-a))$. Thus, the asymptotic covariance matrix of $\sqrt{N}(MS_\tau, MSE)^{'}$ denoted by $\boldsymbol{ACOV}$ is equal to $\boldsymbol{\Gamma\Delta\ \Gamma}$ and its explicit form is given in Eq. (5). □

Theorem 3.2. *The sequences in random vector*

$$\sqrt{N}(MS_\tau - [1+\rho a^{-1}]\sigma^2, MSE - \sigma^2)^{'}$$

converges in distribution to the bivariate normal distribution with zero-mean vector and the covariance matrix $\boldsymbol{ACOV}$ *given in Eq. (5).*

Proof. Define Q_N as $Q_N = SS_\tau + SSE$. Then Q_N is written as $\boldsymbol{Y}^{'}\boldsymbol{P}\boldsymbol{Y}$ where from Eqs. (6) and (8), the matrix $\boldsymbol{P}$ can be written as

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{U}^{'}\boldsymbol{Q}_1\boldsymbol{U} & \boldsymbol{U}^{'}\boldsymbol{Q}_1 \\ \boldsymbol{Q}_1\boldsymbol{U} & \boldsymbol{I}_N - (1/N)\boldsymbol{J}_N \end{bmatrix}$$

Let $\boldsymbol{P} = \{\boldsymbol{P}_{ij}\}_{i=1,j=1}^{i=t,j=t}$. Then with the aid of Eqs. (9) and (10), the $(n_i+1)\times(n_j+1)$ symmetric submatrix $\boldsymbol{P}_{ij}$ of $\boldsymbol{P}$ is written as

$$\boldsymbol{P}_{ii} = \begin{bmatrix} n_i - (1/N)n_i^2 & (1-(1/N))\boldsymbol{1}_{n_i}^{'} \\ (1-(1/N))\boldsymbol{1}_{n_i} & \boldsymbol{I}_{n_i} - (1/N)\boldsymbol{J}_{n_i} \end{bmatrix} \qquad \text{if } \; i = j$$

and

$$\boldsymbol{P}_{ij} = -(1/N)\begin{bmatrix} n_i n_j & \boldsymbol{1}_{n_i}^{'} \\ \boldsymbol{1}_{n_i} & \boldsymbol{J}_{n_i\times n_j} \end{bmatrix} \qquad \text{if } \; i \neq j.$$

Define $\boldsymbol{\epsilon}_i$ as $\boldsymbol{\epsilon}_i^{'} = (\tau_i, e_{i1}, e_{i2}, \ldots, e_{in_i})$. Using the projection method for quadratic forms (see Akritas and Papadatos (2004), van der Vaart (1998) ch. 11), Q_N is decomposed as $Q_N = U_N - V_N$ where

$$U_N = \sum_{i=1}^{t} \boldsymbol{\epsilon}_i^{'}\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i \quad \text{ and } \quad V_N = \sum_{i=1}^{t}\sum_{j\neq i, j=1}^{t} \boldsymbol{\epsilon}_i^{'}\boldsymbol{P}_{ij}\boldsymbol{\epsilon}_j$$

It should be noted that U_N is the sum of independent but not identical random variables and U_N and V_N are uncorrelated.

Observe that

$$E|\boldsymbol{\epsilon}_i^{'}\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i - E[\boldsymbol{\epsilon}_i^{'}\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i]| = tr(\boldsymbol{P}_{ii}E|\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^{'} - E[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^{'}]|)$$
$$\leq tr(\boldsymbol{P}_{ii}\boldsymbol{P}_{ii})^{1/2}(E|\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^{'} - E[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^{'}]|E|\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^{'} - E[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^{'}]|)^{1/2}$$

where the inequality is acquired by using Cauchy-Schwartz inequality. The moment conditions $E|\tau_i|^{4+\delta} < \infty$ and $E|e_{ij}|^{4+\delta} < \infty$ for some positive δ ensure that there exists a finite and positive M such that $(E|\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^{'} - E[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^{'}]|E|\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^{'} - E[\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^{'}]|)^{1/2} \leq M^{1/2}$. On the other hand, $tr(\boldsymbol{P}_{ii}\boldsymbol{P}_{ii}) = (1-(n_i/N))^2(1+n_i)^2 +_i^n -1 \leq 6n_i^4$. It follows from these that

