Simple continuous modules

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Abstract

A module $M$ is called a simple continuous module if it satisfies the conditions $(\text{min} - C_1)$ and $(\text{min} - C_2)$. A module $M$ is called singular simple-direct-injective if for any singular simple submodules $A$, $B$ of $M$ with $A \cong B \mid M$, then $A \mid M$. Various basic properties of these modules are proved, and some well-studied rings are characterized using simple continuous modules and singular simple-direct-injective modules. For instance, it is shown that a ring $R$ is a right $V$-ring if and only if every right $R$-module is a simple continuous module, and that a regular ring $R$ is a right $GV$-ring if and only if every cyclic right $R$-module is a singular simple-direct-injective module.

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1. Introduction and Preliminaries

Throughout this paper, $R$ is an associative ring with identity and all modules are unital right $R$-modules. For a module $M$, we denote by $\text{Soc}(M)$ and $E(M)$ the socle and the injective hull of $M$, respectively. We write $N \leq M$ if $N$ is a submodule of $M$, $N \leq_{e} M$ if $N$ is an essential submodule of $M$, $N \mid M$ if $N$ is a direct summand of $M$, and $N \leq_{c} M$ if $N$ is a closed submodule of $M$.

Recall the following conditions for a module $M$:

$(C_1)$ If each submodule $A$ of $M$ is essential in a direct summand of $M$;

$(C_2)$ If a submodule $A$ of $M$ is isomorphic to a direct summand of $M$, then $A$ is a direct summand of $M$;

$(C_3)$ $K \oplus L$ is a direct summand of $M$ whenever $K$ and $L$ are direct summands of $M$ with $K \cap L = 0$;

$(\text{min} - C_1)$ If each simple submodule $A$ of $M$ is essential in a direct summand of $M$;

$(\text{min} - C_2)$ If a simple submodule $A$ of $M$ is isomorphic to a direct summand of $M$, then $A$ is a direct summand of $M$.

Let $M$ be a module. $M$ is called a CS module if it satisfies the condition $(C_1)$; $M$ is called a direct-injective module if it satisfies the condition $(C_2)$; $M$ is called a continuous module if it satisfies the conditions $(C_1)$ and $(C_2)$; $M$ is called a simple-direct-injective module [5] if it satisfies the condition $(\text{min} - C_2)$.
Extending modules (CS-modules) play important roles in rings and categories of modules, their generalizations and related modules have been studied extensively by many authors. The concept of simple-direct-injective modules was introduced by V. Camillo, Y. Ibrahim, M. Yousif and Y. Q. Zhou [5], and some well-studied rings are characterized using simple-direct-injective modules. Motivated by this, simple continuous modules are given in Section 2 and V-rings are characterized in terms of simple continuous modules. It is shown that a ring $R$ is a right $V$-ring (i.e., every simple right $R$-module is injective) if and only if every right $R$-module is a simple continuous module. In [5], the authors proved that a ring $R$ is a right $V$-ring if and only if every right $R$-module is a simple-direct-injective module. As a proper generalization of $V$-rings, the notion of $GV$-rings was posed by V. S. Ramamurthi, K. M. Rangaswamy [14]. A ring $R$ is called a right $GV$-ring if every singular simple right $R$-module is injective. Inspired by those, singular simple-direct-injective modules are introduced in Section 5. It is shown that a ring $R$ is a right $GV$-ring if and only if every right $R$-module is a singular simple-direct-injective module and a regular ring $R$ is a right $GV$-ring if and only if every cyclic right $R$-module is a singular simple-direct-injective module. For standard definitions we refer to [3, 4, 6–12, 15–17].

2. Simple continuous modules

In this section, the notion of simple continuous modules are introduced and some basic properties of simple continuous modules are proved.

**Definition 2.1.** A module $M$ is called a simple continuous module if it satisfies the conditions $(\min - C_1)$ and $(\min - C_2)$.

**Example 2.2.**

1. $\mathbb{Z}$ is a simple continuous module, but not a continuous module.
2. Let $M = \mathbb{Z}_p \oplus \mathbb{Q}$, where $p$ is a prime. Then $M$ is a simple continuous $\mathbb{Z}$-module, but not continuous.

We do not know whether a direct summand of a simple continuous module is a simple continuous module. We have the following.

Recall that a submodule $X$ of $M$ is called fully invariant if for every $h \in S$, $h(X) \subseteq X$, where $S = \text{End}(M)$, [13].

