INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 12 NO. 2 PAGE 268–275 (2019)

Classification of Flat Lagrangian H-umbilical Submanifolds in Indefinite Complex Euclidean spaces

Shangrong Deng*

(Communicated by Bang-Yen Chen)

ABSTRACT

In this article, we completely characterize flat Lagrangian H-umbilical submanifolds in the indefinite complex Euclidean spaces $\mathbf{C}^n_s$. Consequently, in conjunction with a result from [4], Lagrangian H-umbilical submanifolds in the indefinite complex Euclidean n-space $\mathbf{C}^n_s$ with $n > 2$ are completely classified.

Keywords: *Lagrangian submanifold, H-umbilical submanifold, Indefinite Complex Euclidean space*
AMS Subject Classification (2010): *Primary 53C40, 53C15, 53C25.*

1. Introduction

The notion of Lagrangian H-umbilical submanifolds in Kaehler manifolds was introduced by B.-Y. Chen in [2]. Later he extended this notion in [5] to Lagrangian H-umbilical submanifolds in pseudo-Kaehler manifolds. In particular, he proved that a Lagrangian H-umbilical submanifold in the indefinite complex Euclidean n-space $\mathbf{C}^n_s$ with $n > 2$ and complex index s is locally either a complex extensor, a pseudo-hyperbolic space, a pseudo-Riemannian sphere, or a flat pseudo-Riemannian manifold. In this article, we completely characterize flat Lagrangian H-umbilical submanifolds in $\mathbf{C}^n_s$. Consequently, Lagrangian H-umbilical submanifolds in $\mathbf{C}^n_s$ with $n > 2$ are completely classified. There are new cases in our classification. In order to do so, we have arranged the indices and utilized Legendre curves carefully to make the classification results true and simple.

Notice that for $n = 2$ the complete classification of Lagrangian H-umbilical surfaces of constant curvature in the indefinite complex Euclidean plane was already done earlier in [8], [9] and [10].

2. Preliminaries

We use Chen's book [6] as the general reference for Lagrangian submanifolds in pseudo-Kaehler manifolds.

Let $L : M \to \mathbf{C}^n_s$ be an isometric immersion of an n-dimensional pseudo-Riemannian manifold M into the indefinite complex Euclidean n-space $\mathbf{C}^n_s$. We assume $n > 2$ and $0 < s < n$ in the article. Then M is called a Lagrangian submanifold if the almost complex structure J of $\mathbf{C}^n_s$ interchanges the tangent space and the normal space. Clearly, a Lagrangian submanifold of $\mathbf{C}^n_s$ has real dimension n and real index s. The formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

for tangent vector fields X and Y and normal vector fields ξ, where D is the normal connection. The second fundamental form h is related to A_ξ by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

Received : *25-May-2019,* ***Accepted :*** *07-September-2019*
*** Corresponding author***

The Gauss and Codazzi equations are given by

$$\langle R(X,Y)Z,W\rangle = \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle\,,$$
$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z),$$

where (∇h) is defined by

$$(\nabla h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z).$$

When M is a Lagrangian in $\mathbf{C^n_s}$, we have

$$D_X JY = J\nabla_X Y,$$
$$\langle h(X,Y),JZ\rangle = \langle h(Y,Z),JX\rangle = \langle h(Z,X),JY\rangle\,.$$

We denote the pseudo hypersphere and the pseudo hyperbolic space by

$$S^{2n-1}_{2s}(1) = \{z=(z_1,...,z_n)\in \mathbf{C^n_s} :<\mathbf{z},\mathbf{z}>=\mathbf{1}\},$$
$$H^{2n-1}_{2s-1}(-1) = \{z=(z_1,...,z_n)\in \mathbf{C^n_s} :<\mathbf{z},\mathbf{z}>=-\mathbf{1}\}.$$

