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Research Article

Quasi-n-absorbing and semi-n-absorbing preradicals

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Abstract

The aim of this paper is to introduce the notions of quasi-n-absorbing preradicals and of semi-n-absorbing preradicals. These notions are inspired by applying the concept of n-absorbing preradicals to semiprime preradicals. Also, we study the concepts of quasi-n-absorbing submodules and of semi-n-absorbing submodules and their relations with quasi-n-absorbing preradicals and semi-n-absorbing preradicals.

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1. Introduction

The notion of 2-absorbing ideals of commutative rings was introduced by Badawi in [2], where a proper ideal I of a commutative ring R is called a *2-absorbing ideal of* R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that I is a 2-absorbing ideal of R if and only if whenever I_1, I_2, I_3 are ideals of R with $I_1I_2I_3 \subseteq I$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. Anderson and Badawi [1] generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an *n-absorbing* (resp. *strongly n-absorbing*) *ideal* if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \dots, I_{n+1}$ of R), then there are n of the x_i's (resp. n of the I_i's) whose product is in I. In [20], the concept of 2-absorbing ideals was generalized to submodules of a module over a commutative ring. Let M be an R-module and N be a submodule of M. N is said to be a *2-absorbing submodule of* M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. In [13], Raggi et al. introduced the concepts of prime preradicals and prime submodules over noncommutative rings. The generalized notions of these, "2-absorbing preradicals" and "2-absorbing submodules" were investigated by Yousefian and Mostafanasab in [19]. Raggi et al. [14] defined the notions of semiprime preradicals and semiprime submodules. In this paper, we give the concepts of "quasi-n-absorbing preradicals" and "semi-n-absorbing

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preradicals". Also, investigation of "quasi-n-absorbing submodules" and "semi-n-absorbing submodules" is in this paper.

2. Preliminaries

Throughout this paper, R is an associative ring with identity, and R-Mod denotes the category of all the unitary left R-modules. A ring R is said to be left V-ring if all simple R-modules are injective. We denote by R-simp a complete set of representatives of isomorphism classes of simple left R-modules. We recall that R is a left local ring if and only if $|R\text{-simp}| = 1$. For $M \in R$-Mod, we denote by $\mathrm{E}(M)$ the injective hull of M. Let $U, N \in R$-Mod, we say that N is *generated by* U (or N is U-*generated*) if there exists an epimorphism $U^{(\Lambda)} \to N$ for some index set Λ. Dually, we say that N is *cogenerated by* U (or N is U-*cogenerated*) if there exists a monomorphism $N \to U^{\Lambda}$ for some index set Λ. Also, we say that an R-module X is *subgenerated by* M (or X is M-*subgenerated*) if X is a submodule of an M-generated module. The category of M-subgenerated modules (the Wisbauer category) is denoted $\sigma[M]$ (see [17]). A *preradical* over the ring R is a subfunctor of the identity functor on R-Mod. Denote by R-pr the class of all preradicals over R. There is a natural partial ordering in R-pr given by $\sigma \preceq \tau$ if $\sigma(M) \leq \tau(M)$ for every $M \in R$-Mod. It is proved in [10] that with this partial ordering, R-pr is an atomic and co-atomic big lattice. The smallest and the largest elements of R-pr are denoted 0 and 1, respectively.

Let $M \in R$-Mod. Recall ([6] or [10]) that a submodule N of M is called *fully invariant* if $f(N) \leq N$ for each R-homomorphism $f : M \to M$. In this paper, the notation $N \leq_{fi} M$ means that "N is a fully invariant submodule of M". Obviously the submodule K of M is fully invariant if and only if there exists a preradical τ of R-Mod such that $K = \tau(M)$. If $N \leq M$, then the preradicals α_N^M and ω_N^M are defined as follows: For $K \in R$-Mod,

(1) $\alpha_N^M(K) = \sum\{f(N)|f \in \mathrm{Hom}_R(M, K)\}$.
(2) $\omega_N^M(K) = \bigcap\{f^{-1}(N)|f \in \mathrm{Hom}_R(K, M)\}$.

Using the preradicals α_N^M and ω_N^M, in the works [5], [7] and [13] , two operations were introduced and studied.

(1) α-*product* of submodules $K,\ N \leq M$: $K \cdot N = \alpha_K^M(N)$.
(2) ω-*product* of submodules $K,\ N \leq M$: $K \odot N = \omega_K^M(N)$.

Notice that for $\sigma \in R$-pr and $M,\ N \in R$-Mod we have that $\sigma(M) = N$ if and only if $N \leq_{fi} M$ and $\alpha_N^M \preceq \sigma \preceq \omega_N^M$. We have also that if $K \leq N \leq M$ with $K,\ N \leq_{fi} M$, then $\alpha_K^M \preceq \alpha_N^M$ and $\omega_K^M \preceq \omega_N^M$.

The atoms and coatoms of R-pr are, respectively, $\{\alpha_S^{E(S)}|S \in R\text{-simp}\}$ and $\{\omega_I^R|I \text{ is a maximal ideal of } R\}$ (See [10, Theorem 7]).

There are four classical operations in R-pr, namely, $\wedge, \vee, \cdot$ and $:$ which are defined as follows. For $\sigma,\ \tau \in R$-pr and $M \in R$-Mod:

(1) $(\sigma \wedge \tau)(M) = \sigma M \cap \tau M$,
(2) $(\sigma \vee \tau)(M) = \sigma M + \tau M$,
(3) $(\sigma\tau)(M) = \sigma(\tau M)$ and
(4) $(\sigma : \tau)(M)$ is determined by $(\sigma : \tau)(M)/\sigma M = \tau(M/\sigma M)$.

The meet $\wedge$ and join $\vee$ can be defined for arbitrary families of preradicals as in [10]. The operation defined in (3) is called *product*, and the operation defined in (4) is called *coproduct*. It is easy to show that for $\sigma,\ \tau \in R$-pr, $\sigma\tau \preceq \sigma \wedge \tau \preceq \sigma \vee \tau \preceq (\sigma : \tau)$.

We denote $\sigma\sigma \cdots \sigma$ (n times) by σ^n. Recall that $\sigma \in R$-pr is an *idempotent* if $\sigma^2 = \sigma$, while σ is a *radical* if $(\sigma : \sigma) = \sigma$. We say that σ is *nilpotent* if $\sigma^n = 0$ for some $n \geq 1$. Also σ is called a t-*radical* if $\sigma = \alpha_I^R$ for some ideal I of R. Note that σ is a radical if and only if, $\sigma(M/\sigma(M)) = 0$ for each $M \in R$-Mod. Furthermore, σ is a t-radical if and only if, for each $M \in R$-Mod, $\sigma(M) = \sigma(R)M$.

For any $\sigma \in R$-pr, we will use the following class of R-modules:

$$\mathbb{F}_\sigma = \{M \in R\text{-Mod} \mid \sigma(M) = 0\}.$$

Let $\sigma \in R$-pr. By [10, Theorem 8.2], the following classes of modules are closed under taking arbitrary meets and arbitrary joins:

$\mathcal{A}_a = \{\tau \in R\text{-pr} \mid \tau\sigma = 0\}$.

$\mathcal{A}_c = \{\tau \in R\text{-pr} \mid (\sigma : \tau) = \sigma\}$.

As in [11], we define, for $\sigma \in R$-pr, the following preradicals:

$a(\sigma) = \bigvee\{\tau \in \mathcal{A}_a\}$ = the annihilator of σ.

$c(\sigma) = \bigvee\{\tau \in \mathcal{A}_c\}$ = the co-equalizer of σ.

Clearly, $a(\sigma)\sigma = 0$ and $(\sigma : c(\sigma)) = \sigma$.

In [13], Raggi et al. defined the notions of prime preradicals and prime submodules as follows:

Let $\sigma \in R$-pr. σ is called *prime in* R-pr if $\sigma \neq 1$ and for any $\tau,\ \eta \in R$-pr, $\tau\eta \preceq \sigma$ implies that $\tau \preceq \sigma$ or $\eta \preceq \sigma$. Let $M \in R$-Mod and let $N \neq M$ be a fully invariant submodule of M. The submodule N is said to be *prime in* M if whenever $K,\ L$ are fully invariant submodules of M with $K \cdot L \leq N$, then $K \leq N$ or $L \leq N$. Also, Raggi et al. [14] defined a preradical σ *semiprime in* R-pr if $\sigma \neq 1$ and for any $\tau \in R$-pr, $\tau^2 \preceq \sigma$ implies that $\tau \preceq \sigma$. They said that a proper fully invariant submodule N of M is *semiprime in* M if whenever K is a fully invariant submodule of M with $K \cdot K \leq N$, then $K \leq N$. In the special case, M is a *prime (resp. semiprime) module* if its zero submodule 0 is a prime (resp. semiprime) submodule.

