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SPECTRAL PROPERTIES OF THE SECOND ORDER DIFFERENCE EQUATION WITH SELFADJOINT OPERATOR COEFFICIENTS

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Abstract. In this paper, we consider the second order difference equation defined on the whole axis with selfadjoint operator coefficients. The main objective of this study is to obtain the continuous and discrete spectrum of the discrete operator which is generated by this difference equation. To achieve this, we first obtain the Jost solutions of this equation explicitly and then examine the analytical and asymptotic properties of these solutions. With the help of these properties, we find the continuous and discrete spectrum of this operator. Finally we obtain a sufficient condition which ensures that this operator has a finite number of eigenvalues.

1. Introduction

Difference equations play a very important role on modelling of problems related to physics, chemistry, biology, finance, economics, probability, engineering etc. Difference equations also arise when approximating continuous models and differential equations using numerical methods. Selfadjoint differential operators such as Sturm-Liouville, Dirac and Klein-Gordon operators are used in functional analysis and quantum mechanics and the spectral analysis of these operators have been studied (see [14, 17, 18]). There are also many studies on the spectral analysis of both selfadjoint and non-selfadjoint discrete operators defined by difference equations (see [1, 2, 3] and references therein). Besides, spectral analysis of the selfadjoint differential and difference equations with matrix coefficients are studied in [6, 8, 10]. In particular, in [4] the authors investigated the spectral properties of the discrete operator generated by selfadjoint matrix-valued difference equation of second order defined on the half-axis. Namely, they considered the discrete operator

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L_0 generated by the difference equation with matrix coefficients

$$A_{n-1}Y_{n-1}+B_nY_n+A_nY_{n+1}=\lambda Y_n, n\in\mathbb{N}, \tag{1}$$

and the boundary condition $Y_0=0$, where $A_n\,(n\in\mathbb{N}\cup\{0\})$ and $B_n\,(n\in\mathbb{N})$ are $m\times m$ selfadjoint matrices ($m<\infty$), $detA_n\neq 0\,(n\in\mathbb{N}\cup\{0\})$ and λ is a spectral parameter. The domain of this operator is denoted by $l_2\left(\mathbb{N},\mathbb{C}^m\right)$ which is the Hilbert space of all vector sequences $Y=(Y_n)_{n\in\mathbb{N}}$ such that $Y_n\in\mathbb{C}^m$ and $\sum_{n=1}^{\infty}\|Y_n\|^2<\infty$. The inner product in $l_2\left(\mathbb{N},\mathbb{C}^m\right)$ is defined as

$$(Y,Z):=\sum_{n=1}^{\infty}\left(Y_n,Z_n\right).$$

Note that Equation (1) can be written in Sturm-Liouville form

$$\Delta\left(A_{n-1}\Delta Y_{n-1}\right)+Q_nY_n=\lambda Y_n, n\in\mathbb{N},$$

where $Q_n=A_{n-1}+A_n+B_n$ and Δ is the forward difference operator. The authors obtained the continuous and discrete spectrum of L_0 [4]. Further, in [7] the authors considered the same difference equation with non-selfadjoint matrix coefficients and examined the continuos spectrum, eigenvalues and spectral singularities of the resulting non-selfadjoint discrete operator. They proved the finiteness of the eigenvalues and spectral singularities of the operator under the condition

$$\sum_{n=1}^{\infty}n\left(\|I-A_n\|+\|B_n\|\right)<\infty.$$

Furthermore, in [5] the authors extended the results in [4] to the whole axis by considering the Equation (1) for $n\in\mathbb{Z}$. They obtained the Jost solutions of this equation and also the discrete and continuous spectrum of the discrete operator generated by this equation. They proved that the operator has a finite number of eigenvalues and spectral singularities if the coefficients satisfy