Let $z=z(s)$ be a unit speed time-like curve in $S^{2n-1}_{2s}(1)$ (or space-like curve in $H^{2n-1}_{2s-1}(-1)$). $z=z(s)$ is called a Legendre curve if $\langle z'(s),iz(s)\rangle = 0$ identically. Since $z=z(s)$ is a unit speed curve, $\langle z'(s),z(s)\rangle = 0$. Hence, $z(s),iz(s),z'(s),iz'(s)$ are orthonormal vector fields along the Legendre curve. There exist parallel normal vector fields $P_3,...P_n$ such that

$$z(s),iz(s),z'(s),iz'(s),P_3(s),iP_3(s)...P_n(s),iP_n(s) \tag{2.1}$$

form an orthonormal frame field along the Legendre curve.

By taking the derivatives of $<z'(s),iz(s)>=0$ and $<z'(s),z(s)>=0$, we have $<z''(s),iz(s)>=0$ and $<z''(s),z(s)>=1$ if $z(s)$ is in $S^{2n-1}_{2s}(1)$ (or $<z''(s),z(s)>=-1$ if $z(s)$ is in $H^{2n-1}_{2s-1}(-1)$). In both cases, with respect to an orthonormal frame field (2.1), z'' can be expressed as

$$z''(s) = i\lambda(s)z'(s) + z(s) - \sum_{j=3}^{n} a_j(s)P_j(s) + \sum_{j=3}^{n} b_j(s)iP_j(s) \tag{2.2}$$

for some real functions $\lambda, a_3,...,a_n, b_3,...,b_n$. The Legengre curve is called *special Legendre* if $b_3=...=b_n=0$ (see [4]). Hence (2.2) reduces to

$$z''(s) = i\lambda(s)z'(s) + z(s) - \sum_{j=3}^{n} a_j(s)P_j(s) \tag{2.3}$$

3. Flat Lagrangian H-umbilical Submanifolds in $\mathbf{C^n_s}$

Following [2, 3, 6], a Lagrangian H-umbilical submanifold is a non-totally geodesic Lagrangian submanifold whose second fundamental form takes the following simple form:

$$h(e_1,e_1) = \lambda Je_1, \quad h(e_2,e_2) = \cdots = h(e_n,e_n) = \mu Je_1,$$
$$h(e_1,e_j) = \mu Je_j, \quad h(e_j,e_k) = 0, \quad j\neq k, \quad j,k = 2,\ldots,n$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field. In particular, if $L: M\to \mathbf{C^n_s}$ is a flat Lagrangian H-umbilical submanifold, then the second fundamental form takes the following form:

$$h(e_1,e_1) = \phi Je_1, \quad h(e_1,e_j) = h(e_j,e_k) = 0, \quad j,k = 2,...,n. \tag{3.1}$$

for some orthonormal frame field on M.

We state the following well-known lemma for later use.

Lemma 3.1. *Let $L: M\to \mathbf{C^n_s}$ be a flat Lagrangian H-umbilical submanifold. If $\phi = 0$, L is an open portion of a totally geodesic Lagrangian n-plane in $\mathbf{C^n_s}$.*

Proof. If $\phi = 0$, then the second fundamental form vanishes identically. Hence M is totally geodesic and flat. Therefore, L must be an open portion of a totally geodesic Lagrangian n-plane in $\mathbf{C}^{\mathbf{n}}_{\mathbf{s}}$. ∎

From now on we assume ϕ is nowhere zero. We divide the proof of our classification into two cases; namely, *Case* 1: e_1 *is space-like* and *Case* 2: e_1 *is time-like*.

Case 1: e_1 *is space-like*. In this case, we arrange the indices as follows.

$$\langle e_1, e_1\rangle = \langle e_{s+2}, e_{s+2}\rangle = \cdots = \langle e_n, e_n\rangle = 1; \quad \langle e_2, e_2\rangle = ... = \langle e_{s+1}, e_{s+1}\rangle = -1 \tag{3.2}$$

We put

$$\epsilon_k = -1, k = 2, ..., s+1; \quad \epsilon_k = 1, k = 1, s+2, ..., n$$

Then (3.2) becomes $\langle e_k, e_k\rangle = \epsilon_k$.