Yousefian and Mostafanasab [19] introduced the notions of 2-absorbing preradicals and 2-absorbing submodules. Also, in [18] they defined the notions of co-2-absorbing preradicals and co-2-absorbing submodules. The preradical $\sigma \in R$-pr is called 2-*absorbing* if $\sigma \neq 1$ and, for each $\eta, \mu, \nu \in R$-pr, $\eta\mu\nu \preceq \sigma$ implies that $\eta\mu \preceq \sigma$ or $\eta\nu \preceq \sigma$ or $\mu\nu \preceq \sigma$. More generally, a preradical $1 \neq \sigma$ in R-pr is said to be an *n-absorbing preradical* if whenever $\eta_1\eta_2\dots\eta_{n+1} \preceq \sigma$ for $\eta_1, \eta_2, \dots, \eta_{n+1} \in R$-pr, there are $i_1, i_2, \dots, i_n \in \{1, 2, \dots, n+1\}$ such that $i_1 < i_2 < \cdots < i_n$ and $\eta_{i_1}\eta_{i_2}\dots\eta_{i_n} \preceq \sigma$. They denoted by R-Ass the class of all R-modules M that the operation α-product is associative over fully invariant submodules of M, i.e., for any fully invariant submodules K, N, L of M, $(K \cdot N) \cdot L = K \cdot (N \cdot L)$. So we denote $(K \cdot N) \cdot L$ simply by $K \cdot N \cdot L$. In the special case $K \cdot K \cdots K$ (n times) is denoted by K^n. By Proposition 5.6 of [3], we can see that if an R-module M is projective in $\sigma[M]$, then $M \in R$-Ass; in particular $R \in R$-Ass. Let $M \in R$-Ass and let $N \neq M$ be a fully invariant submodule of M. The submodule N is said to be 2-*absorbing in* M if whenever J, K, L are fully invariant submodules of M with $J \cdot K \cdot L \leq N$, then $J \cdot K \leq N$ or $J \cdot L \leq N$ or $L \cdot K \leq N$. A generalization of 2-absorbing submodules is that the submodule N is said *n-absorbing in* M if whenever $K_1 \cdot K_2 \cdots K_{n+1} \leq N$ for fully invariant submodules $K_1, K_2, \dots, K_{n+1}$ of M, there are $i_1, i_2, \dots, i_n \in \{1, 2, \dots, n+1\}$ such that $i_1 < i_2 < \cdots < i_n$ and $K_{i_1} \cdot K_{i_2} \cdots K_{i_n} \leq N$. We say that a preradical $1 \neq \sigma \in R$-pr is called a *quasi-n-absorbing preradical* if whenever $\mu^n\nu \preceq \sigma$ for $\mu, \nu \in R$-pr, then $\mu^n \preceq \sigma$ or $\mu^{n-1}\nu \preceq \sigma$. A preradical $1 \neq \sigma \in R$-pr is called a *semi-n-absorbing preradical* if whenever $\mu^{n+1} \preceq \sigma$ for $\mu \in R$-pr, then $\mu^n \preceq \sigma$. Let $M \in R$-Ass. We say that a proper fully invariant submodule N of M is *quasi-n-absorbing in* M if for every fully invariant submodules $K,\ L$ of M, $K^n \cdot L \leq N$ implies that $K^n \leq N$ or $K^{n-1} \cdot L \leq N$. A proper fully invariant submodule N of M is called *semi-n-absorbing in* M if for every fully invariant submodule K of M, $K^{n+1} \leq N$ implies that $K^n \leq N$. Notice that for every ideals I_1 and I_2 of R, $I_1 \cdot I_2 = \alpha^R_{I_1}(I_2) = I_1I_2$. Therefore, an ideal I of R is a quasi-n-absorbing submodule of ${}_RR$ if and only if for any ideals $I_1,\ I_2$ of R, $I_1^nI_2 \leq I$ implies that $I_1^n \leq I$ or $I_1^{n-1}I_2 \leq I$. Also, I is a semi-n-absorbing submodule of ${}_RR$ if and only if for any ideal J of R, $J^{n+1} \leq I$ implies that $J^n \leq I$. An R-module M is said to be satisfies the *α-property* if $\tau(M) \cdot \eta(M) = (\tau\eta)(M)$ for every $\tau,\ \eta \in R$-*pr*, [19].

A ring R is called *left hereditary* if all of its left ideals are projective (see [8]).

Corollary 2.1 ([19, Corollary 2.5]). *Let R be a left hereditary ring. Then R satisfies the α-property.*

We recall the definition of relative mono-injectivity (see [16]). Let M and N be modules. N is said to be *mono-M-injective* if, for any submodule K of M and any monomorphism $f : K \to N$, there exists a homomorphism $g : M \to N$ with $g|_K = f$.

Proposition 2.2 ([19, Proposition 2.8(1)]). *Let $M \in R$-Mod. If every fully invariant submodule of M is mono-M-injective, then M satisfies the α-property.*

Proposition 2.3 ([3, Proposition 5.6]). *Let $M \in R$-Mod. Assume that M is projective in $\sigma[M]$, and let K and N be submodules of M. Then $(K \cdot N) \cdot X = K \cdot (N \cdot X)$ for any module ${}_RX \in \sigma[M]$.*

In the next sections we frequently use the following proposition.

Proposition 2.4 ([9, Proposition 1.2]). *Let $\{M_\gamma\}_{\gamma \in I}$ and $\{N_\gamma\}_{\gamma \in I}$ be families of modules in R-Mod such that for each $\gamma \in I$, $N_\gamma \leq M_\gamma$. Let $N = \bigoplus_{\gamma \in I} N_\gamma$, $M = \bigoplus_{\gamma \in I} M_\gamma$, $N' = \prod_{\gamma \in I} N_\gamma$ and $M' = \prod_{\gamma \in I} M_\gamma$.*

(1) *If $N \leq_{fi} M$, then for each $\gamma \in I$, $N_\gamma \leq_{fi} M_\gamma$ and $\alpha_N^M = \bigvee_{\gamma \in I} \alpha_{N_\gamma}^{M_\gamma}$.*

(2) *If $N' \leq_{fi} M'$, then for each $\gamma \in I$, $N_\gamma \leq_{fi} M_\gamma$ and $\omega_{N'}^{M'} = \bigwedge_{\gamma \in I} \omega_{N_\gamma}^{M_\gamma}$.*

3. Quasi-n-absorbing preradicals

Suppose that m, n are positive integers with $m > n$. A preradical $\sigma \neq 1$ is called a *quasi-(m, n)-absorbing preradical* if whenever $\mu^{m-1}\nu \preceq \sigma$ for $\mu, \nu \in R$-pr, then $\mu^n \preceq \sigma$ or $\mu^{n-1}\nu \preceq \sigma$.

Proposition 3.1. *Let $\sigma \in R$-pr and $m > n$ be positive integers. Then σ is quasi-(m, n)-absorbing if and only if σ is quasi-n-absorbing.*

Proof. Assume that σ is quasi-(m, n)-absorbing. Let $\mu^n\nu \preceq \sigma$ for some $\mu, \nu \in R$-pr. Since $n \leq m - 1$, then $\mu^{m-1}\nu \preceq \sigma$. Therefore $\mu^n \preceq \sigma$ or $\mu^{n-1}\nu \preceq \sigma$. Consequently σ is quasi-n-absorbing. Now, suppose that σ is quasi-n-absorbing. Let $\mu^{m-1}\nu \preceq \sigma$ for some $\mu, \nu \in R$-pr. Therefore $\mu^n\mu^{(m-1-n)}\nu \preceq \sigma$. Hence $\mu^n \preceq \sigma$ or $\mu^{n-1}\mu^{(m-1-n)}\nu = \mu^{(m-2)}\nu \preceq \sigma$. Repeating this argument we obtain $\mu^n \preceq \sigma$ or $\mu^{n-1}\nu \preceq \sigma$. Thus σ is quasi-(m, n)-absorbing. □

Remark 3.2. Let $\sigma \in R$-pr.

(1) σ is prime if and only if σ is quasi-1-absorbing if and only if σ is 1-absorbing.
(2) If σ is quasi-n-absorbing, then it is quasi-i-absorbing for all $i \geq n$.
(3) If σ is prime, then it is quasi-n-absorbing for all $n \geq 1$.
(4) If σ is quasi-n-absorbing for some $n \geq 1$, then there exists the least $n_0 \geq 1$ such that σ is quasi-n_0-absorbing. In this case, σ is quasi-n-absorbing for all $n \geq n_0$ and it is not quasi-i-absorbing for $n_0 > i > 0$.

Proposition 3.3. *Let $\mathcal{P}$ be a family of prime preradicals. Then $\bigwedge_{\sigma \in \mathcal{P}} \sigma$ is a quasi-i-absorbing preradical for every $i \geq 2$.*

Proof. Let $\tau = \bigwedge_{\sigma \in \mathcal{P}} \sigma$. By part (2) of Remark 3.2, it is sufficient to show that τ is a quasi-2-absorbing preradical. Suppose that $\mu^2\nu \preceq \tau$ for some $\mu, \nu \in R$-pr. Since every $\sigma \in \mathcal{P}$ is prime and $\mu^2\nu \preceq \sigma$, then $\mu \preceq \sigma$ or $\nu \preceq \sigma$. Therefore $\mu\nu \preceq \tau$, and so, we conclude that τ is a quasi-2-absorbing preradical. □

Let $\rho = \wedge\{\omega_0^S \mid S \in R\text{-simp}\}$. Notice that for every R-module M, $\rho(M) = \mathrm{Rad}(M)$. As in [14], ρ is called the Jacobson radical.

As a direct consequence of Proposition 3.3 we have the following result.

Proposition 3.4. *ρ is a quasi-i-absorbing preradical for every $i \geq 2$.*

Proof. By [13, Corollary 24], for each simple R-module S, ω_0^S is prime. So by Proposition 3.3, we have the claim. □

Proposition 3.5. *If R is a semisimple Artinian ring, then every preradical $1 \neq \sigma \in R$-pr is a quasi-i-absorbing preradical for every $i \geq 2$.*

Proof. Suppose that R is a semisimple Artinian ring. According to [13, Remark 3], every coatom ω_I^R (I is a maximal ideal of R) is a prime preradical. On the other hand, [10, Theorem 11] implies that $\sigma = \bigwedge\{\omega_I^R \mid I \text{ is a maximal ideal of } R,\ \omega_I^R \succeq \sigma\}$. Therefore, every preradical $1 \neq \sigma \in R$-pr is quasi-i-absorbing for every $i \geq 2$, by Proposition 3.3. □

Remark 3.6. Let $S_1, S_2, \ldots, S_{n+1} \in R$-simp be distinct simple modules. Then by Proposition 3.3, $\omega_0^{S_1} \wedge \omega_0^{S_2} \wedge \cdots \wedge \omega_0^{S_{n+1}}$ is a quasi-i-absorbing preradical in R-pr for every $i \geq 2$. But, [19, Corollary 3.6] implies that $\omega_0^{S_1} \wedge \omega_0^{S_2} \wedge \cdots \wedge \omega_0^{S_{n+1}}$ is not an n-absorbing preradical. This remark shows that the two concepts of quasi-n-absorbing preradicals and of n-absorbing preradicals are different in general.