$$\sum_{i=1}^{t}[E|\boldsymbol{\epsilon}_i^{'}\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i - E[\boldsymbol{\epsilon}_i^{'}\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i]|^{2+\delta} \leq M^{1+\delta/2}6^{1+\delta/2}\sum_{i=1}^{t} n_i^{4+2\delta},$$

where $\sum_{i=1}^{t} n_i^{4+2\delta} = O(N)$. Therefore

$$\sum_{i=1}^{t}[E|\boldsymbol{\epsilon}_i^{'}\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i - E[\boldsymbol{\epsilon}_i^{'}\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i]|^{2+\delta} = o(N^b) \tag{22}$$

for $b > 1$ when either the small or large n_i assumption holds. Let $c_N^2 = Var(U_N)$ where $c_N = \sum_{i=1}^{t} Var(\boldsymbol{\epsilon}_i^{'}\boldsymbol{P}_{ii}\boldsymbol{\epsilon}_i)$. By using Lemma 1 of Westfall (1987), c_N^2 is calculated and it is equal to

$$c_N^2 = \sigma^4[2\rho^2\sum_{i=1}^{t}(n_i - (1/N)n_i^2)^2 + 4\rho\sum_{i=1}^{t}n_i(1-(n_i/N))^2 + 2\sum_{i=1}^{t}n_i(1-(1/N))^2$$

$$+k_\tau\rho^2\sum_{i=1}^{t}(n_i - (1/N)n_i^2)^2 + k_e\sum_{i=1}^{t}n_i(1-(n_i/N))^2].$$

Then, using the asymptotic condition in Sec. 2, the following limit is obtained

$$\lim_{N\to\infty}(1/N)c_N^2 = (1/N)Var(U_N) = \sigma^4[2\rho^2\gamma_2 + 4\rho + 2 + k_\tau\rho^2\gamma_2 + k_e]. \tag{23}$$

and consequently $c_N^2 = O(N)$. The facts that Eq. (22) and $c_N^{2+\delta} = O(N^{1+\delta/2})$ together imply that the Liapounov, condition as applied to $\boldsymbol{\epsilon}_1^{'}\boldsymbol{P}_{11}\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2^{'}\boldsymbol{P}_{22}\boldsymbol{\epsilon}_2$, $\ldots$, $\boldsymbol{\epsilon}_t^{'}\boldsymbol{P}_{tt}\boldsymbol{\epsilon}_t$, holds. Thus $Var(U_N)^{-1/2}(U_N - E[U_N])$ converges in distribution to $N(0,1)$.

The expression $\lim_{N\to\infty}(1/N)Var(Q_N)$ can be obtained by Eqs. (18)-(20) since $Q_N = SS_\tau + SSE$ and it is equal to Eq. (23). From the facts that $Q_N = U_N + V_N$ and $Cov(U_N, V_N) = 0$, we get $\lim_{N\to\infty}(1/N)Var(V_N) = \lim_{N\to\infty}(1/N)[Var(Q_N) - Var(U_N)] = 0$. Consequently U_N converges in probability to 0. Thus $Var(Q_N)^{-1/2}$ $(Q_N - E[Q_N])$ converges in distribution to $N(0,1)$ if $Var(U_N)^{-1/2}$ $(U_N - E[U_N])$ converges in distribution to $N(0,1)$. Let $\boldsymbol{SS} = (SS_\tau, SSE)^{'}$ and $\boldsymbol{MS} = (MS_\tau, MSE)^{'}$. Then if $Var(Q_N)^{-1/2}(Q_N - E[Q_N])$ converges in distribution to $N(0,1)$ where $\boldsymbol{\Delta}$ is in Eq. (21). $\sqrt{N}(\boldsymbol{MS} - E[\boldsymbol{MS}])^{'}$ converges in distribution to $N_2(\boldsymbol{0}, \boldsymbol{ACOV})$ if $N^{-1/2}(\boldsymbol{SS} - E[\boldsymbol{SS}])^{'}$ converges in distribution to $N_2(\boldsymbol{0}, \boldsymbol{\Delta})$ where $\boldsymbol{ACOV}$ is in Eq. (5). It should be noted that $E[\boldsymbol{MS}] = (\sigma^2[1 + \rho(N - 1/N\sum_{i=1}^{t} n_i^2)/(t-1)], \sigma^2)^{'}$ and $E[\boldsymbol{MS}]$ converges to $E[\boldsymbol{\Gamma}] = (\sigma^2[1+\rho a^{-1}], \sigma^2)^{'}$ as $N \to \infty$. This completes the proof of Theorem 3.2 □