Theorem 3.2. *Let $\lambda, b, a_3, ..., a_n$ be real-valued functions on an open interval I with λ being nowhere zero and let $z : I \to H^{2n-1}_{2s-1}(-1) \subset \mathbf{C}^{\mathbf{n}}_{\mathbf{s}}$ be a space-like special Legendre curve satisfying (2.3). Put*

$$f(t, u_2, ..., u_n) = b(t) + u_2 + \sum_{j=3}^{n} a_j(t)u_j. \tag{3.3}$$

Denote by $\hat{M}_1(0)$ the twisted product manifold ${}_fI \times E^{n-1}_s$ with twisted product metric given by

$$g = f^2dt^2 - du_2{}^2 - \cdots - du_{s+1}{}^2 + du_{s+2}{}^2 + \cdots + du_n{}^2 \tag{3.4}$$

Then $\hat{M}_1(0)$ is a flat pseudo-Riemannian manifold and

$$L(t, u_2, ..., u_n) = u_2 z(t) + \sum_{j=3}^{n} u_j P_j(t) + \int^t b(t)z'(t)dt \tag{3.5}$$

defines a Lagrangian H-umbilical isometric immersion $L : \hat{M}_1(0) \to \mathbf{C}^{\mathbf{n}}_{\mathbf{s}}$.

Conversely, up to rigid motions of $\mathbf{C}^{\mathbf{n}}_{\mathbf{s}}$, every flat Lagrangian H-umbilical submanifold without totally geodesic points in $\mathbf{C}^{\mathbf{n}}_{\mathbf{s}}$ and with space-like e_1 is locally a Lagrangian cylinder over a curve, or a product of a flat Lagrangian H-umbilical Riemannian submanifold and a time-like s-plane, or a Lagrangian submanifold obtained in the way described above.

Proof. Let $\lambda, b, a_3, ..., a_n$ be real-valued functions on an open interval I with λ nowhere zero and let $z : I \to H^{2n-1}_{2s-1}(-1) \subset \mathbf{C}^{\mathbf{n}}_{\mathbf{s}}$ be a space-like special Legendre curve satisfying (2.3) for some parallel orthonormal vector fields $P_3, ...P_n$ along the curve and we may arrange the indices in such way that $P_{s+2}, ..., P_n$ are space-like. With $P_3, ...P_n$ being parallel, we have

$$P'_j(t) = \eta_j(t)z'(t), \quad j = 3, ..., n. \tag{3.6}$$

for some functions $\eta_3, ..., \eta_n$.

Let $L(t, u_2, ..., u_n)$ be given by (3.5). By taking the partial derivatives of L with respect to $t, u_2, ..., u_n$, we find

$$\begin{aligned} L_t &= u_2 z'(t) + \sum_{j=3}^{n} u_j P'_j(t) + b(t)z'(t), \\ L_{u_2} &= z(t), \\ &\cdots \\ L_{u_j} &= P_j(t). \end{aligned} \tag{3.7}$$

From (3.7) and the special Legendre curve we have

$$\left\langle L_t, L_{u_j}\right\rangle = 0, \quad \left\langle L_{u_j}, L_{u_k}\right\rangle = \epsilon_k \delta_{jk}, \quad j, k = 2, ..., n. \tag{3.8}$$

Since $z'(t)$ and $P_j(t)$ are perpendicular, (3.6) yields

$$P'_j(t) = a_j(t)z'(t), \quad j = 3, ..., n. \tag{3.9}$$

From (3.7) and (3.9) we get

$$L_t(t) = fz'(t) \tag{3.10}$$

It follows from (3.3), (3.4), (3.8) and (3.10) that $L(t, u_2, ..., u_n)$ is an isometric immersion of $\hat{M}_1(0)$ in $\mathbf{C^n_s}$. From the definition of special Legendre curves, L is Lagrangian.