Corollary 3.7. *If R is a ring such that every quasi-n-absorbing preradical in R-pr is n-absorbing, then $|R\text{-simp}| \leq n$.*

Proposition 3.8. *Let R be a ring. The following statements are equivalent:*

(1) *For every preradicals $\mu, \nu \in R$-pr, $\mu^n\nu = \mu^n$ or $\mu^n\nu = \mu^{n-1}\nu$;*
(2) *For every preradicals $\sigma_1, \sigma_2, \ldots, \sigma_{n+1} \in R$-pr, $(\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n)^n \preceq \sigma_1\sigma_2\cdots\sigma_{n+1}$ or $(\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n)^{n-1}\sigma_{n+1} \preceq \sigma_1\sigma_2\cdots\sigma_{n+1}$;*
(3) *Every preradical $1 \neq \sigma \in R$-pr is quasi-n-absorbing.*

Proof. (1)⇒(2) If $\sigma_1, \sigma_2, \ldots, \sigma_{n+1} \in R$-pr, then we get from (1),

$$(\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n)^n = (\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n)^n\sigma_{n+1} \preceq \sigma_1\sigma_2\cdots\sigma_{n+1},$$

or

$$(\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n)^{n-1}\sigma_{n+1} = (\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n)^n\sigma_{n+1} \preceq \sigma_1\sigma_2\cdots\sigma_{n+1}.$$

(2)⇒(1) For preradicals $\mu, \nu \in R$-pr, we have from (2), $\mu^n = (\overbrace{\mu \wedge \cdots \wedge \mu}^{n \text{ times}})^n \preceq \mu^n\nu$ or $\mu^{n-1}\nu = (\overbrace{\mu \wedge \cdots \wedge \mu}^{n \text{ times}})^{n-1}\nu \preceq \mu^n\nu$. So we have that $\mu^n\nu = \mu^n$ or $\mu^n\nu = \mu^{n-1}\nu$.
(1)⇔(3) is trivial. □

Proposition 3.9. *Let $1 \neq \sigma \in R$-pr be an idempotent radical.*

(1) *If σ is such that for any $\mu, \nu \in R$-pr, we have $\mu^n\nu \preceq \sigma \preceq \mu \wedge \nu \Rightarrow [\mu^n \preceq \sigma$ or $\mu^{n-1}\nu \preceq \sigma]$, then σ is quasi-n-absorbing.*
(2) *If σ is such that for any $\mu_1, \mu_2, \ldots, \mu_{n+1} \in R$-pr, we have*
$$\mu_1\mu_2\cdots\mu_{n+1} \preceq \sigma \preceq \mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_{n+1} \Rightarrow$$
$$[\mu_1\cdots\widehat{\mu_i}\cdots\mu_{n+1} \preceq \sigma, \quad \textit{for some } 1 \leq i \leq n+1]$$
then σ is an n-absorbing preradical.

Proof. (1) Let $\sigma \neq 1$ be an idempotent radical that satisfies the hypothesis stated in (1). Let $\tau^n\lambda \preceq \sigma$ for some $\tau, \lambda \in R$-pr. Then, by [10, Theorem 8(3)] we have

$$(\sigma : \tau)^n(\sigma : \lambda) = (\sigma : \tau^n\lambda) \preceq (\sigma : \sigma) = \sigma \preceq (\sigma : \tau) \wedge (\sigma : \lambda).$$

So, by hypothesis we have $\tau^n \preceq (\sigma : \tau^n) = (\sigma : \tau)^n \preceq \sigma$ or $\tau^{n-1}\lambda \preceq (\sigma : \tau^{n-1}\lambda) = (\sigma : \tau)^{n-1}(\sigma : \lambda) \preceq \sigma$. Therefore σ is quasi-n-absorbing.

(2) The proof is similar to that of (1).

□

Proposition 3.10. *Let $\mathcal{P}$ be a chain of quasi-n-absorbing preradicals, that is, a subclass of quasi-n-absorbing preradicals which is linearly ordered. Then $\bigwedge_{\sigma\in\mathcal{P}}\sigma$ is a quasi-n-absorbing preradical.*

Proof. Let $\tau = \bigwedge_{\sigma\in\mathcal{P}}\sigma$ and suppose that $\mu^n\nu \preceq \tau$ for some $\mu, \nu \in R$-pr. If $\mu^n \preceq \sigma$ for each $\sigma \in \mathcal{P}$, then $\mu^n \preceq \tau$. If there is $\sigma_0 \in \mathcal{P}$ such that $\mu^n \npreceq \sigma_0$, then $\mu^n \npreceq \sigma$ for each $\sigma \preceq \sigma_0$. Since all preradicals in $\mathcal{P}$ are quasi-n-absorbing, it follows that $\mu^{n-1}\nu \preceq \sigma$ for each $\sigma \preceq \sigma_0$. Thus $\mu^{n-1}\nu \preceq \sigma$ for each $\sigma \in \mathcal{P}$, so that $\mu^{n-1}\nu \preceq \tau$. We conclude that τ is a quasi-n-absorbing preradical. □

Theorem 3.11. *Let $M \in R$-Ass and N be a fully invariant submodule of M. Consider the following statements:*

(1) *N is n-absorbing in M.*
(2) *ω_N^M is an n-absorbing preradical.*
Then (2) ⇒ (1), and if M satisfies the α-property, then (1) ⇒ (2).

Proof. Similar to the proof of [19, Theorem 4.2]. □

We recall that the commutative hereditary domains are precisely the Dedekind domains.

The following remark shows that the two concepts of quasi-$(n+1)$-absorbing preradicals ($(n+1)$-absorbing preradicals) and of quasi-n-absorbing preradicals are different in general. Also, in this remark we can observe that the intersection of two quasi-n-absorbing preradicals may not be quasi-n-absorbing.

Remark 3.12. Let p, q be distinct prime numbers. By [13, Theorem 15], $\omega_{p\mathbb{Z}}^{\mathbb{Z}}$ is a prime preradical in $\mathbb{Z}$-pr. On the other hand, $\omega_{p^n\mathbb{Z}}^{\mathbb{Z}}$ is an n-absorbing preradical, by [1, p. 1650] and Theorem 3.11. Hence, [19, Proposition 3.5] implies that $\omega_{p^n\mathbb{Z}}^{\mathbb{Z}} \wedge \omega_{q\mathbb{Z}}^{\mathbb{Z}}$ is an $(n+1)$-absorbing preradical, and so it is quasi-$(n+1)$-absorbing preradical. If $\omega_{p^n\mathbb{Z}}^{\mathbb{Z}} \wedge \omega_{q\mathbb{Z}}^{\mathbb{Z}}$ is a quasi-n-absorbing preradical, $\left(\omega_{p\mathbb{Z}}^{\mathbb{Z}}\right)^n \omega_{q\mathbb{Z}}^{\mathbb{Z}} \preceq \omega_{p^n\mathbb{Z}}^{\mathbb{Z}} \wedge \omega_{q\mathbb{Z}}^{\mathbb{Z}}$ implies that either $\left(\omega_{p\mathbb{Z}}^{\mathbb{Z}}\right)^n \preceq \omega_{q\mathbb{Z}}^{\mathbb{Z}}$ or $\left(\omega_{p\mathbb{Z}}^{\mathbb{Z}}\right)^{n-1} \omega_{q\mathbb{Z}}^{\mathbb{Z}} \preceq \omega_{p^n\mathbb{Z}}^{\mathbb{Z}}$. Therefore, by Corollary 2.1 we have that $p^n \in q\mathbb{Z}$ or $p^{n-1}q \in p^n\mathbb{Z}$. These contradictions show that $\omega_{p^n\mathbb{Z}}^{\mathbb{Z}} \wedge \omega_{q\mathbb{Z}}^{\mathbb{Z}}$ is not quasi-n-absorbing.

Proposition 3.13. *If σ_i is a quasi-n_i-absorbing preradical in R-pr for every $1 \le i \le k$, then $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k$ is a quasi-n-absorbing preradical for $n = n_1 + \cdots + n_k$.*

Proof. Let $\mu, \nu \in R$-pr be such that $\mu^n\nu \preceq \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k$. Note that σ_i is quasi-n_i-absorbing, for every $1 \le i \le k$. Then for every $1 \le i \le k$, σ_i is $(n+1, n_i)$-absorbing, by Proposition 3.1. Hence, for every $1 \le i \le k$, either $\mu^{n_i} \preceq \sigma_i$ or $\mu^{n_i-1}\nu \preceq \sigma_i$. If for every $1 \le i \le k$, $\mu^{n_i} \preceq \sigma_i$, then $\mu^n \preceq \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k$. If for every $1 \le i \le k$, $\mu^{n_i-1}\nu \preceq \sigma_i$, then $\mu^{n-1}\nu \preceq \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k$. Otherwise, without loss of generality we may assume that there exists $1 \le j < k$ such that $\mu^{n_i} \preceq \sigma_i$ for every $1 \le i \le j$ and $\mu^{n_i-1}\nu \preceq \sigma_i$ for every $j+1 \le i \le k$. Hence, $\mu^{n-1}\nu \preceq \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k$ which shows that $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k$ is a quasi-n-absorbing preradical. □

Proposition 3.14. *Let $\sigma_1, \sigma_2, \ldots, \sigma_t \in R$-pr.*

(1) *If σ_1 is a quasi-n-absorbing preradical and σ_2 is a quasi-m-absorbing preradical for $m < n$, then $\sigma_1 \wedge \sigma_2$ is a quasi-$(n+1)$-absorbing preradical.*
(2) *If $\sigma_1, \sigma_2, \ldots, \sigma_t$ are quasi-n-absorbing preradicals, then $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_t$ is a quasi-$(n+t-1)$-absorbing preradical.*
(3) *If σ_i is a quasi-n_i-absorbing preradical for every $1 \le i \le t$ with $n_1 < n_2 < \cdots < n_t$ and $t > 2$, then $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_t$ is a quasi-(n_t+1)-absorbing preradical.*

Proof. (1) Let $\mu, \nu \in R$-pr be such that $\mu^{n+1}\nu \preceq \sigma_1 \wedge \sigma_2$. Since σ_1 is quasi-n-absorbing, then, by Proposition 3.1, σ_1 is quasi-$(n+2, n)$-absorbing. Hence, either $\mu^n \preceq \sigma_1$ or $\mu^{n-1}\nu \preceq \sigma_1$. Also, σ_2 is quasi-m-absorbing, so, again by Proposition 3.1, either $\mu^m \preceq \sigma_2$ or $\mu^{m-1}\nu \preceq \sigma_2$. There are four cases.
Case 1. Suppose that $\mu^n \preceq \sigma_1$ and $\mu^m \preceq \sigma_2$. Then $\mu^n \preceq \sigma_1 \wedge \sigma_2$.
Case 2. Suppose that $\mu^n \preceq \sigma_1$ and $\mu^{m-1}\nu \preceq \sigma_2$. Then $\mu^n\nu \preceq \sigma_1 \wedge \sigma_2$.
Case 3. Suppose that $\mu^{n-1}\nu \preceq \sigma_1$ and $\mu^m \preceq \sigma_2$. Then $\mu^{n-1}\nu \preceq \sigma_1 \wedge \sigma_2$.
Case 4. Suppose that $\mu^{n-1}\nu \preceq \sigma_1$ and $\mu^{m-1}\nu \preceq \sigma_2$. Then $\mu^{n-1}\nu \preceq \sigma_1 \wedge \sigma_2$. Consequently $\sigma_1 \wedge \sigma_2$ is quasi-$(n+1)$-absorbing.