Theorem 3.3. *Suppose the asymptotic condition established in Sec. 2 holds. Then*

$$\sqrt{N}(F_N - [1 + \rho a^{-1}])$$

converges in distribution to normal distribution with 0*-mean and variance* σ_F^2 *as* $N \to \infty$ *where* F_N *is as in Eq. (2),* σ_F^2 *is:*

$$\sigma_F^2 = \frac{2(\rho^2\gamma_2 + 2\rho + a)}{a^2} + \frac{2(1+\rho a^{-1})}{(1-a)} + k_\tau\frac{\rho^2\gamma_2}{a^2}$$

$$+ k_e(\frac{\gamma_{-1}}{a^2} - \frac{2(a-\gamma_{-1})(1+\rho a^{-1})}{a(1-a)} + \frac{(1-2a+\gamma_{-1})(1+\rho a^{-1})^2}{(1-a)^2}), \tag{24}$$

and the limits a *and* γ_p *for* $p = -1, 2$ *are in Eq. (4).*

Proof. Let $\bigtriangledown F_N$ denote the vector of the partial derivatives of F_N with respect to MS_τ and MSE at their expectations. Then $\bigtriangledown F_N = (1/\sigma^2, -[1+\rho a^{-1}]/\sigma^2)'$. From the delta method, $\sqrt{N}(F_N - [1+\rho a^{-1}])$ converges in distribution to normal distribution with zero mean and the variance $\sigma_F^2 = \bigtriangledown' F_N \boldsymbol{ACOV} \bigtriangledown F_N$ where $\boldsymbol{ACOV}$ is in (5). The explicit form of σ_F^2 is given in Eq. (24). □

4. The Proposed Test

The α sized approximate test rejects $H_0 : \rho = 0$ when $F_N > u_\alpha$ where F_N is in Eq. (2) and u_α is the upper $1-\alpha$ quantile of the asymptotic null distribution of F_N. Then, we have

$$P(F_N > u_\alpha | \rho = 0) = \alpha$$

The asymptotic null distribution of $\sqrt{N}(F_N - 1)$ determined from Theorem 3.3 is the normal distribution with zero mean and variance σ_0^2 where it is written as

$$\sigma_0^2 = \frac{2}{a(1-a)} + k_e \frac{\gamma_{-1} - a^2}{a^2(1-a)^2} \tag{25}$$

after some algebraic operation on Eq. (24). One finds u_α and it is given by

$$u_\alpha = \frac{\sigma_0}{\sqrt{N}} z_\alpha + 1 \tag{26}$$

where z_α is the upper $1-\alpha$ quantile of the standard normal distribution.

Finally the approximate power of the proposed test for a finite sample size is:

$$P(F_N > u_\alpha | \rho > 0) = 1 - \Phi\left(\frac{\sqrt{N}(u_\alpha - [1+\rho a^{-1}])}{\sigma_F}\right) \tag{27}$$

where Φ denotes the cumulative standard normal distribution, σ_F^2 and u_α are in Eqs. (24) and (26) respectively.

5. Robustness of the Test

The robustness of the asymptotic distribution of the F_N statistic is valid only for the balanced models and it is defined as follows. The asymptotic null distribution of F_N does not depend on the fourth moment of error.

Corollary 5.1. *The asymptotic null distribution of F_N does not depend on the fourth moment of error in the balanced models.*

Proof. To show this, it is enough to show that the asymptotic null variance σ_0^2 in Eq. (25) is free of the kurtosis k_e of error. When $n_i = n$ for all i, where n is fixed, we have $N = tn$ and then

$$\gamma_{-1} - a^2 = \lim_{N\to\infty} \{(1/N) \sum_{i=1}^{t} 1/n_i - t^2/N^2\} = 1/n^2 - 1/n^2 = 0.$$

where γ_{-1} and a are in Eq. (4). The coefficient of k_e appearing in the asymptotic null variance σ_0^2 is equal to 0. So σ_0^2 does not include k_e. □

As indicated by (Akritas and Arnold 2000, p.221), (Scheffe 1959, p.344), and Güven (2014) the asymptotic null distribution of F_N is asymptotically robust with respect to departure from normality of error. So, for the balanced case, the size of the test is asymptotically robust to nonnormal error.

6. Numerical and Simulation Study

The power values of the approximate test are compared with the simulated power values for some selected distributions to τ_i and e_{ij} in order to check accuracy of the power of the approximate test.