From (2.3), (3.7), (3.10) and the definition of special Legendre curves, we find

$$L_{tt} = f_t z'(t) + fz''(t), \quad L_{tu_j} = a_j(t)z'(t), \quad L_{u_j u_k} = 0, \quad j,k = 2, ..., n. \tag{3.11}$$

From (2.3), (3.7), (3.9), (3.10), (3.11) and the formula of Gauss, we have

$$h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \lambda(t)J\left(\frac{\partial}{\partial t}\right), \quad h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial u_j}\right) = h\left(\frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\right) = 0, \quad j,k = 2, ..., n.$$

which implies that $L : M \to \mathbf{C^n_s}$ is Lagrangian H-umbilical.

Conversely, assume that $L : M \to \mathbf{C^n_s}$ is a flat Lagrangian H-umbilical isometric immersion of a flat pseudo-Riemannian manifold M into $\mathbf{C^n_s}$ with e_1 space-like and without totally geodesic points. Since M is flat, the second fundamental form h of L satisfies

$$h(e_1, e_1) = \phi J e_1, \quad h(e_1, e_j) = h(e_j, e_k) = 0, \quad j,k = 2, ..., n. \tag{3.12}$$

for some nowhere zero function ϕ with respect to some orthonormal frame field $e_1, ..., e_n$. Without loss of generality, we may assume $\phi > 0$. Since e_1 is space- like, we may arrange the indices as the following

$$\langle e_1, e_1\rangle = \langle e_{s+2}, e_{s+2}\rangle = \cdots = \langle e_n, e_n\rangle = 1; \quad \langle e_2, e_2\rangle = \cdots = \langle e_{s+1}, e_{s+1}\rangle = -1 \tag{3.13}$$

or

$$< e_k, e_k >= \epsilon_k, \quad \epsilon_k = -1, k = 2, ..., s+1; \quad \epsilon_k = 1, k = 1, s+2, ..., n$$

From Codazzi's equation and (3.12), we have (see also [5, page 174])

$$e_j \ln \phi = \omega_1^j(e_1), \quad \omega_1^j(e_k) = 0, \quad j,k = 2, ..., n. \tag{3.14}$$

Let $\mathcal{D}$ and $\mathcal{D}^\perp$ be the distributions of M spanned by $\{e_1\}$ and $\{e_2, ...e_n\}$, respectively. Being one dimensional, $\mathcal{D}$ is integrable and space-like. From (3.13) and (3.14), $\mathcal{D}^\perp$ is also integrable and the leaves of $\mathcal{D}^\perp$ are totally geodesic submanifolds of $\mathbf{C^n_s}$ with real index s. Because $\mathcal{D}$ and $\mathcal{D}^\perp$ are both integrable and they are perpendicular, there exist local coordinates $\{x_1, x_2, ..., x_n\}$ such that $\frac{\partial}{\partial x_1}$ spans $\mathcal{D}$ and $\{\frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\}$ spans $\mathcal{D}^\perp$. We may assume that $x_2, ..., x_{s+1}$ are time-like. Since $\mathcal{D}$ is one dimensional, we may choose x_1 such that $\frac{\partial}{\partial x_1} = \phi^{-1} e_1$. Then (3.12) becomes

$$h\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right) = J\left(\frac{\partial}{\partial x_1}\right), \quad h\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j}\right) = h\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = 0, \quad j,k = 2, ..., n. \tag{3.15}$$