(2) We use induction on t. For $t = 1$ there is nothing to prove. Let $t > 1$ and assume that for $t-1$ the claim holds. Then $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_{t-1}$ is quasi-$(n+t-1)$-absorbing. Since σ_t is quasi-n-absorbing, then it is quasi-$(n+t-2)$-absorbing, by Remark 3.2(2). Therefore $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_t$ is quasi-$(n+t-1)$-absorbing by part (1).

(3) Induction on t. For $t = 3$ apply parts (1) and (2). Let $t > 3$ and suppose that for $t-1$ the claim holds. Hence $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_{t-1}$ is quasi-$(n_{t-1}+1)$-absorbing. We consider the following cases:
Case 1. Let $n_{t-1}+1 < n_t$. In this case $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_t$ is quasi-(n_t+1)-absorbing by part (1).
Case 2. Let $n_{t-1}+1 = n_t$. Thus $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_t$ is quasi-(n_t+1)-absorbing by part (2).

□

Proposition 3.15. *Let $\sigma \in R$-pr be idempotent. If σ is quasi-n-absorbing, then $c(\sigma)$ is quasi-n-absorbing.*

Proof. Assume that σ is quasi-n-absorbing, and let $\mu^n\nu \preceq c(\sigma)$ for some $\mu, \nu \in R$-pr. Then $(\sigma : \mu)^n(\sigma : \nu) \preceq (\sigma : \mu^n\nu) \preceq (\sigma : c(\sigma)) = \sigma$. Since σ is quasi-n-absorbing and idempotent either $(\sigma : \mu)^n = (\sigma : \mu^n) \preceq \sigma$ or $(\sigma : \mu)^{n-1}(\sigma : \nu) = (\sigma : \mu^{n-1}\nu) \preceq \sigma$, and so either $(\sigma : \mu^n) = \sigma$ or $(\sigma : \mu^{n-1}\nu) = \sigma$. By definition of co-equalizer either $\mu^n \preceq c(\sigma)$ or $\mu^{n-1}\nu \preceq c(\sigma)$. Consequently, $c(\sigma)$ is quasi-n-absorbing. □

The annihilator operator can be generalized to a relative annihilator, which can be considered also as an operator $\mathbf{r.a}_\tau : R\text{-pr} \to R\text{-pr}$.

Definition 3.16. Let $\sigma,\ \tau \in R$-pr. The right annihilator of σ relative to τ is $\mathbf{r.a}_\tau(\sigma) = \bigvee\{\rho \in R\text{-pr} \mid \sigma\rho \preceq \tau\}$. The operator $\mathbf{r.a}_0$ is denoted by $\mathbf{r.a}$, and $\mathbf{r.a}(\sigma)$ is called the right annihilator of σ.

Each $\sigma \in R$-pr has a unique pseudocomplement $\sigma^\perp$ such that if $\tau \in R$-pr and $\sigma \wedge \tau = 0$ then $\tau \preceq \sigma^\perp$, [12, Theorem 4]. This pseudocomplement can be described as $\sigma^\perp = \bigwedge\{\omega_0^{E(S)} \mid S \in R\text{-simp } \sigma(E(S)) \neq 0\}$ (see [11]).

Proposition 3.17. *Let $\sigma \in R$-pr. If σ is quasi-n-absorbing, then for each $\tau \in R$-pr with $\tau^n \npreceq \sigma$, $\mathbf{r.a}_\sigma(\tau^n) = \mathbf{r.a}_\sigma(\tau^{n-1})$. Moreover $\tau^{n-1}(\tau^n)^\perp \preceq \sigma$.*

Proof. Suppose that σ is quasi-n-absorbing and let $\tau \in R$-pr such that $\tau^n \npreceq \sigma$. If $\rho \in R$-pr is such that $\tau^n\rho \preceq \sigma$, then $\tau^{n-1}\rho \preceq \sigma$, since σ is quasi-n-absorbing. Therefore $\mathbf{r.a}_\sigma(\tau^n) \preceq \mathbf{r.a}_\sigma(\tau^{n-1})$. On the other hand, $\mathbf{r.a}_\sigma(\tau^{n-1}) \preceq \mathbf{r.a}_\sigma(\tau^n)$. So the equality holds. Note that $\tau^n(\tau^n)^\perp \preceq \tau^n \wedge (\tau^n)^\perp = 0$. Thus $\tau^{n-1}(\tau^n)^\perp \preceq \sigma$, since σ is quasi-n-absorbing and $\tau^n \npreceq \sigma$. □

Corollary 3.18. *Let R be a ring. If 0 is a quasi-n-absorbing preradical in R-pr, then for each $\tau \in R$-pr, either $\tau^n = 0$ or $\mathbf{r.a}(\tau^n) = \mathbf{r.a}(\tau^{n-1})$.*

Proof. By Proposition 3.17. □

4. Semi-n-absorbing preradicals

Suppose that $m,\ n$ are positive integers with $n > m$. A more general concept than semi-n-absorbing preradicals is the concept of semi-(n, m)-absorbing preradicals. A preradical $\sigma \neq 1$ is called a *semi-(n, m)-absorbing preradical* if whenever $\mu^n \preceq \sigma$ for $\mu \in R$-pr, then $\mu^m \preceq \sigma$.

Note that a semiprime preradical is just a semi-1-absorbing preradical.

Theorem 4.1. *Let $\sigma \in R$-pr and $m,\ n$ be positive integers with $n > m$.*

(1) *If σ is quasi-(n, m)-absorbing, then it is semi-(n, m)-absorbing.*
(2) *σ is semi-(n, m)-absorbing if and only if σ is semi-(n, k)-absorbing for each $n > k \geq m$ if and only if σ is semi-(i, j)-absorbing for each $n \geq i > j \geq m$.*
(3) *If σ is semi-(n, m)-absorbing, then it is semi-(nk, mk)-absorbing for every positive integer k.*
(4) *If σ is semi-(n, m)-absorbing and semi-(r, s)-absorbing for some positive integers $r > s$, then it is semi-(nr, ms)-absorbing.*

Proof. (1) Is trivial.
(2) Straightforward.
(3) Assume that σ is semi-(n, m)-absorbing. Let $\mu \in R$-pr and let k be a positive integer such that $\mu^{nk} \preceq \sigma$. Then $\left(\mu^k\right)^n \preceq \sigma$. Since σ is semi-(n, m)-absorbing, $\left(\mu^k\right)^m = \mu^{mk} \preceq \sigma$, and so σ is semi-(nk, mk)-absorbing.
(4) Suppose that σ is semi-(n, m)-absorbing and semi-(r, s)-absorbing for some positive integers $r > s$. Let $\mu^{nr} \preceq \sigma$. Since σ is semi-(n, m)-absorbing, $\mu^{mr} \preceq \sigma$, and since σ is semi-(r, s)-absorbing, $\mu^{ms} \preceq \sigma$. Hence σ is semi-(nr, ms)-absorbing.

□

Corollary 4.2. *Let $\sigma \in R$-pr and n be a positive integer.*

(1) *If σ is quasi-n-absorbing, then it is semi-n-absorbing.*
(2) *Let $t \leq n$ be an integer. If σ is semi-$(n+1, t)$-absorbing, then it is semi-$(nk+i, tk)$-absorbing for all $k \geq i \geq 1$.*
(3) *If σ is semi-n-absorbing, then it is semi-$(nk + i, nk)$-absorbing for all $k \geq i \geq 1$.*
(4) *If σ is semi-n-absorbing, then it is semi-$(nk + j)$-absorbing for all $k > j \geq 0$.*
(5) *If σ is semi-n-absorbing, then it is semi-(nk)-absorbing for every positive integer k.*
(6) *If σ is semiprime, then it is semi-k-absorbing for every positive integer k.*
(7) *If σ is semiprime, then for every $k \geq 1$ and every $\mu \in R$-pr, $\mu^k \preceq \sigma$ implies that $\mu \preceq \sigma$.*
(8) *If σ is semi-n-absorbing, then it is semi-$((n + 1)^t, n^t)$-absorbing for all $t \geq 1$.*
(9) *If σ is semiprime, then it is quasi-k-absorbing for every $k > 1$.*

Proof. (1) By Theorem 4.1(1).
(2) Let σ be semi-$(n+1, t)$-absorbing. Then, by Theorem 4.1(3), σ is semi-$(nk+k, tk)$-absorbing, for every positive integer k. Hence, by Theorem 4.1(2), σ is semi-$(nk + i, tk)$-absorbing for every $k \geq i \geq 1$.
(3) In part (2) get $t = n$.
(4) By part (3).
(5) Is a special case of (4).
(6) Is a direct consequence of (5).
(7) By part (6).
(8) By Theorem 4.1(4).
(9) Assume that σ is semiprime. Let $\mu^k\nu \preceq \sigma$ for some $\mu, \nu \in R$-pr and some $k > 1$. Then $(\mu\nu)^k \preceq \mu^k\nu \preceq \sigma$. Therefore $\mu\nu \preceq \sigma$, by part (7). So σ is quasi-k-absorbing.