$$\sum_{n=-\infty}^{\infty}|n|\left(\|I-A_n\|+\|B_n\|\right)<\infty.$$

Let H be a separable Hilbert space ($\dim H\leq\infty$) and $L_2\left(\mathbb{R}_+,H\right)$ denote the space of vector-valued functions $f(x)$ ($0\leq x<\infty$) which are strongly-integrable in each finite subinterval of $[0,\infty)$ such that $\int_0^\infty|f(x)|^2\,dx<\infty$. Consider the differential expression in $L_2\left(\mathbb{R}_+,H\right)$

$$l_0(Y)=-Y^{''}+Q(x)Y,\ 0<x<\infty, \tag{2}$$

where $Q(x)$ is a selfadjoint, completely continuous operator in H for each $x\in(0,\infty)$. In [9, 12, 13, 15], the authors have studied the discrete spectrum of the Sturm-Liouville operator l_0 generated by (2) and the boundary condition $Y\left(0\right)=0$.

In this paper, we consider the discrete analogue of the operator l_0 and call it the discrete Sturm-Liouville operator which will be denoted by L hereafter. We investigate the spectral properties of the discrete Sturm-Liouville operator L on the whole axis with selfadjoint operator coefficients. In particular, we find Jost solutions of L and obtain the continuous and point spectrum of L. We also show that L has a finite number of eigenvalues under a condition on the coefficients.

2. Some properties and Jost solutions of the operator L

In this section we specify the properties of the discrete Sturm-Liouville operator on the whole axis. Let H be a separable Hilbert space and $H_1 = l_2(\mathbb{N}, H)$ denote the space of vector sequences $y = (y_n)_{n\in\mathbb{N}}$ $(y_n \in H,\ n \in \mathbb{N}\ )$ such that $\|y\|_1 := \sum_{n=-\infty}^{\infty} \|y_n\|_H^2 < \infty$. H_1 is a Hilbert space with inner product

$$(y, z)_1 = \sum_{n=-\infty}^{\infty} (y_n, z_n)_H \,.$$

Consider the difference expression in H_1

$$l(y)_n = A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1},\ n \in \mathbb{Z}, \tag{3}$$

where $A_n, B_n\,(n \in \mathbb{Z})$ are selfadjoint operators in H and $A_n - I, B_n\,(n \in \mathbb{Z})$ are completely continuous operators in H. We also assume A_n is invertible for each $n \in \mathbb{Z}$. We consider the operator L generated by (3). We can also define the operator L by using the infinite Jacobi matrix

$$(J)_{ij} = \begin{cases} B_i, & i = j, \\ A_{i-1}, & i = j+1, \\ A_i, & i = j-1, \\ 0, & \text{otherwise} \end{cases}$$

It is obvious that the operator L is selfadjoint in H_1. We will examine the difference equation

$$A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N}. \tag{4}$$

We shall also consider the equation

$$A_{n-1}Y_{n-1} + B_n Y_n + A_n Y_{n+1} = \lambda Y_n,\ n \in \mathbb{N}, \tag{5}$$

where Y_n is an operator sequence i.e, Y_n is an operator in H for each $n \in \mathbb{N}$.

Lemma 1. *Every sequence of solutions of (4) can be represented as an operator sequence which satisfies (5). Conversely, one can construct a sequence of vector sequences which satisfies (4) for a given operator solution of (5).*

Proof. Since H is a separable Hilbert space, there exists an orthonormal basis $(u_m)_{m\in\mathbb{N}}$. Suppose vector sequences $y_m = \left(y_m^i\right)_{i\in\mathbb{Z}}$ satisfy Equation (4) for each $m \in \mathbb{N}$. We can construct an operator sequence $Y = (Y_n)_{n\in\mathbb{Z}}$ such that $Y_n u_m = (y_m^n)_{n\in\mathbb{Z}}$ for every $m \in \mathbb{N}$. It is obvious that $Y_n u_m = y_m$ and Y satisfies the Equation (5).