A power value of the approximate test is obtained from Eq. (27) for a given positive variance ratio ρ after calculation of the upper percentile point u_α in Eq. (26) for a given α and of variance σ_F^2 in (24). The limit values a, γ_{-1} and γ_2 appearing in σ_F^2 are replaced with their sample encounter values.

The simulated power value is the ratio of the number of generated F_N value in (2) exceeding u_α to the number of simulation runs. Generation of the F_N value is as follow

1)Set μ equal to any constant.

2)Generate τ_i for $i = 1, 2, \ldots, t$ from one of three different distributions: $\sqrt{\rho}N(0, 1)$, $\sqrt{\rho}(exp(1)-1)$ and $\sqrt{\rho}U(-\sqrt{3}, \sqrt{3})$ for a given ρ where $\rho = 0.0, 0.5, 0.7, 1.0, 1.5, 1.7$.

3) Generate e_{ij} for $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, n_i$ from one of three different distributions: $N(0, 1)$, $exp(1) - 1$ and $U(-\sqrt{3}, \sqrt{3})$. The generation of e_{ij}'s is separated from the generation of τ_i's.

It should be noted that the distributions $N(0, 1)$, $exp(1) - 1$ and $U(-\sqrt{3}, \sqrt{3})$ have zero mean and unit variance. Also note that the distributions $N(0, 1)$, $exp(1) - 1$ and $U(-\sqrt{3}, \sqrt{3})$ have the null (0), positive (6) and negative ($-6/5$) kurtosis respectively.

4) Generated Y_{ij}, values $i = 1, 2, \ldots t$ and $j = 1, 2, \ldots n_i$ are obtained where $Y_{ij} = \mu + \tau_i + e_{ij}$ and then the ratio F_N is obtained.

Two different design are considered. One is a small n_i design for which $t = 20$, $n_1 = \ldots = n_5 = 2$, $n_6 = \ldots = n_{10} = 3$, $n_{11} = \ldots = n_{15} = 4$ and $n_{16} = \ldots = n_{20} = 5$. The other one is a large n_i design for which $t = 4$, $n_1 = n_2 = 20$ and $n_3 = n_4 = 25$.

Simulation is based on 1000 runs. In each run, F_N is calculated from generating data. The number of F_N exceeding u_α is divided by 1000 to get a power value of the approximate test. The simulated level of significance of the test is obtained in getting simulated power value of the approximate test when $\rho = 0$. Simply we skip the step 2 in generation of F_N It is equivalently to simulate the level of significance of the test for testing hypothesis of no fixed treatment effects in the one-way ANOVA model.

In Table 1. through 6, sizes and power values of the approximate test are very closer to simulated sizes and power values of the test for small values of ρ. However,the differences between them values slightly increase as the value of ρ increases. It is also observed that both approximated and simulated power values of the test are higher for a large n_i design than for a small n_i design. So according to the simulation results, the test is more appropriate for a small variance ratio and large group sizes.

Table 1 and 4 are for the null kurtosis case while the rest of the tables are for either the positive or negative kurtosis case. It is not detected any significant rise or decline of power values of the approximate test in departing from the null kurtosis case. In comparison Tables 2 and 5 with Table 3 and 6, the power values of the test are higher for the negative kurtosis case than for the positive kurtosis case.

7. Conclusion

In the present paper we establish the approximate test for the hypothesis of zero variance ratio in the unbalanced one way random effects model from non normal universes. As shown in Sec. 4. calculation of both the upper percentile point and a power value of the test can easily be accomplished. The test is robust for the balanced one way random effects model. In the balanced case the null distribution of the test statistics F_N ratio does not depend on the fourth moment of the error term.

The differences between the calculated and generated sizes and power values are closer to a small design and lower variance ratios than a large design and higher variance ratios. It follows that the approximate test is more accurate for a small design and lower variance ratios. It is not detected any significant rise or descend of the power from null to non null kurtosis. Thus, departing from null kurtosis does not have an impact to the power of the approximate test.

Table 1. Approximation to power values of the α sized test for a small n_i design and the null kurtosis case where the numbers in parentheses are simulated values.

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
0	0	0.01	0.63	0.76	0.85	0.92	0.93
		(0.03)	(0.61)	(0.77)	(0.89)	(0.97)	(0.99)
		0.05	0.73	0.83	0.89	0.94	0.95
		(0.08)	(0.74)	(0.86)	(0.94)	(0.99)	(0.99)
		0.10	0.78	0.86	0.91	0.95	0.95
		(0.13)	(0.80)	(0.90)	(0.96)	(0.99)	(0.99)

Table 2. Approximation to power values of the α sized test for a small n_i design and the positive kurtosis case where the numbers in parentheses are simulated values.