□

Proposition 4.3. *Let $\sigma_1, \sigma_2, \dots, \sigma_n \in R$-pr. If for every $1 \leq i \leq n$, σ_i is a semiprime preradical, then $\sigma_1\sigma_2\cdots\sigma_n$ is a semi-n-absorbing preradical. In particular, if σ is a semiprime preradical, then σ^n is a semi-n-absorbing preradical.*

Proof. Use Corollary 4.2 (7). □

Lemma 4.4. *Let $\sigma \in R$-pr. If σ^{n+1} is a semi-n-absorbing preradical, then $\sigma^{n+1} = \sigma^n$. In particular, if σ^2 is a semiprime preradical, then σ is idempotent.*

The following remark shows that the two concepts of semi-n-absorbing preradicals and of semi-$(n+1)$-absorbing preradicals are different in general.

Remark 4.5. Let $n > 1$, R be a left hereditary ring and I be a two-sided prime ideal of R. Since ω_I^R is a prime preradical, $(\omega_I^R)^{n+1}$ is a semi-$(n+1)$-absorbing preradical, by Proposition 4.3. If $(\omega_I^R)^{n+1}$ is a semi-n-absorbing preradical, then $(\omega_I^R)^{n+1} = (\omega_I^R)^n$, and so $I^{n+1} = I^n$, by Corollary 3.1. Consequently, for any prime number p, $(\omega_{p\mathbb{Z}}^{\mathbb{Z}})^{n+1}$ is a semi-$(n+1)$-absorbing preradical in $\mathbb{Z}$-pr which is not a semi-n-absorbing preradical.

Proposition 4.6. *Let $\sigma \in R$-pr, $\sigma \neq 1$ be an idempotent radical. If σ is such that for any $\mu \in R$-pr, we have $\mu^{n+1} \preceq \sigma \preceq \mu \Rightarrow \mu^n \preceq \sigma$, then σ is semi-n-absorbing.*

Proof. The proof is similar to that of Proposition 3.9(1). □

Proposition 4.7. *Let $\sigma_1, \sigma_2, \dots, \sigma_n \in R$-pr be semi-2-absorbing preradicals. Then $\sigma_1\sigma_2\cdots\sigma_n$ is a semi-(3^n-1)-absorbing preradical.*

Proof. Suppose that $\mu^{3^n} \preceq \sigma_1\sigma_2\cdots\sigma_n$ for some $\mu \in R$-pr. For every $1 \leq i \leq n$, $\mu^{3^n} \preceq \sigma_i$ and σ_i is semi-2-absorbing, then $\mu^{2^n} \preceq \sigma_i$. Therefore $\mu^{n2^n} \preceq \sigma_1\sigma_2\cdots\sigma_n$. On the other hand, $n2^n \leq 3^n - 1$. So $\mu^{3^n-1} \preceq \sigma_1\sigma_2\cdots\sigma_n$ which shows that $\sigma_1\sigma_2\cdots\sigma_n$ is semi-(3^n-1)-absorbing. □

Theorem 4.8. *If σ_i is a semi-n_i-absorbing preradical in R-pr for every $1 \leq i \leq k$, then $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k$ is a semi-$(n-1)$-absorbing preradical for $n = \prod_{i=1}^{k}(n_i+1)$.*

Proof. Let $\mu \in R$-pr be such that $\mu^n \preceq \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k$. Then for every $1 \leq i \leq k$, $(\mu^m)^{(n_i+1)} \preceq \sigma_i$, where $m = \prod_{j=1, j\neq i}^{k}(n_j+1)$). Since σ_i's are semi-n_i-absorbing, then, for each $1 \leq i \leq k$, $\mu^{n_i m} \preceq \sigma_i$. Note that for every $1 \leq i \leq k$, $n_i m \leq \prod_{i=1}^{k}(n_i+1) - 1 = n-1$. So we have $\mu^{n-1} \preceq \sigma_i$ for every $1 \leq i \leq k$. Hence $\mu^{n-1} \preceq \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k$ which implies that $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k$ is a semi-$(n-1)$-absorbing preradical. □

Proposition 4.9. *Let $\sigma_1, \sigma_2 \in R$-pr and m, n be positive integers.*

(1) *If σ_1 is quasi-m-absorbing and σ_2 is semi-n-absorbing, then $\sigma_1\sigma_2$ is semi-$(n(m+1)+m)$-absorbing.*
(2) *If σ_1 is quasi-$(2m)$-absorbing and σ_2 is semi-m-absorbing, then $\sigma_1\sigma_2$ is semi-$(m(m+2))$-absorbing.*

Proof. (1) Assume that $\mu^{(n+1)(m+1)} \preceq \sigma_1\sigma_2$ for some $\mu \in R$-pr. Since σ_1 is quasi-m-absorbing and $\mu^{(n+1)(m+1)} \preceq \sigma_1$, then $\mu^m \preceq \sigma_1$. On the other hand, σ_2 is semi-n-absorbing and $\mu^{(n+1)(m+1)} \preceq \sigma_2$, then $\mu^{n(m+1)} \preceq \sigma_2$. Consequently $\mu^{n(m+1)+m} \preceq \sigma_1\sigma_2$, and so $\sigma_1\sigma_2$ is semi-$(n(m+1)+m)$-absorbing.

(2) Suppose that $\mu^{(m+1)^2} \preceq \sigma_1\sigma_2$ for some $\mu \in R$-pr. Since σ_1 is quasi-$(2m)$-absorbing and $\mu^{(m+1)^2} \preceq \sigma_1$, then $\mu^{2m} \preceq \sigma_1$. Since σ_2 is semi-m-absorbing and $\mu^{(m+1)^2} \preceq \sigma_2$,

then $\mu^{m^2} \preceq \sigma_2$. Hence $\mu^{m^2+2m} \preceq \sigma_1\sigma_2$ which shows that $\sigma_1\sigma_2$ is semi-$(m(m+2))$-absorbing.

□

Proposition 4.10. *Let R be a ring. The following statements are equivalent:*

(1) *For every preradical $\sigma \in R$-pr, $\sigma^{n+1} = \sigma^n$;*
(2) *For all preradicals $\sigma_1\sigma_2, \ldots, \sigma_{n+1} \in R$-pr we have $(\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_{n+1})^n \preceq \sigma_1\sigma_2 \cdots \sigma_{n+1}$;*
(3) *Every preradical $1 \neq \sigma \in R$-pr is semi-n-absorbing.*

Proof. (1)⇒(2) If $\sigma_1, \sigma_2, \ldots, \sigma_{n+1} \in R$-pr, then from (1),

$$(\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_{n+1})^n = (\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_{n+1})^{n+1} \preceq \sigma_1\sigma_2 \cdots \sigma_{n+1}.$$

(2)⇒(1) For a preradical $\sigma \in R$-pr, we get from (2), $\sigma^n = (\overbrace{\sigma \wedge \cdots \wedge \sigma}^{n+1 \text{ times}})^n \preceq \sigma^{n+1}$. So we have that $\sigma^{n+1} = \sigma^n$.
(1)⇔(3) It is obvious. □

Remark 4.11. Let $\{\sigma_\alpha\}_{\alpha \in I} \subseteq R$-pr. If σ_α is semi-n-absorbing for every $\alpha \in I$, then $\bigwedge_{\alpha \in I} \sigma_\alpha$ is semi-n-absorbing.

The following remark shows that the two concepts of semi-n-absorbing preradicals and of quasi-n-absorbing (n-absorbing) preradicals are different in general.

Remark 4.12. Let p, q be distinct prime numbers. By Remark 4.11, $\omega^{\mathbb{Z}}_{p^n\mathbb{Z}} \wedge \omega^{\mathbb{Z}}_{q\mathbb{Z}}$ is a semi-n-absorbing preradical, but it is not quasi-n-absorbing, by Remark 3.12.

Proposition 4.13. *Let $\sigma \in R$-pr be idempotent. If σ is semi-n-absorbing, then $c(\sigma)$ is semi-n-absorbing.*

Proof. Is similar to the proof of Proposition 3.15. □

In Proposition 17 of [14], it was shown that $\sigma_0 := \bigwedge\{\sigma \in R\text{-pr} \mid \sigma \text{ is semiprime}\}$ is the unique least semiprime preradical.

Proposition 4.14. *There exists in R-pr a unique least semi-n-absorbing preradical.*

Proof. Set $\sigma_0^{(n)} = \bigwedge\{\sigma \in R\text{-pr} \mid \sigma \text{ is semi-}n\text{-absorbing}\}$. By Remark 4.11, $\sigma_0^{(n)}$ is the least semi-n-absorbing preradical. □

By notation in the the proof of the previous proposition we have that $\sigma_0^{(1)} = \sigma_0$.

Remark 4.15. As ρ is a semiprime preradical, then $\sigma_0 \preceq \rho$. Also ρ^n is a semi-n-absorbing preradical, by Proposition 4.3. Therefore, $\sigma_0^{(n)} \preceq \rho^n$.

Proposition 4.16. *The following statements hold:*

(1) $\sigma_0 = \bigvee_{n \geq 1} \sigma_0^{(n)}$.
(2) $\sigma_0^{(nk)} \preceq \sigma_0^{(n)}$ *for every positive integer k.*
(3) $\sigma_0^{(n)} \preceq \sigma^n$ *for every semiprime preradical σ.*

Proof. (1) By Corollary 4.2(6), every semiprime preradical is semi-n-absorbing for every $n \geq 1$. Then $\sigma_0^{(n)} \preceq \sigma_0$ for every $n \geq 1$.
(2) By Corollary 4.2(5).
(3) By Proposition 4.3. □

In Proposition 20 of [14] it was shown that $\nu^0 \preceq \sigma_0$, where $\nu^0 = \bigvee\{\tau \mid \tau \in R\text{-pr}, \ \tau \text{ is nilpotent}\}$.

The following proposition is straightforward.

Proposition 4.17. *Suppose that* $\nu^0_{(n)} = \bigvee\{\tau^n \mid \tau \in R\text{-}pr,\ \tau^{n+1} = 0\}$. *Then:*

(1) $\nu^0_{(n)} \preceq \sigma_0^{(n)}$.
(2) $\nu^0_{(1)} \preceq \nu^0$.

Corollary 4.18. *The following statements hold:*

(1) *If* $\rho^{n+1} = 0$, *then* $\nu^0_{(n)} = \sigma_0^{(n)} = \sigma_0^n = \rho^n$.
(2) *If* $\rho^2 = 0$, *then* $\nu^0_{(1)} = \sigma_0 = \rho = \nu^0$.