Let N_s^{n-1} be an integral submanifold of $\mathcal{D}^\perp$. Then N_s^{n-1} is a totally geodesic submanifold of $\mathbf{C^n_s}$. Thus, N_s^{n-1} is an open portion of an indefinite Euclidean (n-1)-space E_s^{n-1}. Hence, M is an open portion of the twisted product manifold ${}_fI \times E_s^{n-1}$ with twisted product metric (see [1, page 66] or [12]; and we arrange the indices as in (3.2)):

$$g = f^2 dx_1^2 - {dx_2}^2 - \cdots - {dx_{s+1}}^2 + {dx_{s+2}}^2 + \cdots + {dx_n}^2 \tag{3.16}$$

where $f = \phi^{-1}$ and I is the interval on which ϕ is defined. (3.16) implies

$$\begin{aligned} \nabla_{\partial/\partial x_1} \frac{\partial}{\partial x_1} &= \frac{f_1}{f}\frac{\partial}{\partial x_1} - f\sum_{k=2}^{n} \epsilon_k f_k \frac{\partial}{\partial x_k}, \\ \nabla_{\partial/\partial x_1} \frac{\partial}{\partial x_j} &= \frac{f_j}{f}\frac{\partial}{\partial x_1}, \quad \nabla_{\partial/\partial x_j} \frac{\partial}{\partial x_k} = 0, \quad j,k = 2, ..., n \end{aligned} \tag{3.17}$$

where $f_i = \partial f/\partial x_i, i = 1, ..., n$.

From (3.17) we have

$$R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_1} = f\sum_{k=2}^{n} \epsilon_k f_{jk} \frac{\partial}{\partial x_k}, \quad j = 2, ..., n \tag{3.18}$$

Since M is flat, (3.18) implies $f_{jk} = 0, j, k = 2, ..., n$. Therefore, f is given by

$$f = \beta(x_1) + \sum_{j=2}^{n} \alpha_j(x_1)x_j. \tag{3.19}$$

for some functions $\beta, \alpha_2, ..., \alpha_n$. By (3.19), (3.17) becomes

$$\begin{aligned} \nabla_{\partial/\partial x_1}\frac{\partial}{\partial x_1} &= \frac{1}{f}\{\beta'(x_1) + \sum_{j=2}^{n}\alpha_j'(x_1)x_j\}\frac{\partial}{\partial x_1} - f\sum_{k=2}^{n}\epsilon_k\alpha_k\frac{\partial}{\partial x_k}, \\ \nabla_{\partial/\partial x_1}\frac{\partial}{\partial x_j} &= \frac{\alpha_j}{f}\frac{\partial}{\partial x_1}, \quad \nabla_{\partial/\partial x_j}\frac{\partial}{\partial x_k} = 0, \quad j, k = 2, ..., n \end{aligned} \tag{3.20}$$

By (3.15), (3.20) and the formula of Gauss, we obtain

$$L_{x_1x_1} = \frac{1}{f}\{\beta'(x_1) + \sum_{j=2}^{n}\alpha_j'(x_1)x_j\}L_{x_1} - f\sum_{k=2}^{n}\epsilon_k\alpha_k L_{x_k} + iL_{x_1} \tag{3.21}$$

$$L_{x_1x_j} = \frac{\alpha_j}{f}L_{x_1} \tag{3.22}$$

$$L_{x_jx_k} = 0, \quad j, k = 2, ..., n. \tag{3.23}$$

Integrating (3.23) yields

$$L = \sum_{j=2}^{n} P_j(x_1)x_j + D(x_1), \tag{3.24}$$

for some $\mathbf{C^n_s}$ valued functions $P_2, ..., P_n, D$ of x_1. Hence,

$$L_{x_1} = \sum_{j=2}^{n} P_j'(x_1)x_j + D'(x_1), \tag{3.25}$$

$$L_{x_j} = P_j(x_1), \quad j = 2, ..., n \tag{3.26}$$

From (3.16) and (3.26), we know that $P_2, ..., P_n$ are orthonormal tangent vectors on M with $P_2, ..., P_{s+1}$ being time-like. Applying (3.22), (3.25) and (3.26), we have

$$\alpha_j(x_1)D'(x_1) = \beta(x_1)P_j'(x_1), \tag{3.27}$$

$$\alpha_j(x_1)P_k'(x_1) = \alpha_k(x_1)P_j'(x_1), \quad j, k = 2, ..., n. \tag{3.28}$$