**Proposition 2.3.** Any fully invariant direct summand of a simple continuous module is a simple continuous module.

**Proof.** Let $M$ be a simple continuous module and $K$ a fully invariant direct summand of $M$. It is easy to see that $K$ satisfies the condition $(\min - C_2)$. Next we shall show that $K$ satisfies the condition $(\min - C_1)$. Let $S$ be a simple submodule of $K$. Since $M$ satisfies the condition $(\min - C_1)$, there is a direct summand $H$ of $M$ such that $S \leq e H$. Write $M = H \oplus H'$, then $S \oplus H' \leq e M$, and hence $S \oplus (H' \cap K) \leq e K$. So $S \leq e H \cap K$. Since $K$ is a fully invariant direct summand of $M$ and $M = H \oplus H'$, $K = (H \cap K) \oplus (H' \cap K)$ by [13, Lemma 2.1], as required. \[\Box\]

Recall that a module $M$ is called a (weakly) duo module if any (direct summand) submodule is a fully invariant submodule of $M$, [13].

**Corollary 2.4.** Any direct summand of a simple continuous (weakly) duo is a simple continuous module.

A module $M$ is said to be a UC-module if every submodule of $M$ has a unique closure in $M$, [16].

**Proposition 2.5.** Let $M$ be a simple continuous UC module. Then any summand of $M$ is a simple continuous module.
Proof. Let $M$ be a simple continuous UC module and $K$ a direct summand of $M$. It is easy to see that $K$ satisfies the condition $(\text{min} - C_2)$. Next we shall show that $K$ satisfies the condition $(\text{min} - C_1)$. Let $S$ be a simple submodule of $K$. Since $M$ satisfies the condition $(\text{min} - C_1)$, there exists a direct summand $H$ of $M$ such that $S \leq H$. Let $L$ denote the closure of $S$ in $K$. So that $S \leq L \leq K$, and hence $L \leq M$. Thus $S \leq L \leq M$ and $S \leq H \leq M$. Since $M$ is a UC module, $L = H$. Since $H$ is a direct summand of $M$, $L$ is a direct summand of $K$. Therefore $S$ is essential in a direct summand $L$ of $K$, as desired. \qed

Example 2.6. $\mathbb{Z}_2$ and $\mathbb{Z}_8$ are simple continuous $\mathbb{Z}$-modules, but $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not a simple continuous $\mathbb{Z}$-module because the non-summand $0 \oplus \mathbb{Z}(4 + 8\mathbb{Z})$ is isomorphic to the simple summand $\mathbb{Z}_2 \oplus 0$.

Example 2.7. ([11, Example 2.9]) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is any field. Let $A = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. It is clear that $A$ and $B$ are simple continuous as $R$-modules. However $R = A \oplus B$ is not simple continuous.

The above two examples show that a direct sum of simple continuous modules need not be a continuous module, so we have the following.

Proposition 2.8. Let $M = M_1 \oplus M_2$, where $M_1$ and $M_2$ satisfy the condition $(\text{min} - C_1)$ and $M_1$ is $M_2$-injective, then $M$ satisfies the condition $(\text{min} - C_1)$.

Proof. Let $S$ be a simple submodule of $M$. We shall prove that $S$ is essential in a direct summand of $M$ by considering two cases.

Case 1: $S \cap M_1 = 0$. In this case, since $M_1$ is $M_2$-injective, there exists a direct summand $N$ of $M$ such that $N \cong M_2$, $S \leq N$ and $M = M_1 \oplus N$. Then $N$ satisfies the condition $(\text{min} - C_1)$, and so there is a direct summand $K$ of $N$ such that $S \leq K$, as required.

Case 2: $S \cap M_1 \neq 0$. Since $S$ is simple, $S \leq M_1$. The rest is obvious. \qed

Lemma 2.9 ([5, Lemma 3.3]). If $M$ is an indecomposable module that is not simple, then $M \oplus E(M)$ is simple-direct-injective.

Corollary 2.10. If $M$ is a uniform module that is not simple, then $M \oplus E(M)$ is a simple continuous module.

Proof. It follows by Proposition 2.8 and Lemma 2.9. \qed

The following examples reveal the relationships among simple-direct-injective modules, modules satisfying the condition $(\text{min} - C_1)$ and modules satisfying the condition $(C_1)$.