Proof. (1) By Remark 4.15 and Proposition 4.17 we have that $\nu^0_{(n)} \preceq \sigma_0^{(n)} \preceq \sigma_0^n \preceq \rho^n$. If $\rho^{n+1} = 0$, then $\rho^n \preceq \nu^0_{(n)}$, and so $\nu^0_{(n)} = \sigma_0^{(n)} = \sigma_0^n = \rho^n$.

(2) By part (1) and [14, Corollary 21]. □

Proposition 4.19. *For a ring R the following statements are equivalent:*

(1) *For every* $\mu \in R\text{-}pr$, $\mu^{n+1} = 0$ *implies that* $\mu^n = 0$;
(2) 0 *is a semi-n-absorbing preradical;*
(3) $\sigma_0^{(n)} = 0$;
(4) $\nu^0_{(n)} = 0$.

Proof. It can be easily proved. □

Notation 4.20. Let $\tau \in R$-pr. Define

$$S^{(n)}(\tau) = \bigwedge\{\sigma \in R\text{-pr} \mid \tau \preceq \sigma,\ \sigma \text{ semi-}n\text{-absorbing}\},$$

which is the unique least semi-n-absorbing preradical greater than or equal to τ. Notice that in [14], $S^{(1)}$ is denoted by S.

Proposition 4.21. *Let R be a ring.*

(1) $\sigma_0^{(n)} = S^{(n)}(0) = \bigwedge_{\tau \in R\text{-}pr} S^{(n)}(\tau)$.
(2) *For each* $\tau \in R\text{-}pr$, $\tau \preceq S^{(n)}(\tau)$.
(3) *For each* $\tau, \sigma \in R\text{-}pr$ *we have* $\tau \preceq \sigma \Rightarrow S^{(n)}(\tau) \preceq S^{(n)}(\sigma)$.
(4) *For each* $\tau \in R\text{-}pr$, $S^{(n)}(\tau^{n+1}) = S^{(n)}(\tau^n)$.
(5) *For each* $\tau \in R\text{-}pr$, τ *is semi-n-absorbing if and only if* $\tau = S^{(n)}(\tau)$.
(6) $\{\tau \in R\text{-}pr \mid \tau \text{ is semi-}n\text{-absorbing}\} = Im\ S^{(n)} = \{S^{(n)}(\sigma) \mid \sigma \in R\text{-}pr\}$.
(7) $\left[S^{(n)}\right]^2 = S^{(n)}$. *Then,* $S^{(n)}$ *is a closure operator on R-pr.*
(8) *For each family* $\{\tau_\alpha\}_{\alpha \in I} \subseteq R\text{-}pr$, *we have* $S^{(n)}(\bigvee_{\alpha \in I} \tau_\alpha) = S^{(n)}(\bigvee_{\alpha \in I} S^{(n)}(\tau_\alpha))$.
(9) $S^{(n)} = \bigvee_{k \geq 1} S^{(nk)}$, *in particular* $S = \bigvee_{k \geq 1} S^{(k)}$.
(10) $S^{(n)}(\sigma^{n+1}) = S^{(n)}(\sigma^n) = \sigma^n$ *for every semiprime preradical* σ.

Proof. (1), (2), (3), (5) and (6) are evident.

(4) For every $\tau \in R$-pr, part (3) implies that $S^{(n)}(\tau^{n+1}) \preceq S^{(n)}(\tau^n)$. Since $S^{(n)}(\tau^{n+1})$ is semi-n-absorbing (by Remark 4.11) and $\tau^{n+1} \preceq S^{(n)}(\tau^{n+1})$, then $\tau^n \preceq S^{(n)}(\tau^{n+1})$. Hence $S^{(n)}(\tau^n) \preceq S^{(n)}(\tau^{n+1})$. Consequently the equality holds.

(7) Is a direct consequence of part (5).

(8) The proof is similar to that of [14, Proposition 25](5).

(9) Use Corollary 4.2(5).

(10) By Proposition 4.3 and parts (4), (5). □

Now consider the operator $\widehat{(\_)}$ in R-pr that assigns to each preradical σ the greatest idempotent below σ (see [15, p. 137]).

Lemma 4.22. *Let $\sigma, \tau \in R$-pr such that σ is idempotent and τ is semi-n-absorbing. Then:*

(1) $\sigma \preceq \widehat{S^{(n)}(\sigma)} \preceq S^{(n)}(\sigma)$.
(2) $S^{(n)}(\sigma) = S^{(n)}(\widehat{S^{(n)}(\sigma)})$.
(3) $\widehat{\tau} \preceq S^{(n)}(\widehat{\tau}) \preceq \tau$.
(4) $\widehat{\tau} = \widehat{S^{(n)}(\widehat{\tau})}$.

Proof. Similar to the proof of [14, Lemma 26]. □

The following result is a direct consequence of the previous properties.

Proposition 4.23. *Let R be a ring.*

(1) *The operator $\widehat{S^{(n)}(\_)}$ defines a closure operator on the ordered class of idempotent preradicals.*
(2) *The operator $S^{(n)}(\widehat{(\_)})$ defines an interior operator on the ordered class of semi-n-absorbing preradicals.*

Notice that the "closed" idempotent preradicals associated with the closure operator $\widehat{S^{(n)}(\_)}$ are

$$\mathcal{C}_{id}^{(n)} = \{\sigma \text{ idempotent} \mid \sigma = \widehat{\tau} \text{ for some semi-}n\text{-absorbing } \tau\}.$$

The "open" semi-n-absorbing preradicals associated with the interior operator $S^{(n)}(\widehat{(\_)})$ are

$$\mathcal{O}_{sa}^{(n)} = \{\tau \text{ semi-}n\text{-absorbing} \mid \tau = S^{(n)}(\sigma) \text{ for some idempotent } \sigma\}.$$

The following result is immediate.

Corollary 4.24. *For a ring R the operators $S^{(n)}(\_)$ and $\widehat{(\_)}$ restrict to mutually inverse maps between $\mathcal{C}_{id}^{(n)}$ and $\mathcal{O}_{sa}^{(n)}$.*

Definition 4.25. Let $\tau \in R$-pr. Define $S_1^{(n)}(\tau) = \bigvee\{\sigma^n \mid \sigma \in R\text{-pr}, \sigma^{n+1} \preceq \tau\}$.

Proposition 4.26. *Let R be a ring.*

(1) *For each $\tau \in R$-pr, $\tau^n \preceq S_1^{(n)}(\tau)$.*
(2) *For each $\tau \in R$-pr, τ is semi-n-absorbing if and only if $S_1^{(n)}(\tau) \preceq \tau$.*
(3) 0 *is a semi-n-absorbing preradical if and only if $S_1^{(n)}(0) = 0$.*
(4) *Let $\tau,\ \sigma \in R$-pr. If $\tau \preceq \sigma$, then $S_1^{(n)}(\tau) \preceq S_1^{(n)}(\sigma)$.*
(5) *For each family $\{\tau_\alpha\}_{\alpha \in I} \subseteq R$-pr, $S_1^{(n)}(\bigwedge_{\alpha \in I} \tau_\alpha) \preceq \bigwedge_{\alpha \in I} S_1^{(n)}(\tau_\alpha)$ and $\bigvee_{\alpha \in I} S_1^{(n)}(\tau_\alpha) \preceq S_1^{(n)}(\bigvee_{\alpha \in I} \tau_\alpha)$.*

Proof. The assertions have straightforward verifications. □

We apply an "Amitsur construction" to $S_1^{(n)}$ as follows:

Definition 4.27. Let $\tau \in R$-pr. We define recursively the preradical $S_\lambda^{(n)}(\tau)$ for each ordinal λ as follows:

(1) $S_0^{(n)}(\tau) = \tau$.
(2) $S_{\lambda+1}^{(n)}(\tau) = S_1^{(n)}(S_\lambda^{(n)}(\tau))$.
(3) If λ is a limit ordinal, then $S_\lambda^{(n)}(\tau) = \bigvee_{\beta<\lambda} S_\beta^{(n)}(\tau)$.
(4) $S_\Omega^{(n)}(\tau) = \bigvee_{\lambda \text{ ordinal}} S_\lambda^{(n)}(\tau)$.

Proposition 4.28. *Let $\tau \in R$-pr. Then the following statements are equivalent:*

(1) *τ is semi-n-absorbing;*
(2) *For each ordinal λ, $S_\lambda^{(n)}(\tau) \preceq \tau$;*
(3) $S_\Omega^{(n)}(\tau) = \tau$.

Proof. By Proposition 4.26 and applying transfinite induction we have the claim. □

As is the case with $S_1^{(n)}$, all of the operators $S_\lambda^{(n)}$ preserve order between preradicals.

Proposition 4.29. *Let $\tau, \sigma \in R$-pr be such that $\tau \preceq \sigma$. Then:*

(1) *For each ordinal λ, $S_\lambda^{(n)}(\tau) \preceq S_\lambda^{(n)}(\sigma)$.*
(2) $S_\Omega^{(n)}(\tau) \preceq S_\Omega^{(n)}(\sigma)$.

Proposition 4.30. *For each $\tau \in R$-pr, $S_\Omega^{(n)}(\tau) \preceq S^{(n)}(\tau)$.*

Proof. Let $\tau \in R$-pr. By transfinite induction, we have that $S_0^{(n)}(\tau) = \tau \preceq S^{(n)}(\tau)$. Assume that λ is an ordinal such that $S_\lambda^{(n)}(\tau) \preceq S^{(n)}(\tau)$. Since $S^{(n)}(\tau)$ is semi-n-absorbing, $S_{\lambda+1}^{(n)}(\tau) = S_1^{(n)}(S_\lambda^{(n)}(\tau)) \preceq S_1^{(n)}(S^{(n)}(\tau)) \preceq S^{(n)}(\tau)$, by parts (2) and (4) of Proposition 4.26. If λ is a limit ordinal and $S_\beta^{(n)}(\tau) \preceq S^{(n)}(\tau)$ for each $\beta < \lambda$, then $S_\lambda^{(n)}(\tau) = \bigvee_{\beta<\lambda} S_\beta^{(n)}(\tau) \preceq S^{(n)}(\tau)$. □

In the following result we give equivalent conditions for the equality $S_\Omega^{(n)}(\tau) = S^{(n)}(\tau)$ to hold.