Case (i): $\alpha_2 = \cdots = \alpha_n = 0$. From (3.27) we have $P_2'(x_1) = \cdots = P_n'(x_1) = 0$, since $\beta \neq 0$ by (3.19). Hence, $P_2, ..., P_n$ are constant vectors in $\mathbf{C^n_s}$. Therefore, (3.24) becomes

$$L(x_1, ..., x_n) = \sum_{j=2}^{n} P_j x_j + D(x_1)$$

for some function $D = D(x_1)$ and orthonormal constant vectors $P_2, ..., P_n$ in $\mathbf{C^n_s}$. This means that L is a Lagrangian cylinder over the curve $D = D(x_1)$ whose ruling are $(n-1)$-planes parallel to the totally real $x_2, ..., x_n$–plane in $\mathbf{C^n_s}$.

Case (ii): $\alpha_2 = \cdots = \alpha_{s+1} = 0$. (3.27) yields $P_2'(x_1) = \cdots = P_{s+1}'(x_1) = 0$, since $\beta \neq 0$ by (3.19). Hence, $P_2, ..., P_{s+1}$ are constant vectors in $\mathbf{C^n_s}$. Constant vectors $P_2, iP_2, ..., P_{s+1}, iP_{s+1}$ span the time-like subspace of $\mathbf{C^n_s}$. Now within the space-like subspace of $\mathbf{C^n_s}$, the function f in (3.19) becomes

$$f = \beta(x_1) + \sum_{j=s+2}^{n} \alpha_j(x_1)x_j. \tag{3.29}$$

By Theorem 5.1 in [3], the immersion is definite flat Lagrangian H-umbilical. Therefore, M is locally the product of a flat Lagrangian H-umbilical Riemannian submanifold and a time-like s-plane.

Case (iii): *At least one of the* $\alpha_2, ..., \alpha_{s+1}$ *is nonzero*. In this case, we may assume that $\alpha_2 \neq 0$. By changing the variables:

$$t = \int_0^{x_1} \alpha_2(x_1)dx_1, \quad u_2 = x_2, ..., u_n = x_n, \tag{3.30}$$

we have

$$g = \hat{f}^2 dt^2 - du_2{}^2 - \cdots - du_{s+1}{}^2 + du_{s+2}{}^2 + \cdots + du_n{}^2 \tag{3.31}$$

where

$$\hat{f} = b(t) + u_2 + \sum_{j=3}^{n} a_j(t)u_j, \tag{3.32}$$

for some functions $b(t), a_3(t), ..., a_n(t)$.
From (3.15) and (3.30) we obtain

$$h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \lambda(t)J\left(\frac{\partial}{\partial t}\right), \quad h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial u_j}\right) = h\left(\frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\right) = 0, \quad j, k = 2, ..., n. \tag{3.33}$$

where $\lambda = (\alpha_2)^{-1}$ is a function of t. By applying (3.17), (3.31), (3.32), (3.33) and the formula of Gauss, we have

$$L_{tt} = \frac{1}{\hat{f}}(b'(t) + \sum_{j=3}^{n} a_j'(t)u_j)L_{x_1} - \hat{f}\sum_{k=2}^{n} \epsilon_k a_k L_{u_k} + i\lambda L_t \tag{3.34}$$

$$L_{tu_j} = \frac{a_j}{\hat{f}} L_t \tag{3.35}$$

$$L_{u_j u_k} = 0, \quad j, k = 2, ..., n. \tag{3.36}$$

where $a_2 = 1$. Solving (3.36), we get

$$L = \sum_{j=2}^{n} P_j(t)u_j + D(t), \tag{3.37}$$

for some $\mathbf{C^n_s}$ valued functions $P_2, ..., P_n, D$ of t. Hence,

$$L_t = \sum_{j=2}^{n} P_j'(t)u_j + D'(t), \tag{3.38}$$

$$L_{u_j} = P_j(t), \quad j = 2, ..., n \tag{3.39}$$

(3.31) and (3.39) implies that $P_2, ..., P_n$ are orthonormal tangent vectors on M with $P_2, ..., P_{s+1}$ being time-like. Applying (3.35), (3.38) and (3.39), we have