Example 2.11.

1. Let $p$ be any rational prime and $M_1 = \mathbb{Z}_p$, $M_2 = \mathbb{Z}_\infty$. Then $M = M_1 \oplus M_2$ satisfies the condition $(\text{min} - C_1)$, but not the condition $(C_1)$.

2. Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{pmatrix}$ be the upper triangular generalized triangular matrix ring. Then $R_R$ satisfies the condition $(\text{min} - C_1)$, but not the condition $(C_1)$.

3. $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ satisfies the condition $(\text{min} - C_1)$, but it is not a simple-direct-injective module because the non-summand $0 \oplus \mathbb{Z}(4 + 8\mathbb{Z})$ is isomorphic to the simple summand $\mathbb{Z}_2 \oplus 0$.

4. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is any field. Then $R_R$ satisfies the condition $(\text{min} - C_1)$, but it is not a simple-direct-injective module. As $\text{Soc}(R_R)$ is projective, if $R_R$ is a simple-direct-injective module, then $R$ is a mininjective ring by [5, P44]. It is impossible.
The following conditions are equivalent for a ring $R$:

1. $R$ is a right $V$-ring.
2. Every right $R$-module is a simple continuous module.
3. Every finitely cogenerated right $R$-module is a simple continuous module.
4. Direct sums of simple continuous modules are simple continuous modules.
5. Every 2-generated right $R$-module is a simple continuous module.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (5). They are clear.

(3) $\Rightarrow$ (1) Let $S$ be a simple right $R$-module. Since $S \oplus E(S)$ is finitely cogenerated, it is a simple continuous module by hypothesis. Thus $S \oplus E(S)$ is simple-direct-injective, and hence $S = E(S)$ by [5, Proposition 2.1]. Therefore $S$ is injective and $R$ is a right $V$-ring.

(4) $\Rightarrow$ (1) Let $S$ be a simple right $R$-module. Since $S$ and $E(S)$ are simple continuous modules, $S \oplus E(S)$ is a simple continuous module by hypothesis. Thus $S \oplus E(S)$ is simple-direct-injective, and hence $S = E(S)$ by [5, Proposition 2.1]. Therefore $S$ is injective and $R$ is a right $V$-ring.

(5) $\Rightarrow$ (1) Let $S = xR$ be a simple right $R$-module and $0 \neq y \in E(S)$. Then $xR \leq yR$. By hypothesis, $xR \oplus yR$ is a simple continuous module, and so it is simple-direct-injective. Thus, $xR = yR$ by [5, Proposition 2.1] and hence $S = E(S)$. Therefore $S$ is injective and $R$ is a right $V$-ring.

It is well known that a ring $R$ is semisimple if and only if every right $R$-module is a continuous module. From Theorem 3.1, if a ring $R$ is a right $V$-ring which is not semisimple, then there is a simple continuous module which is not a continuous module. See the following example.

Example 3.2. Let $\mathbb{F}$ be a field and $\mathfrak{J}$ be an infinite index set. Let $R = \Pi_{i \in \mathfrak{J}} F_i$, where $F_i = \mathbb{F}$ for each $i \in \mathfrak{J}$. Then $R$ is a right $V$-ring which is not semisimple, and hence there is a simple continuous module which is not a continuous module.

Proposition 3.3. A regular ring $R$ is a right $V$-ring if and only if every cyclic right $R$-module is a simple continuous module.

Proof. “$\Rightarrow$”. This is clear by Theorem 3.1.

“$\Leftarrow$”. Since every cyclic right $R$-module is a simple continuous module, it is simple-direct-injective. The rest is obvious by [5, Theorem 4.4].

Lemma 3.4. Any direct sum of injective modules is a simple continuous module.

Proof. It is clear by [5, Lemma 3.1].

A module $M$ is called strongly soc-injective if for any module $N$ and any semisimple submodule $K$ of $N$, every homomorphism $f : K \to M$ extends to $N$. [2].

Lemma 3.5 ([2, Proposition 16]). A module $M$ is strongly soc-injective if and only if $M = E \oplus T$, where $E$ is injective and $\text{Soc}(T) = 0$. 

3. Simple continuous modules and $V$-rings

In this section, some connections between right $V$-rings and simple continuous modules are presented.