Proposition 4.31. *For each $\tau \in R$-pr the following statements are equivalent:*

(1) *$S_\Omega^{(n)}(\tau)$ is semi-n-absorbing;*
(2) $S_1^{(n)}(S_\Omega^{(n)}(\tau)) \preceq S_\Omega^{(n)}(\tau)$;
(3) *For each ordinal λ we have $S_\lambda^{(n)}(S_\Omega^{(n)}(\tau)) \preceq S_\Omega^{(n)}(\tau)$;*
(4) $S_\Omega^{(n)}(S_\Omega^{(n)}(\tau)) = S_\Omega^{(n)}(\tau)$;
(5) $S_\Omega^{(n)}(\tau) = S^{(n)}(\tau)$.

Proof. (1)⇒(2) By Proposition 4.26(2).
(2)⇒(3) It follows by transfinite induction on λ.
(3)⇒(4) Is easy.
(4)⇒(1) By Proposition 4.28.
(1)⇒(5) Assume that $S_\Omega^{(n)}(\tau)$ is semi-n-absorbing. Since $\tau \preceq S_\Omega^{(n)}(\tau)$, the definition of $S^{(n)}(\tau)$ implies that $S^{(n)}(\tau) \preceq S_\Omega^{(n)}(\tau)$. On the other hand, $S_\Omega^{(n)}(\tau) \preceq S^{(n)}(\tau)$, by Proposition 4.30. So the equality holds.
(5)⇒(1) Is straightforward. □

5. Quasi-n-absorbing and semi-n-absorbing submodules

Remark 5.1. Let $M \in R$-Ass and N be a proper fully invariant submodule of M. Then, the following conditions hold:

(1) N is n-absorbing in $M \Rightarrow N$ is quasi-n-absorbing in $M \Rightarrow N$ is semi-n-absorbing in M.
(2) N is a quasi-1-absorbing submodule of M if and only if N is a prime submodule of M.
(3) N is a semi-1-absorbing submodule of M if and only if N is a semiprime submodule of M.

Proposition 5.2. *Let $\sigma \in R$-pr. If for every $M \in R$-Mod, $\sigma(M)$ is a semiprime submodule of M, then σ is a semiprime preradical.*

Proof. By hypothesis, [14, Theorem 14] implies that $\omega^M_{\sigma(M)}$ is a semiprime preradical. So $\sigma = \bigwedge\{\omega^M_{\sigma(M)} \mid M \in R\text{-Mod}\}$ (see [12, Remark 1]) is a semiprime preradical. □

Corollary 5.3. *Let R be a ring. If every R-module is semiprime, then 0 is a semiprime preradical in R-pr.*

Lemma 5.4 ([7, Lemma 3.4])**.** *Let $M \in R$-Mod. Then for any submodules N, K of M, $\omega^M_{N\cap K} = \omega^M_N \wedge \omega^M_K$.*

Proposition 5.5. *Let $M \in R$-Mod. Suppose that $\{N_i\}_{i\in I}$ is a family of semiprime submodules of M. Then $N = \cap_{i\in I} N_i$ is a semiprime submodule of M.*

Proof. Let $\{N_i\}_{i\in I}$ be a family of semiprime submodules of M. Then, by [14, Proposition 14], $\omega^M_{N_j}$'s are semiprime preradicals. Thus $\omega^M_N = \bigwedge_{i\in I} \omega^M_{N_i}$ (see Lemma 5.4) is a semiprime preradical. Again, by [14, Proposition 14], $N = \cap_{i\in I} N_i$ is a semiprime submodule of M. □

Proposition 5.6. *Let R be a ring and $\{M_i\}_{i\in I}$ be a family of semiprime R-modules. Then $M = \bigoplus_{i\in I} M_i$ is a semiprime R-module.*

Proof. Since for every $i \in I$, M_i is a semiprime R-module, thus for every $i \in I$, $\omega^{M_i}_0$ is a semiprime preradical by [14, Proposition 14]. Therefore $\bigwedge_{i\in I} \omega^{M_i}_0 = \omega^M_0$ is a semiprime preradical, and so, again by [14, Proposition 14], $M = \bigoplus_{i\in I} M_i$ is a semiprime R-module. □

Proposition 5.7. *For a ring R the following statements are equivalent:*

(1) *R is a left V-ring;*
(2) *0 is a semiprime preradical;*
(3) *$\bigoplus_{S\in R\text{-simp}} E(S)$ is a semiprime R-module.*

Proof. (1)⇔(2) By [14, Theorem 23].
(2)⇔(3) Set $C = \bigoplus_{S\in R\text{-simp}} E(S)$. Notice that $\omega^C_0 = 0$, by [10, Lemma 6]. Now apply [14, Theorem 14]. □

The following result shows that the injective hull of a semiprime R-module may not be semiprime.

Corollary 5.8. *Let R be a ring that is not a left V-ring. Then there exists a simple R-module $S \in R$-simp such that $E(S)$ is not semiprime.*

Proof. By Proposition 5.6 and Proposition 5.7. □

Theorem 5.9. *Let $M \in R$-Ass and N be a fully invariant submodule of M. Consider the following statements:*

(1) *N is quasi-n-absorbing (resp. semi-n-absorbing) in M.*
(2) *ω^M_N is a quasi-n-absorbing (resp. semi-n-absorbing) preradical.*

Then (2) ⇒ (1), and if M satisfies the α-property, then (1) ⇒ (2).

Proof. (1) ⇒ (2) Assume that N is quasi-n-absorbing in M and that $\eta(M)\cdot\mu(M) = (\eta\mu)(M)$ for every $\eta,\ \mu \in R$-pr. Since $N \neq M$ we have $\omega^M_N \neq 1$. Now let $\eta, \mu \in R$-pr be such that $\eta^n\mu \preceq \omega^M_N$. In this case we have

$$\eta(M)^n \cdot \mu(M) = (\eta^n\mu)(M) \leq \omega^M_N(M) = N.$$

Since N is quasi-n-absorbing in M, by hypothesis we get $\eta^n(M) = \eta(M)^n \leq N$ or $(\eta^{n-1}\mu)(M) = \eta(M)^{n-1} \cdot \mu(M) \leq N$. It follows from [10, Proposition 5] that $\eta^n \preceq \omega_N^M$ or $\eta^{n-1}\mu \preceq \omega_N^M$, that is ω_N^M is quasi-n-absorbing.
(2) $\Rightarrow$ (1) Assume that ω_N^M is a quasi-n-absorbing preradical. Since $\omega_N^M \neq 1$, we have $N \neq M$. Suppose that J, K are fully invariant submodules of M such that $J^n \cdot K \leq N$. Then we have

$$J^n \cdot K = \left(\alpha_J^M\right)^n (K) = \left(\alpha_J^M\right)^n \alpha_K^M(M).$$

By [10, Proposition 5], we get $\left(\alpha_J^M\right)^n \alpha_K^M \preceq \omega_{J^n \cdot K}^M \preceq \omega_N^M$. Since ω_N^M is quasi-n-absorbing, we have $\left(\alpha_J^M\right)^n \preceq \omega_N^M$ or $\left(\alpha_J^M\right)^{n-1} \alpha_K^M \preceq \omega_N^M$. Therefore $J^n = \left(\alpha_J^M\right)^n (M) \leq N$ or $J^{n-1} \cdot K = \left(\alpha_J^M\right)^{n-1} \alpha_K^M(M) \leq N$.
A similar proof can be stated for semi-n-absorbing preradicals. □

Remark 5.10. Note that in Theorem 5.9, for the case $n = 2$ we can omit the condition $M \in R$-Ass, by the definition of quasi-2-absorbing (semi-2-absorbing) submodules.

Definition 5.11. Let $M \in R$-Ass. We say that M is a quasi-n-absorbing (resp. semi-n-absorbing) module if its zero submodule 0 is a quasi-n-absorbing (resp. semi-n-absorbing) submodule of M.

Corollary 5.12. *Let R be a ring. If R is a semisimple Artinian ring, then every R-module is quasi-i-absorbing for every $i \geq 2$.*

Proof. By Proposition 3.5 and Theorem 5.9. □

Example 5.13. Let R be a semisimple Artinian ring and $S_1, S_2, \dots, S_{n+1} \in R$-simp be distinct. Then $\bigoplus_{i=1}^{n+1} S_i$ is quasi-n-absorbing by Corollary 5.12. But note that, by [19, Corollary 3.6] and Theorem 3.11, $\bigoplus_{i=1}^{n+1} S_i$ is not n-absorbing. This example shows that the two concepts of quasi-n-absorbing modules and of n-absorbing modules are different in general.

Proposition 5.14. *Let $M_1, M_2, \dots, M_t$ be projective R-modules. Suppose that $M_1, M_2, \dots, M_t$ are quasi-n-absorbing R-modules that satisfy the α-property. Then $M = \bigoplus_{i=1}^t M_i$ is a quasi-$(n+t-1)$-absorbing R-module.*

Proof. Let $M_1, M_2, \dots, M_t$ be quasi-n-absorbing R-modules. Then, by Theorem 5.9, $\omega_{M_1}^{M_1}, \omega_{M_2}^{M_2}, \dots, \omega_{M_t}^{M_t}$ are quasi-n-absorbing preradicals, and so $\omega_M^M = \omega_{M_1}^{M_1} \wedge \omega_{M_2}^{M_2} \wedge \cdots \wedge \omega_{M_t}^{M_t}$ is a quasi-$(n+t-1)$-absorbing preradical by Proposition 3.14(2). Again, by Theorem 5.9, $M = \bigoplus_{i=1}^t M_i$ is a quasi-$(n+t-1)$-absorbing R-module. □

Lemma 5.15. *Let $M \in R$-Mod, $N \leq_{fi} M$ and $K_1, K_2, K_3 \leq M$.*

1. *Suppose that $N \leq K_i$ such that $K_i/N \leq_{fi} M/N$ for every $1 \leq i \leq 3$. If $[(K_1/N) \cdot (K_2/N)] \cdot (K_3/N) = 0$, then $[K_1 \cdot K_2] \cdot K_3 \leq N$. In particular, if $(K_1/N) \cdot (K_2/N) = 0$, then $K_1 \cdot K_2 \leq N$.*
2. *Let $K_i \leq_{fi} M$ and $K_i^* = (K_i + N)/N$ for every $1 \leq i \leq 3$. If M is quasi-projective and $[K_1 \cdot K_2] \cdot K_3 \leq N$, then $[K_1^* \cdot K_2^*] \cdot K_3^* = 0$. In particular, if $K_1 \cdot K_2 \leq N$, then $K_1^* \cdot K_2^* = 0$.*