Conversely, suppose an operator sequence $Y = (Y_n)_{n\in\mathbb{Z}}$ satisfies (5). Let $z_m := (z_m^n)_{n\in\mathbb{Z}} = Y_n u_m$ for every $m \in \mathbb{N}$. Then it is clear that $z_m = (z_m^n)_{n\in\mathbb{Z}}$ satisfy Equation (4) for every $m \in \mathbb{N}$. □

Note that from Lemma 1, we have one-to-one correspondence between the operator solutions of Equation (5) and sequences of solutions of Equation (4). Hence we can consider and examine both equations.
Let us assume

$$\sum_{n=-\infty}^{\infty} \left(\|I - A_n\| + \|B_n\|\right) < \infty. \tag{6}$$

Let $E(z) := (E_n(z))_{n\in\mathbb{Z}}$ and $F(z) := (F_n(z))_{n\in\mathbb{Z}}$ denote the operator solutions of the equation

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = \left(z + \frac{1}{z}\right) Y_n,\ n \in \mathbb{Z}, \tag{7}$$

satisfying the conditions

$$\lim_{n\to\infty} E_n(z)z^{-n} = I,\ z \in D_0 := \{z \in \mathbb{C} :\ |z| = 1\},$$

and

$$\lim_{n\to\infty} F_n(z)z^{n} = I,\ z \in D_0,$$

respectively. $E(z)$ and $F(z)$ are called the Jost solutions of Equation (7). Note that these solutions are bounded.

Theorem 2. *Under the condition (6), the solutions $E(z)$ and $F(z)$ exist and have the representations*

$$\begin{aligned} E_n(z) &= z^n I + \\ &\sum_{k=n+1}^{\infty} \frac{z^{k-n} - z^{n-k}}{z - z^{-1}} \left[(I - A_{k-1})\, E_{k-1}(z) - B_k E_k(z) + (I - A_k)\, E_{k+1}(z)\right], \end{aligned}$$

$$\begin{aligned} F_n(z) &= z^{-n} I + \\ &\sum_{k=-n+1}^{\infty} \frac{z^{k+n} - z^{-n-k}}{z - z^{-1}} \left[(I - A_{k-1})\, F_{k-1}(z) - B_k F_k(z) + (I - A_k)\, F_{k+1}(z)\right]. \end{aligned}$$

Now, suppose that

$$\sum_{n=-\infty}^{\infty} |n| \left(\|I - A_n\| + \|B_n\|\right) < \infty, \tag{8}$$

holds.

Theorem 3. *Under the condition (8), the Jost solutions* $(E_n(z)), (F_n(z))$ $(n \in \mathbb{Z})$ *have the represantations* ********************

$$E_n(z) = T_n z^n \left[I + \sum_{m=1}^{\infty} K_{n,m} z^m \right], \quad n \in \mathbb{Z},$$

$$F_n(z) = R_n z^{-n} \left[I + \sum_{-\infty}^{m=-1} L_{n,m} z^{-m} \right], \quad n \in \mathbb{Z},$$

where $T_n, R_n, K_{n,m}$ *and* $L_{n,m}$ *are obtained in terms of* A_n *and* B_n*. Further*

$$\|K_{n,m}\| \leq c \sum_{p=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(\|I - A_p\| + \|B_p\| \right), m \in \mathbb{Z}_+,$$

$$\|L_{n,m}\| \leq d \sum_{-\infty}^{p=n+\left[\left|\frac{m}{2}\right|\right]} \left(\|I - A_p\| + \|B_p\| \right), m \in \mathbb{Z}_-,$$

hold where $c, d > 0$ *are constants. Thus,* $(E_n(z))$ *and* $(F_n(z))$ *have analytic continuations from* D_0 *to* $D_1 := \{z \in \mathbb{C} : \ |z| < 1\} \setminus \{0\}$.