Theorem 3.1. The following conditions are equivalent for a ring $R$:

1. $R$ is a right $V$-ring.
2. Every right $R$-module is a simple continuous module.
3. Every finitely cogenerated right $R$-module is a simple continuous module.
4. Direct sums of simple continuous modules are simple continuous modules.
5. Every 2-generated right $R$-module is a simple continuous module.
**Proposition 3.6.** The following are equivalent for a ring $R$:

1. $R$ is a right noetherian right $V$-ring;
2. Every simple continuous module is strongly Soc-injective.

**Proof.** Similar to [5, Proposition 4.3].

**4. When are simple continuous modules continuous?**

We characterize the rings whose simple continuous modules are continuous.

**Lemma 4.1** ([1, Corollary 2.4 and 2.6]).

1. If $M = A_1 \oplus A_2$ is a $C_3$-module and $f : A_1 \rightarrow A_2$ is an $R$-monomorphism, then $\text{Im}f$ is a direct summand of $A_2$.
2. If $M \oplus M$ is a $C_3$-module, then $M$ is a $C_2$-module.

A module is uniserial if the lattice of its submodules is totally ordered under inclusion. A ring $R$ is called left uniserial if $R R$ is a uniserial module. A ring $R$ is called serial if both modules $R R$ and $R R$ are direct sums of uniserial modules.

A ring $R$ is said to satisfy the condition $(\ast)$ if every finitely generated right $R$-module satisfies the condition $(\text{min} - C_1)$. For instance, a dedekind domain satisfies the condition $(\ast)$.

**Theorem 4.2.** The following are equivalent for a ring $R$ with the condition $(\ast)$:

1. Every simple continuous right $R$-module is a $C_3$-module.
2. Every simple continuous right $R$-module is continuous.
3. Every simple continuous right $R$-module is quasi-injective.
4. Every right $R$-module is a direct sum of a semisimple module and a family of injective uniserial modules of length 2.
5. $R$ is an artinian serial ring with $J(R)^2 = 0$.

**Proof.** (3) $\Rightarrow$ (2) $\Rightarrow$ (1) They are clear.

(1) $\Rightarrow$ (4) We claim that $R$ is right artinian. First we show that $R$ is right semiartinian. Assume on the contrary that $M$ is a right $R$-module with $\text{Soc}(M) = 0$. If $0 \neq N \leq M$, then $\text{Soc}(N \oplus M) = 0$ and $N \oplus M$ is a simple continuous module. Thus $N \oplus M$ is a $C_3$-module by hypothesis, and the inclusion map $i : N \rightarrow M$ splits by Lemma 4.1. This shows that $M$ is semisimple, a contradiction. Thus $\text{Soc}(M) \neq 0$ for every right $R$-module $M$, and hence $R$ is right semiartinian. Next we show that $R$ is right noetherian. It suffices to show that, for any family $K_i (i \in I)$ of simple right $R$-modules, $M = \bigoplus_{i \in I} E(K_i)$ is injective. By Lemma 3.4, $M \oplus E(M)$ is a simple continuous module, so $M \oplus E(M)$ is a $C_3$-module by hypothesis. By Lemma 4.1, the inclusion map $i : M \rightarrow E(M)$ splits, and hence $M = E(M)$ is injective, as required. So $R$ is right noetherian, and hence $R$ is right artinian.

We next show that every indecomposable injective right $R$-module $E$ has a unique composition series of length at most 2. Note that $E$ has a simple socle $X$ and $E = E(X)$. If $E = X$, we are done. Suppose that $X \subset Y \subset E$. It suffices to show that $Y = E$. Let $M = Y \oplus E$. Then $M$ is a simple continuous module by Corollary 2.10, and hence $M$ is a $C_3$-module. So $Y = E$ by Lemma 4.1, as desired.