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
6	0	0.01	0.60	0.70	0.77	0.82	0.84
		(0.03)	(0.54)	(0.69)	(0.82)	(0.91)	(0.93)
		0.05	0.69	0.76	0.81	0.85	0.85
		(0.08)	(0.68)	(0.79)	(0.88)	(0.94)	(0.96)
		0.10	0.73	0.78	0.83	0.86	0.86
		(0.13)	(0.74)	(0.84)	(0.91)	(0.97)	(0.97)
0	6	0.01	0.52	0.65	0.76	0.84	0.86
		(0.06)	(0.50)	(0.68)	(0.84)	(0.93)	(0.95)
		0.05	0.64	0.74	0.82	0.87	0.89
		(0.17)	(0.66)	(0.80)	(0.91)	(0.97)	(0.98)
		0.10	0.70	0.78	0.83	0.86	0.86
		(0.25)	(0.74)	(0.86)	(0.93)	(0.98)	(0.99)
6	6	0.01	0.51	0.62	0.71	0.77	0.79
		(0.05)	(0.45)	(0.58)	(0.71)	(0.85)	(0.89)
		0.05	0.62	0.70	0.76	0.80	0.82
		(0.08)	(0.58)	(0.69)	(0.81)	(0.91)	(0.93)
		0.10	0.67	0.73	0.78	0.82	0.83
		(0.12)	(0.65)	(0.76)	(0.86)	(0.93)	(0.95)

Table 3. Approximation to power values of the α sized test for a small n_i design and the negative kurtosis case where the numbers in parentheses are simulated values

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
-6/5	0	0.01	0.63	0.78	0.88	0.95	0.96
		(0.03)	(0.62)	(0.80)	(0.92)	(0.99)	(0.99)
		0.05	0.75	0.85	0.92	0.96	0.97
		(0.08)	(0.76)	(0.89)	(0.96)	(0.99)	(0.99)
		0.10	0.80	0.88	0.93	0.97	0.97
		(0.13)	(0.83)	(0.98)	(0.91)	(0.99)	(0.99)
0	-6/5	0.01	0.66	0.78	0.88	0.93	0.95
		(0.03)	(0.63)	(0.80)	(0.92)	(0.98)	(0.98)
		0.05	0.76	0.85	0.91	0.95	0.96
		(0.07)	(0.77)	(0.89)	(0.96)	(0.99)	(0.99)
		0.10	0.80	0.88	0.93	0.96	0.96
		(0.13)	(0.83)	(0.92)	(0.97)	(0.98)	(0.99)
-6/5	-6/5	0.01	0.67	0.81	0.91	0.96	0.97
		(0.03)	(0.64)	(0.85)	(0.94)	(0.99)	(0.99)
		0.05	0.77	0.87	0.94	0.97	0.98
		(0.07)	(0.80)	(0.91)	(0.97)	(0.99)	(0.99)
		0.10	0.82	0.90	0.95	0.98	0.98
		(0.13)	(0.86)	(0.94)	(0.98)	(0.99)	(0.99)

Table 4. Approximations to power values of the α sized test for a large n_i design and the null kurtosis case where the numbers in parentheses are simulated values.

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
0	0	0.01	0.86	0.88	0.89	0.90	0.90
		(0.05)	(0.89)	(0.93)	(0.96)	(0.98)	(0.98)
		0.05	0.87	0.88	0.89	0.90	0.90
		(0.10)	(0.93)	(0.96)	(0.97)	(0.99)	(0.99)
		0.10	0.88	0.86	0.91	0.95	0.95
		(0.14)	(0.93)	(0.96)	(0.97)	(0.99)	(0.99)

Table 5. Approximation to power values of the α sized test for a large n_i design and the positive kurtosis case where the numbers in parentheses are simulated values.