$$D'(t) = b(t)P_2'(t), \quad P_k'(t) = a_k(t)P_2'(t), \quad k = 2, ..., n. \tag{3.40}$$

Substituting (3.40) into (3.38) yields

$$L_t = \hat{f}^2 P_2'(t), \tag{3.41}$$

If we put $z(t) = P_2(t)$, then $z = (t)$ can be regarded as a unit speed space-like curve $z : I \to H^{2n-1}_{2s-1}(-1) \subset \mathbf{C^n_s}$ defined on some interval I. Since L is Lagrangian, it follows from (3.38),(3.39) and (3.40) that $z = z(t)$ is a Legendre curve in $H^{2n-1}_{2s-1}(-1) \subset \mathbf{C^n_s}$. Now, by applying (3.38), (3.39) and (3.40) we see that $z(s), iz(s), z'(s), iz'(s), P_3(s), iP_3(s)...P_n(s), iP_n(s)$ form an orthonormal frame field, where $P_3, ..., P_{s+1}$ are time-like and $P_3, ..., P_n$ are parallel normal vector fields along the Legendre curve. Moreover, (3.37) and (3.41) imply that, up to rigid motions of $\mathbf{C^n_s}$, L is given by

$$L(t, u_2, ..., u_n) = u_2 z(t) + \sum_{j=3}^{n} u_j P_j(t) + \int^t b(t)z'(t)dt \tag{3.42}$$

Finally, we conclude from (3.34), (3.38), (3.39), (3.40) and (3.42) that $z = z(t)$ satisfies (2.3). Hence $z = z(t)$ in (3.42) is a special Legendre curve in $H^{2n-1}_{2s-1}(-1) \subset \mathbf{C^n_s}$. ■

Case 2: e_1 *is time-like*. In this case, we arrange the indices as the following

$$\langle e_1, e_1\rangle = \langle e_{n-s+2}, e_{n-s+2}\rangle = \cdots = \langle e_n, e_n\rangle = -1; \quad \langle e_2, e_2\rangle = \cdots = \langle e_{n-s+1}, e_{n-s+1}\rangle = 1 \tag{3.43}$$

If we put

$$\epsilon_k = 1, k = 2, ..., n-s+1; \quad \epsilon_k = -1, k = 1, n-s+2, ..., n,$$

then (3.43) becomes $< e_k, e_k >= \epsilon_k$.

Theorem 3.3. *Let* $\lambda, b, a_3, ..., a_n$ *be real-valued functions on an open interval* I *with* λ *being nowhere zero and let* $z : I \to S^{2n-1}_{2s}(1) \subset \mathbf{C^n_s}$ *be a time-like special Legendre curve satisfying (2.3). Put*

$$f(t, u_2, ..., u_n) = b(t) + u_2 + \sum_{j=3}^{n} a_j(t)u_j. \tag{3.44}$$

Denote by $\hat{M}_2(0)$ *the twisted product manifold* ${}_fI \times E^{n-1}_{s-1}$ *with twisted product metric given by*

$$g = -f^2dt^2 + du_2{}^2 + \cdots + du_{n-s+1}{}^2 - du_{n-s+2}{}^2 - \cdots - du_n{}^2 \tag{3.45}$$

Then $\hat{M}_2(0)$ *is a flat pseudo-Riemannian manifold and*

$$L(t, u_2, ..., u_n) = u_2z(t) + \sum_{j=3}^{n} u_jP_j(t) + \int^t b(t)z'(t)dt \tag{3.46}$$

defines a Lagrangian H-umbilical isometric immersion $L : \hat{M}_2(0) \to \mathbf{C^n_s}$.