We now show that every finitely generated indecomposable right $R$-module has a unique composition series of length at most 2. To see this, let $M$ be a finitely generated indecomposable right $R$-module. If $M$ is simple, we are done. If $M$ is not simple, since $R$ satisfies the condition $(\ast)$, $M$ satisfies the condition $(\text{min} - C_1)$, and hence $M \oplus E(M)$ satisfies the condition $(\text{min} - C_1)$ by Proposition 2.8. Therefore $M \oplus E(M)$ is a simple continuous module by Lemma 2.9. Thus $M \oplus E(M)$ is a $C_3$-module by hypothesis, and so $M = E(M)$ is injective by Lemma 4.1. Thus $M$ is an indecomposable injective right $R$-module, and, as above, it has a unique composition series of length at most 2.
Finally, consider an arbitrary right $R$-module $M$. Since $R$ is right noetherian, $M$ contains a maximal injective submodule $N$. Write $M = N \oplus K$, where $K$ contains no nonzero injective submodules. The injective module $N$ is a direct sum of indecomposable injective modules each of which has a unique composition series of length at most 2. Thus there is a decomposition $N = E_1 \oplus E_2$, where $E_1$ is semisimple and $E_2$ is a direct sum of injective uniserial modules of length 2. So, to finish the proof, it suffices to show that $K$ is semisimple. Without loss of generality, we may assume that $K$ is a cyclic module. Since $R$ is right artinian, $K$ is artinian, so it is a direct sum of indecomposable modules. Therefore we can further assume that $K$ is a cyclic indecomposable module. As above, $K$ is a uniserial module of length at most 2. If $K$ is of length 2, then $K = E(K)$ because $E(K)$ is a uniserial module of length at most 2. This contradicts the fact that $K$ contains no nonzero injective submodules. Hence $K$ is simple, as desired.

The rest follow by [5, Theorem 3.4].

**Corollary 4.3.** A dedekind domain $R$ is semisimple artinian if and only if every simple continuous module is injective.

**Proof.** “$\Rightarrow$” is clear.

“$\Leftarrow$” if every simple continuous module is injective, then $R$ is a $V$-ring. But $R$ is artinian by Theorem 4.2, so $R$ is semisimple artinian. □

5. **Singular simple-direct-injective modules and GV-rings**

In [5], the authors proved that a ring $R$ is a right $V$-ring if and only if every right $R$-module is a simple-direct-injective module. As a generalization of $V$-rings, the notion of $GV$-rings was posed by V. S. Ramamurthi, K. M. Rangaswamy [14]. A ring $R$ is called a right $GV$-ring if every singular simple right $R$-module is injective. Inspired by those, singular simple-direct-injective modules are introduced in this Section. It is shown that a ring $R$ is a right $GV$-ring if and only if every right $R$-module is a singular simple-direct-injective module and a regular ring $R$ is a right $GV$-ring if and only if every cyclic right $R$-module is a singular simple-direct-injective module.

**Definition 5.1 ([14]).** A ring $R$ is a right $GV$-ring if each simple right $R$-module is either projective or injective if and only if every singular simple right $R$-module is injective.

**Proposition 5.2.** The following are equivalent for a module $M$:

1. For any singular simple submodules $A, B$ of $M$ with $A \cong B \mid M, A \mid M$.
2. For any singular simple summands $A, B$ of $M$ with $A \cap B = 0, A \oplus B \mid M$.
3. If $M = A_1 \oplus A_2$ with $A_1$ singular simple and $f : A_1 \rightarrow A_2$ an $R$-homomorphism, then $\text{Im} f \mid A_2$.

**Proof.** (1) $\Rightarrow$ (2) Let $A, B$ be singular simple summands of $M$ with $A \cap B = 0$. Write $M = A \oplus T$ for a submodule $T \leq M$, and let $\pi : A \oplus T \rightarrow T$ be the canonical projection. Clearly $A \oplus B = A \oplus \pi(B)$. Since $\pi(B) \cong B$ and $B$ is a singular simple summand of $M$, $\pi(B) \mid M$ by hypothesis, and so $\pi(B) \mid T$. Thus $M = A \oplus T = A \oplus \pi(B) \oplus K = A \oplus B \oplus K$ for a submodule $K \leq T \leq M$. Therefore $A \oplus B \mid M$.

(2) $\Rightarrow$ (3) Without loss of generality we may assume that $f \neq 0$. This means that $f$ is an $R$-monomorphism. Let $T = \{a + f(a) : a \in A_1\}$ be the graph submodule of $M$. We claim that $M = T \oplus A_2$. For, if $x \in M$, then $x = a + b$, where $a \in A_1$, $b \in A_2$. Now $x = a + f(a) - f(a) + b \in T + A_2$, and so $M = T + A_2$. If $x \in T \cap A_2$, then $x = a + f(a)$ for some $a \in A_1$, and consequently $a = x - f(a) \in A_1 \cap A_2 = 0$. This shows that $x = 0$, so $M = T \oplus A_2$, and $T \mid M$. Next we show that $A_1 \cap T = 0$. For, if $x \in A_1 \cap T$, then $x = a + f(a)$ for some $a \in A_1$, and consequently $x - a = f(a) \in A_1 \cap A_2 = 0$. Now, since $f$ is monic, $a = 0$, and hence $x = 0$. Since $T \cong M/A_2 \cong A_1$ is singular simple, $A_1 \oplus T \mid M$ by hypothesis. Finally we show that $A_1 \oplus T = A_1 \oplus \text{Im} f$. For, if $x \in \text{Im} f$, then $x = f(a)$