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
6	0	0.01	0.72	0.73	0.74	0.75	0.75
		(0.03)	(0.75)	(0.81)	(0.85)	(0.90)	(0.91)
		0.05	0.73	0.74	0.74	0.75	0.75
		(0.10)	(0.80)	(0.84)	(0.88)	(0.92)	(0.93)
		0.10	0.73	0.74	0.75	0.75	0.75
		(0.14)	(0.82)	(0.86)	(0.89)	(0.92)	(0.93)
0	6	0.01	0.85	0.86	0.88	0.88	0.89
		(0.06)	(0.87)	(0.91)	(0.94)	(0.97)	(0.98)
		0.05	0.86	0.87	0.88	0.89	0.89
		(0.10)	(0.90)	(0.93)	(0.96)	(0.98)	(0.99)
		0.10	0.86	0.88	0.88	0.89	0.89
		(0.14)	(0.91)	(0.94)	(0.97)	(0.99)	(0.99)
6	6	0.01	0.72	0.73	0.74	0.74	0.74
		(0.06)	(0.76)	(0.82)	(0.87)	(0.90)	(0.92)
		0.05	0.73	0.73	0.74	0.74	0.75
		(0.10)	(0.81)	(0.85)	(0.89)	(0.93)	(0.94)
		0.10	0.73	0.74	0.74	0.75	0.75
		(0.14)	(0.83)	(0.87)	(0.91)	(0.94)	(0.95)

Table 6. Approximations to power values of the α sized test for a large n_i design and the negative kurtosis case where the numbers in parentheses are simulated values.

k_τ	k_e	α	$\rho = 0.5$	0.7	1.0	1.5	1.7
-6/5	0	0.01	0.93	0.95	0.96	0.97	0.97
		(0.05)	(0.92)	(0.95)	(0.97)	(0.99)	(0.99)
		0.05	0.94	0.96	0.96	0.97	0.97
		(0.10)	(0.94)	(0.97)	(0.98)	(0.99)	(0.99)
		0.10	0.95	0.96	0.97	0.97	0.97
		(0.14)	(0.96)	(0.97)	(0.91)	(0.99)	(0.99)
0	-6/5	0.01	0.86	0.88	0.89	0.90	0.90
		(0.05)	(0.89)	(0.93)	(0.96)	(0.97)	(0.98)
		0.05	0.87	0.89	0.90	0.90	0.91
		(0.10)	(0.92)	(0.95)	(0.96)	(0.98)	(0.99)
		0.10	0.88	0.89	0.90	0.91	0.91
		(0.15)	(0.94)	(0.96)	(0.97)	(0.99)	(0.99)
-6/5	-6/5	0.01	0.94	0.95	0.97	0.97	0.97
		(0.05)	(0.92)	(0.95)	(0.98)	(0.99)	(0.99)
		0.05	0.95	0.96	0.97	0.97	0.98
		(0.10)	(0.95)	(0.96)	(0.98)	(0.99)	(0.99)
		0.10	0.95	0.96	0.97	0.98	0.98
		(0.15)	(0.95)	(0.97)	(0.99)	(0.99)	(0.99)

References

[1] Akritas, M and Arnold, S., Asymptotic for analysis of variance when the number of levels is large, *J. Amer Statist Assoc.* 95, (2000), 212-226.

[2] Akritas, M.G. and Papdatos, N., Heteroscedastic one-way ANOVA and lack-of-fit tests, *J. Amer Statist. Assoc.,* 99, (2004) , 368-382.

[3] Güven, B., Testing for random effect in the Fuller-Battese model, *Statistics,* 48, (2014) , 802-814.

[4] Rudin, W., Principles of Mathematical Analysis, New York, McGraw-Hill, 1976.

[5] Searle, S. R., Linear Models for Unbalanced Data, New York, Wiley, 1988.

[6] Scheffe, H., The Analysis of Variance, New York Wiley, 1959

[7] Westfall, P., A comparison of variance component estimates for arbitrary underlying distributions, *J. Amer. Statist. Assoc.,* 62, (1987) , 866-874.

[8] Westfall, P., Robustness and power of tests for a null variance ratio, *Biometrika*, 75; (1988) , 207-214.

[9] van der Vaart, A.W., Asymptotic statistics, New York, Cambridge 1998.

Current address: Sevgi Demircioğlu (Corresponding Author): Department of Mathematics, Çanakkale Onsekiz Mart University Çanakkale, Turkey

E-mail address: sevgidemircioglu@stu.comu.edu.tr

ORCID Address: http://orcid.org/0000-0003-3900-0713

Current address: Bilgehan Güven: Department of Mathematics, Çanakkale Onsekiz Mart University Çanakkale, Turkey

E-mail address: bguven@comu.edu.tr

ORCID Address: http://orcid.org/0000-0001-8661-1408