Theorem 4. *Under the condition (8), the Jost solutions satisfy the following asymptotic relations for* $z \in D := \{z \in \mathbb{C} : \ |z| \leq 1\} \setminus \{0\}$

$$E_n(z) = z^n \left[I + o(1) \right] \ , \ n \to \infty,$$
$$F_n(z) = z^{-n} \left[I + o(1) \right] \ , \ n \to -\infty.$$

Remark 5. *The proofs of above theorems are omitted since they are similar to the matrix coefficient case which have been obtained in* [4, 5].

3. Continuous and discrete spectrum of L

Let us introduce the equation

$$Y_{n-1}A_{n-1} + Y_n B_n + Y_{n+1} A_n = \left(z + z^{-1} \right) Y_n, \ n \in \mathbb{N}. \tag{9}$$

It can be shown similarly that equation (9) has a solution $H(z) := (H_n(z))_{n \in \mathbb{Z}}$ such that

$$\lim_{n \to \infty} H_n(z) z^n = I, \ z \in D_0$$

holds. Indeed, the solution $H(z)$ is the adjoint of the operator solution $F(z)$ i.e., $H_n(z) = (F_n(z))^*, \ n \in \mathbb{Z}$.

Definition 6. *Let* U_n *and* V_n *be operator solutions of the Equations (5) and (9), respectively. The Wronskian of* U_n *and* V_n *is defined by*

$$(W[U,V])_n := V_{n-1} A_{n-1} U_n - V_n A_{n-1} U_{n-1}.$$

Lemma 7. *Let* U_n *be an operator solution of (7) and* V_n *be an operator solution of (9). Then, the Wronskian of these solutions is constant i.e., independent of* n.

Proof. We have the equalities

$$A_{n-1}U_{n-1} + B_nU_n + A_nU_{n+1} = \left(z + z^{-1}\right) U_n,$$

$$V_{n-1}A_{n-1} + V_nB_n + V_{n+1}A_n = \left(z + z^{-1}\right) V_n.$$

If we multiply the first equality with V_n from the left and the second equality with $-U_n$ from the right we get

$$V_nA_{n-1}U_{n-1} - V_{n-1}A_{n-1}U_n + V_nA_nU_{n+1} - V_{n+1}A_nU_n = 0, \tag{10}$$

by adding two equalities. Let $H_n := (W[U,V])_n = V_{n-1}A_{n-1}U_n - V_nA_{n-1}U_{n-1}$. From (10), we have

$$\Delta H_n = H_{n+1} - H_n = V_nA_nU_{n+1} - V_{n+1}A_nU_n - V_{n-1}A_{n-1}U_n + V_nA_{n-1}U_{n-1} = 0,$$

which implies $W[U,V]$ is constant. □

From Lemma 7 it easily follows that

$$W[E(z), H(z)] = G_0(z)\, A_0E_1(z) - G_1(z)\, A_0E_0(z).$$

Let us define $T(z) := W[E(z), H(z)]$ for $z \in D$. $T(z)$ is called the Jost function of L. Now we obtain the continuous spectrum of L.

Theorem 8. *Under the condition (8), the continuous spectrum of L is $\sigma_c(L) = [-2,2]$.*

Proof. Let L_0 and L_1 denote the operators generated in $H_1 = l_2(\mathbb{Z}, H)$ by the difference expressions

$$L_0(y)_n = y_{n-1} + y_{n+1},\ n \in \mathbb{Z},$$

and

$$L_1(y)_n = (A_{n-1} - I)\, y_{n-1} + B_ny_n + (A_n - I)\, y_{n+1},\ n \in \mathbb{Z},$$

respectively. We can also define the operators L_0 and L_1 by using the infinite Jacobi matrices

$$(J_0)_{ij} = \begin{cases} I, & i = j+1 \text{ } or \text{ } i = j-1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$(J_1)_{ij} = \begin{cases} B_i, & i = j, \\ A_i - I, & i = j-1, \\ A_{i-1} - I, & i = j+1, \\ 0, & \text{otherwise} \end{cases}$$

respectively. We have $L = L_0 + L_1, L_0 = L_0^*$ and $\sigma(L_0) = \sigma_c(L_0) = [-2,2]$ (see [19]). It is well known that L_1 is a compact operator iff L_1 is bounded and the set $R = \{L_1y : \|y\|_1 \leq 1\}$ is compact in H_1. It is obvious that L_1 is bounded. Moreover, if we use the compactness criteria in l_p spaces (see [16] (p. 167)) we obtain the compactness of R. Indeed, let $\|y\|_1 \leq 1$. Then (8) implies that for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\sum_{i=n+1}^{\infty} \left(\|(A_i - I)\| + \|B_i\|\right) < \frac{\epsilon}{C}.$$