Proof. (1) Assume that $[(K_1/N) \cdot (K_2/N)] \cdot (K_3/N) = 0$. Notice that by [13, Lemma 17], $K_i/N \leq_{fi} M/N$ implies that $K_i \leq_{fi} M$. Since $[(K_1/N) \cdot (K_2/N)] \cdot (K_3/N) = 0$, then $f((K_1/N) \cdot (K_2/N)) = 0$ for every $f \in \mathrm{Hom}_R(M/N, K_3/N)$. We get $g : M \to K_3$. Since $N \leq_{fi} M$, $g(N) \leq N$, thus g induces $\bar{g} : M/N \to K_3/N$ such that $\bar{g}((K_1/N) \cdot (K_2/N)) = 0$. Now, let $h : M \to K_2$, similarly h induces $\bar{h} : M/N \to K_2/N$. Therefore $\bar{g}(\bar{h}(K_1/N)) = 0$, and thus $gh(K_1) \leq N$. Consequently,

$$[K_1 \cdot K_2] \cdot K_3 = \sum \{g(h(K_1)) \mid g \in \mathrm{Hom}_R(M, K_3), h \in \mathrm{Hom}_R(M, K_2)\} \leq N.$$

(2) Assume that M is quasi-projective and $[K_1 \cdot K_2] \cdot K_3 \leq N$. By [13, Lemma 17], $K_i \leq_{fi} M$ implies that $K_i^* \leq_{fi} M/N$. Let $f : M/N \to K_3^*$ and $g : M/N \to K_2^*$. Let $\pi : M \to M/N$ be the canonical projection and $\pi_i : K_i \to K_i^*$ be its restriction to K_i for $i = 2, 3$. Since M is quasi-projective, M is K_i-projective, for $i = 2, 3$. So there exist $h : M \to K_3$ and $t : M \to K_2$ such that $\pi_3 h = f\pi$ and $\pi_2 t = g\pi$. Since $[K_1 \cdot K_2] \cdot K_3 \leq N$, then $ht(K_1) \leq N$. Therefore $fg(K_1^*) = 0$. Consequently,
$[K_1^* \cdot K_2^*] \cdot K_3^* = \sum\{f(g(K_1^*)) \mid f \in \text{Hom}_R(M/N, K_3^*), g \in \text{Hom}_R(M/N, K_2^*)\} = 0.$ □

Proposition 5.16. *Let M be a quasi-projective R-module and let $N \neq M$ be a fully invariant submodule of M. Then N is quasi-2-absorbing (resp. semi-2-absorbing) in M if and only if M/N is a quasi-2-absorbing (resp. semi-2-absorbing) module.*

Proof. ($\Rightarrow$) Assume that N is quasi-2-absorbing in M and let $J/N,\ K/N$ be fully invariant submodules of M/N such that $(J/N)^2 \cdot (K/N) = 0$. By [13, Lemma 17], $J,\ K$ are fully invariant submodules of M. We deduce from Lemma 5.15 that $J^2 \cdot K \leq N$. Since N is quasi-2-absorbing in M, we have $J^2 \leq N$ or $J \cdot K \leq N$. So $(J/N)^2 = 0$ or $(J/N) \cdot (K/N) = 0$, by Lemma 5.15. Hence M/N is a quasi-2-absorbing module.
($\Leftarrow$) Let $J,\ K$ be fully invariant submodules of M such that $J^2 \cdot K \leq N$. Then, by [13, Lemma 17], $J^* = (J + N)/N,\ K^* = (K + N)/N$ are fully invariant submodules of M/N. By Lemma 5.15, $J^{*2} \cdot K^* = 0$. Since M/N is assumed to be a quasi-2-absorbing module, we get $J^{*2} = 0$ or $J^* \cdot K^* = 0$. Hence $J^2 \leq N$ or $J \cdot K \leq N$, by Lemma 5.15. Consequently, N is quasi-2-absorbing in M. □

Theorem 5.17. *Let $M \in R$-Ass that satisfies the α-property. The following statements are equivalent:*

(1) *M is quasi-n-absorbing;*
(2) *ω_0^M is quasi-n-absorbing;*
(3) *For each fully invariant submodule K of M and $\alpha \in R$-pr, $\alpha^n \preceq \omega_0^K \Rightarrow \alpha^{n-1} \preceq \omega_0^K$ or $\alpha^n \preceq \omega_0^M$;*
(4) *For each fully invariant submodule K of M and $\alpha \in R$-pr, $\alpha^n(K) = 0 \Rightarrow \alpha^{n-1}(K) = 0$ or $\alpha^n(M) = 0$;*
(5) *For each $\tau, \eta \in R$-pr, $M \in \mathbb{F}_{\tau^n\eta} \Rightarrow M \in \mathbb{F}_{\tau^n}$ or $M \in \mathbb{F}_{\tau^{n-1}\eta}$.*

Proof. (1) $\Leftrightarrow$ (2) Is clear by Theorem 5.9.
(2) $\Rightarrow$ (3) Assume that K is a fully invariant submodule of M and $\alpha \in R$-pr such that $\alpha^n \preceq \omega_0^K$ and $\alpha^n \npreceq \omega_0^M$. Then $\alpha^n(K) \leqslant \omega_0^K(K) = 0$, and so $\alpha^n\omega_K^M(M) = 0$ which shows that $\alpha^n\omega_K^M \preceq \omega_0^M$. Now, since ω_0^M is quasi-n-absorbing and $\alpha^n \npreceq \omega_0^M$, then $\alpha^{n-1}\omega_K^M \preceq \omega_0^M$. Hence $\alpha^{n-1}(K) = \alpha^{n-1}\omega_K^M(M) = 0$, and thus $\alpha^{n-1} \preceq \omega_0^K$.
(3) $\Leftrightarrow$ (4) Is obvious.
(4) $\Rightarrow$ (5) Let $\tau, \eta \in R$-pr such that $\tau^n\eta(M) = 0$. Suppose that $\tau^{n-1}\eta(M) \neq 0$. By setting $K := \eta(M)$ we have $\tau^n(K) = 0$, $\tau^{n-1}(K) \neq 0$. Consequently, $\tau^n(M) = 0$, by (4).
(5) $\Rightarrow$ (2) Let $\tau, \eta \in R$-pr such that $\tau^n\eta \preceq \omega_0^M$. Then, $\tau^n\eta(M) = 0$, so by hypothesis $\tau^n(M) = 0$ or $\tau^{n-1}\eta(M) = 0$. Consequently, $\tau^n \preceq \omega_0^M$ or $\tau^{n-1}\eta \preceq \omega_0^M$, so ω_0^M is quasi-n-absorbing. □

Similarly we can prove the following theorem.

Theorem 5.18. *Let $M \in R$-Ass that satisfies the α-property. The following statements are equivalent:*

(1) *M is semi-n-absorbing;*
(2) *ω_0^M is semi-n-absorbing;*
(3) *For each $\tau \in R$-pr, $M \in \mathbb{F}_{\tau^{n+1}} \Rightarrow M \in \mathbb{F}_{\tau^n}$.*

Theorem 5.19. *Let $M \in R$-Mod be such that, for each pair K, L of fully invariant submodules of M, we have $\alpha_K^M \alpha_L^M = \alpha_{K\cdot L}^M$. Then, for each quasi-n-absorbing (resp. semi-n-absorbing) preradical σ such that $\sigma(M) \neq M$, we have that $\sigma(M)$ is quasi-n-absorbing (resp. semi-n-absorbing) in M.*

Proof. Let σ be a quasi-n-absorbing preradical such that $\sigma(M) \neq M$. If K, L are fully invariant submodules of M such that $K^n \cdot L \leq \sigma(M)$, then

$$\left(\alpha_K^M\right)^n \alpha_L^M = \alpha_{K^n\cdot L}^M \preceq \alpha_{\sigma(M)}^M \preceq \sigma.$$

Since σ is quasi-n-absorbing, then $\alpha_{K^n}^M = \left(\alpha_K^M\right)^n \preceq \sigma$ or $\alpha_{K^{n-1}\cdot L}^M = \left(\alpha_K^M\right)^{n-1} \alpha_L^M \preceq \sigma$. In the first case we have $K^n = \alpha_{K^n}^M(M) \leq \sigma(M)$; in the second case we have $K^{n-1} \cdot L = \alpha_{K^{n-1}\cdot L}^M(M) \leq \sigma(M)$. Consequently, $\sigma(M)$ is quasi-n-absorbing. □

Lemma 5.20. *Let $M \in R$-Mod. If M is projective in $\sigma[M]$, then $\alpha_K^M \alpha_L^M = \alpha_{K\cdot L}^M$ for any fully invariant submodules K and N of M.*

Proof. It follows from Proposition 2.3. □

Corollary 5.21. *Let σ be a quasi-n-absorbing (resp. semi-n-absorbing) preradical. Then $\sigma(R)$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R.*

Proof. Notice that if $\sigma(R) = R$, then by [4, Proposition 4(v)], $\sigma = 1$ which is a contradiction. Now apply Theorem 5.19 and Lemma 5.20. □

For two R-modules U, N, the submodule

$$\mathrm{Rej}(N, U) = \bigcap\{\mathrm{Ker}\ f | f \in \mathrm{Hom}_R(N, U)\} \leq N$$

is called the *reject of U in N*.

Corollary 5.22. *Let $M \in R$-Ass that satisfies the α-property. If M is quasi-n-absorbing (resp. semi-n-absorbing), then $Ann_R(M)$ is a quasi-n-absorbing (resp. semi-n-absorbing) ideal of R.*

Proof. Note that for any R-module M, $\omega_0^M(R) = \mathrm{Rej}(R, M) = \mathrm{Ann_R}(M)$. Now apply Theorem 5.17 and Corollary 5.21. □

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