for some \(a \in A_1\), and so \(x = -a + a + f(a) \in A_1 + T\), and hence \(A_1 \oplus T = A_1 \oplus Imf\). Since \(A_1 \oplus T \mid M\), \(A_1 \oplus Imf \mid M\), and so \(Imf \mid A_2\), as required.

(3) \(\Rightarrow\) (1) Let \(A, B\) be singular simple submodules of \(M\) with \(B \cong A \mid M\). We need to show that \(B \mid M\). If \(A \cap B \neq 0\), there is nothing to prove. Otherwise, assume that \(A \cap B = 0\), and write \(M = A \oplus T\) for some submodule \(T\) of \(M\). If \(\pi : A \oplus T \to T\) be the canonical projection, then clearly \(A \oplus B = A \oplus \pi(B)\) and \(\pi(B) \cong B\) is singular simple. Now, since \(A\) is singular simple, \(M = A \oplus T\), and \(\pi \mid_{B} \sigma^{-1} : A \to T\) is monic with \(Im(\pi \mid_{B} \sigma^{-1}) = \pi(B)\). By hypothesis, \(\pi(B) \mid T\). If \(T = \pi(B) \oplus K\) for some submodule \(K\) of \(T\), then \(M = A \oplus T = A \oplus \pi(B) \oplus K = A \oplus B \oplus K\) and \(B \mid M\), as desired.

\[\square\]

**Definition 5.3.** A module \(M\) is called singular simple-direct-injective if \(M\) satisfies the equivalent conditions of Proposition 5.2.

**Theorem 5.4.** The following conditions are equivalent for a ring \(R\):

1. \(R\) is a right GV-ring.
2. Every right \(R\)-module is a singular simple-direct-injective module.
3. Every finitely cogenerated right \(R\)-module is a singular simple-direct-injective module.
4. Direct sums of singular simple-direct-injective modules are singular simple-direct-injective modules.
5. Every 2-generated right \(R\)-module is a singular simple-direct-injective module.

**Proof.** Similar to Theorem 3.1. \(\square\)

**Example 5.5.** Let \(R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}\), where \(F\) is any field. Then \(R\) is a right GV-ring and not a right \(V\)-ring. Since a ring \(R\) is a right \(V\)-ring if and only if every right \(R\)-module is simple-direct-injective, there is a singular simple-direct-injective module that is not simple-direct-injective by Theorem 5.4.

**Theorem 5.6.** A regular ring \(R\) is a right GV-ring if and only if every cyclic right \(R\)-module is singular simple-direct-injective.

**Proof.** 

\(\Rightarrow\) is clear by Theorem 5.4.

\(\Leftarrow\) Let \(S\) be a singular simple right \(R\)-module and \(E = E(S)\) the injective hull of \(S\). Assume, there is a nonzero element \(x \in E\) such that \(x \not\in S\). Clearly, \(S \leq e_x R\). Define the epimorphism \(f : R \to xR\) by \(f(r) = xr\), \(r \in R\), and set \(X = Ker f\).

Now the map \(f\) induces an isomorphism \(\sigma : xR \to R/X\). If \(T/X = \sigma(S)\) is singular simple, then \(T/X = (tR + X)/X\) for some nonzero element \(t \in R\). Since \(R\) is regular, there is \(s \in R\) such that \(ts = t\). If we set \(e = ts\), then \(e^2 = e\) and \(T/X = (tR + X)/X = (eR + X)/X\). Inasmuch as \(S \leq e_x X\), we infer that \(T/X\) is a minimal essential right ideal of \(R/X\). If \(M = \{r \in R : er \in X\}\), then \(R/M \cong T/X\) and \(M\) is a maximal right ideal of \(R\).