Now we have

$$\begin{aligned}
\sum_{i=n+1}^{\infty} \|(L_1 y)_i\|_H^2 &= \sum_{i=n+1}^{\infty} \|(A_{i-1} - I)\, y_{i-1} + B_i y_i + (A_i - I)\, y_{i+1}\|_H^2 \\
&\leq \sum_{i=n+1}^{\infty} \left(\|(A_{i-1} - I)\|^2 \|y_{i-1}\|_H^2 + \|B_i\|^2 \|y_i\|_H^2 + \|(A_i - I)\|^2 \|y_{i+1}\|^2\right) \\
&\leq \|y\|_1^2 \sum_{i=n+1}^{\infty} \left(\|(A_{i-1} - I)\|^2 + \|B_i\|^2 + \|(A_i - I)\|^2\right) \\
&\leq \sum_{i=n+1}^{\infty} \left(2\,\|(A_i - I)\|^2 + \|B_i\|^2\right) \\
&\leq \sum_{i=n+1}^{\infty} \left(C_1 \|(A_i - I)\| + C_2 \|B_i\|\right) \\
&\leq \sum_{i=n+1}^{\infty} C\left(\|(A_i - I)\| + \|B_i\|\right) \\
&< \varepsilon,
\end{aligned}$$

where

$$C_1 = \frac{1}{2} \sup_{i \in \mathbb{N}} \|(A_i - I)\|, \quad C_2 = \sup_{i \in \mathbb{N}} \|B_i\|$$

and $C = C_1 + C_2$. Thus, we proved L_1 is a compact operator in H_1. By Weyl Theorem of Compact Perturbation [11], we have

$$\sigma_c(L) = \sigma_c(L_0) = [-2, 2]\,.$$

□

Since the operator L is selfadjoint, all eigenvalues of L are real. Note that from the definition of discrete spectrum and Theorem 8 we have

$$\sigma_d(L) \subset (-\infty, -2] \cup [2, \infty)\,. \tag{11}$$

Further, from the definition of eigenvalues we find

$$\sigma_d(L) = \left\{\lambda :\ \lambda = z + \frac{1}{z},\ z \in (-1, 0) \cup (0, 1)\,,\ T(z) \text{ is not invertible}\right\}.$$

Theorem 9. *Under the condition (8), L has a finite number of eigenvalues.*

Proof. From (11), it follows that the limit points of the set $\sigma_d(L)$ could only be $\pm 2, \pm\infty$. If $\lambda = \pm\infty$ is a limit point of $\sigma_d(L)$ then it implies that L is unbounded operator which gives a contradiction. On the other hand, if $\lambda = 2$ is a limit point of $\sigma_d(L)$ then there exists an eigenvalue in the neighbourhood $[2-\varepsilon, 2)$ for sufficiently small $\varepsilon > 0$. From Theorem 8 we have $\sigma_c(L) = [-2, 2]$ and it is well known that for a selfadjoint operator $\sigma_d(L) \nsubseteq \sigma_c(L)$. Hence there can't be any eigenvalue in $[2-\varepsilon, 2)$ which means $\lambda = 2$ is not a limit point of $\sigma_d(L)$. Similarly, $\lambda = -2$ can not be a limit point of $\sigma_d(L)$. As a result, the set of eigenvalues has no limit point and therefore should have a finite number of elements by Bolzano-Weierstrass Theorem. □

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