Conversely, up to rigid motions of $\mathbf{C^n_s}$*, every flat Lagrangian H-umbilical submanifold without totally geodesic points in* $\mathbf{C^n_s}$ *and with time-like* e_1 *is locally either a Lagrangian cylinder over a curve, a product of a definite flat Lagrangian H-umbilical submanifold and a (n-s)-plane, or a Lagrangian submanifold obtained in the way described above.*

Proof. If we follow the indices as given in (3.43), then the proof is almost the same as the proof in Theorem 3.2. For simplicity we omit the details. ■

Remark 3.1. Flat Lagrangian H-umbilical submanfolds in complex Euclidean spaces C^n were completely classified in [3] and [4].

Remark 3.2. Legendre curves $z : I \to S^3_2(1) \subset C^2_1$ (or $z : I \to H^3_1(-1) \subset C^2_1$) are special and they can be considered as special Legendre curves in $\mathbf{C^n_s}$. The proof of the existence of special Legendre curves in [4] is also true for indefinite case.

Remark 3.3. Lemma 3.1, Theorem 3.2 and Theorem 3.3 provide the complete classification of flat Lagrangian H-umbilical submanifolds in $\mathbf{C^n_s}$. The proof is also valid for $n = 2$, the results correspond to Case (5) and Case (6) in [8] except for the order of the variables.

References

[1] Chen, B.-Y., Geometry of submanifolds and its applications, (Science University of Tokyo, Tokyo, Japan 1981).
[2] Chen, B.-Y., Interaction of Legendre curves and Lagrangian submanifolds, Israel J. Math. **99** (1997), 69–108.
[3] Chen, B.-Y., Complex extensors and Lagrangian submanifolds in complex Euclidean spaces, Tohoku Math. J. **49** (1997), 277–297.
[4] Chen, B.-Y., Representation of flat Lagrangian H-umbilical submanifolds in complex Euclidean spaces, Tohoku Math. J. **51** (1999), 13–20.
[5] Chen, B.-Y., Complex extensors and Lagrangian submanifolds in indefinite complex Euclidean spaces, Bulletin Math. Inst. Academia Sinica **31** (2003), 151-179.
[6] Chen, B.-Y., Pseudo-Riemannian geometry, δ-invariants and applications, (World Scientific Publications, Hackensack, New Jersey, 2011).
[7] Chen, B.-Y., A construction method of Lagrangian surfaces in complex pseudo Euclidean plane C^2_1 and its applications, Int. Electron. J. Geom. **7** (2014), 4-25.
[8] Chen, B.-Y. and Fastenakels, J., Classification of flat Lagrangianl surfaces in complex Lorentzian plane, Acta Mathematica Sinica, **23**, No.12 (2007), 2111-2144.

[9] Deng, S., Lagrangian H-umbilical surfaces in complex Lorentzian plane, Int. Electron. J. Geom. **9**, No. 2 (2016), 87-93.
[10] Deng, S., Classification of Lagrangian H-umbilical surfaces of constant curvature in complex Lorentzian plane, Int. Electron. J. Geom. **10**, No. 1 (2017), 48-57.
[11] B. O'Neill, Semi-Riemannian geometry with applications to relativity, (Academic Press, New York, 1983).
[12] R. Ponge and H. Reckziegel, Twisted products in pseudo-Riemannian geometry, Geometriae Dedicata **48**, (1993), 15-25.
[13] K. Yano and M. Kon, Structures on manifolds, (World Scientific Publishing Co., Singapore, 1984).

Affiliations

SHANGRONG DENG
ADDRESS: Department of Mathematics, Kennesaw State University, 1100 South Marietta Parkway, Marietta, GA 30060, U.S.A.
E-MAIL: sdeng@kennesaw.edu