Now we claim that, for \(N = M \cap X\), \(X/N \cong R/M\). To see this, observe first since \((eR + X)/X\) is a singular simple essential submodule of \(R/X\) and \(((1 - e)R + X)/X\) is a nonzero submodule of \(R/X\), it follows that \((eR + X)/X \cong ((1 - e)R + X)/X\), and hence \(e + X = (1 - e)(-r) + X\) for some \(r \in R\). Therefore \(e = e + (1 - e)r \in X\), and if we multiply on the left by \(e\), we get \(ey = e\). Now \(N = M \cap X \subseteq X \subseteq T\), and if \(y \in N\), then \(y \in M\), which implies that \(ey \in X\), and so \(e \in X\), a contradiction. Thus \(y \not\in N\), and it follows that \(X\) is not contained in \(M\). Now \(X/N = X/(M \cap X) \cong (X + M)/M = R/M\).

Next we show that \((eR + X)/N \cong R/M\). If \(g : R \to (eR + N)/N\) is given by \(g(r) = er + N\), where \(r \in R\), then \(g\) is a well-defined \(R\)-epimorphism. Since \(M\) is a maximal right ideal of \(R\) and \(M \subseteq Ker g\), we infer that \(M = Ker g\) and \((eR + N)/N \cong R/M\), as required.

Next we show that \((1 - e)yR + N)/N \cong R/M\). Observe first that if \(m \in M\), then it follows, from the definition of \(M\) and the fact \(ey = e\), that \(em = emy \in X\), and hence...
ym ∈ M. Therefore ym ∈ M ∩ X = N, and so yM ⊆ N. Since eM ⊆ N and ey = e, it follows that eyM ⊆ N, and consequently (1−e)ym ⊆ yM + eyM ⊆ N. Now if we define h : R → ((1−e)yR+N)/N by h(r) = (1−e)yr+N, where r ∈ R, then h is a well-defined R-epimorphism. Since (1−e)ym ⊆ N, it follows that R/M ≅ ((1−e)yR+N)/N, as desired. Therefore ((1−e)yR+N)/N ≅ (eR+N)/N ⊆ R/M ⊆ X/N ∼ (eR+X)/X ∼ T/X ∼ S are singular simple.

As eM ⊆ N, eN ⊆ eM ⊆ N and N is invariant under left multiplication by e. Therefore R/N = (eR+N)/N ⊕ ((1−e)R+N)/N. Since ((1−e)yR+N)/N ∼ (eR+N)/N and (eR+N)/N is a singular simple summand of R/N, ((1−e)yR+N)/N is a singular simple summand of R/N by hypothesis, and hence ((1−e)yR+N)/N is a singular simple summand of ((1−e)R+N)/N. Thus R/N = (eR+N)/N ⊕ ((1−e)yR+N)/N ⊕ A/N, where A/N ≤ R/N.

Finally, we only need to show that R/N = (eR+N)/N ⊕ X/N. Since if this happens, then (R/N)_R has uniform dimension 2. So A/N must be zero and R/N = (eR+N)/N ⊕ ((1−e)yR+N)/N, and consequently R/X ∼ (eR+N)/N is singular simple, a contradiction. First, we have (eR+N)/N ∩ X/N = 0. To see this, let er + N = x + N ∈ (eR+N)/N ∩ X/N, r ∈ R, x ∈ X, then er − x ∈ N, and since N ⊆ X, it follows that er ∈ X. This means r ∈ M, and hence er ∈ N. Therefore (eR+N)/N ∩ X/N = 0. Since (eR+N)/N and X/N are singular simple submodules of R/N and X/N ∼ (eR+N)/N | R/N, (eR+N)/N ⊕ X/N is a direct summand of R/N by hypothesis. Hence it suffices to show that (eR+N)/N ⊕ X/N ≤ e R/N. Now, let (aR+N)/N be a nonzero submodule of R/N. If a ∈ X, then 0 + N ̸= a + N ∈ X/N ̸⊆ (eR+N)/N ⊕ X/N. Otherwise, assume that a ∈ X. In this case (aR+X)/X is a nonzero submodule of R/X. Consequently, since (eR+X)/X is a singular simple essential submodule of R/X, it follows that (eR+X)/X ⊆ (aR+X)/X. Therefore, e + X = ar + X, r ∈ R. Thus ar = e + l for some l ∈ X and 0 + N ̸= ar + N = (e+N) + (l+N) ∈ (eR+N)/N ⊕ X/N, as desired. □

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