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Research Article

On the locally socle of $C(X)$ whose local cozeroset is cocountable (cofinite)

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Abstract

Let $C_F(X)$ be the socle of $C(X)$ (i.e., the sum of minimal ideals of $C(X)$). We introduce and study the concept of colocally socle of $C(X)$ as $C_\mu S_\lambda(X) = \left\{ f \in C(X) : |X \backslash S_f^\lambda| < \mu \right\}$, where S_f^λ is the union of all open subsets U in X such that $|U \backslash Z(f)| < \lambda$. $C_\mu S_\lambda(X)$ is a z-ideal of $C(X)$ containing $C_F(X)$. In particular, $C_{\aleph_0} S_{\aleph_0}(X) = CC_F(X)$ and $C_{\aleph_1} S_{\aleph_1}(X) = CS_c(X)$ are investigated. For each of the containments in the chain $C_F(X) \subseteq CC_F(X) \subseteq C_\mu S_\lambda(X) \subseteq C(X)$, we characterize the spaces X for which the containment is actually an equality. We determine the conditions such that $CC_F(X)$ ($CS_c(X)$) is not prime in any subrings of $C(X)$ which contains the idempotents of $C(X)$. The primeness of $CC_F(X)$ in some subrings of $C(X)$ is investigated.

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1. Introduction

Unless otherwise mentioned all topological spaces X are infinite completely regular Hausdorff and we will employ the definitions and notations used in [4, 9]. $C(X)$ denotes the ring of all real valued continuous functions on a topological space X. Let $C_c(X) = \{f \in C(X) : |f(X)| \leq \aleph_0\}$, $C^F(X) = \{f \in C(X) : |f(X)| < \aleph_0\}$. A topological space X is called functionally countable whenever $C(X) = C_c(X)$, see [6, 7]. Motivated by the fact that $C_c(X)$ is the largest subring of $C(X)$ whose elements have countable image, the subrings $L_c(X)$, $L_{cc}(X)$ of $C(X)$ where $C_c(X) \subseteq L_{cc}(X) \subseteq L_c(X) \subseteq C(X)$ are introduced. Let C_f be the union of all open subsets $U \subseteq X$ such that $|f(U)| \leq \aleph_0$, $L_c(X) = \{f \in C(X) : \overline{C_f} = X\}$ and $L_{cc}(X) = \{f \in C(X) : |X \backslash C_f| \leq \aleph_0\}$, see [12, 14]. For each $f \in C(X)$, the zero-set of f, denoted by $Z(f)$, is the set of zeros of f and $X \backslash Z(f) = coz(f)$ is the cozero-set of f and the set of all zero-sets in X is denoted by $Z(X)$. An ideal I in $C(X)$ is called a z-ideal if whenever $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$. The socle of $C(X)$ (i.e., $C_F(X)$) which is in fact a direct sum of minimal ideals of $C(X)$ is characterized topologically in [13, Proposition 3.3], and it turns out that $C_F(X) = \{f \in C(X) : |X \backslash Z(f)| < \aleph_0\}$ is a useful object in the context of $C(X)$. We know that one of the main objectives of working in the context of $C(X)$ is to characterize topological properties of a given space X in terms of a suitable algebraic properties of $C(X)$ and $C_F(X)$

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is an important object in this way, see [1–3,5,13]. Let λ, μ be two arbitrary infinite ordinal numbers. In [11], the λ-super socle of $C(X)$, $S_\lambda(X) = \{f \in C(X) : |X\backslash Z(f)| < \lambda\}$ which includes $C_F(X) = S_{\aleph_0}(X)$ is investigated, see also [8]. This motivates us to investigate the locally socle of $C(X)$. We define $LS_\lambda(X) = \left\{f \in C(X) : \overline{S_f^\lambda} = X\right\}$, where S_f^λ is the union of all open subsets U in X such that $|U\backslash Z(f)| < \lambda$. $LS_\lambda(X)$ is called the locally λ-super socle of $C(X)$ and it is a z-ideal of $C(X)$ containing $C_F(X) = S_{\aleph_0}(X)$ and $S_\lambda(X)$. Let us put $LS_{\aleph_0} = LC_F(X)$, we characterize spaces X for which the equality in the relation $C_F(X) \subseteq LC_F(X) \subseteq C(X)$ is hold. In fact, it is shown that X is an almost discrete space if and only if $LC_F(X) = C(X)$. We note that if X is an infinite space, then $C_F(X) \subsetneq C(X)$. It is also observed that $|I(X)| < \infty$ if and only if $C_F(X) = LC_F(X)$, see [15]. We state this facts for locally λ-super socle. The importance of the role of $C_F(X)$ in the context of $C(X)$, and the subalgebra $C_c(X) \subseteq L_{cc}(X) \subseteq L_c(X) \subseteq C(X)$ motivated us to define and study the colocally socle of $C(X)$, $C_\mu S_\lambda(X)$ and in particular $C_{\aleph_0} S_{\aleph_0}(X) = CC_F(X)$, cofinite locally socle of $C(X)$, and $C_{\aleph_1} S_{\aleph_1}(X) = CS_c(X)$, cocountable locally socle of $C(X)$ are investigated. The equality in the relation $C_F(X) \subseteq CC_F(X) \subseteq C(X)$ is characterized. It is shown that $CC_F(X)$ ($CS_c(X)$) is an intersection of essential ideals of $C(X)$. The conditions such that $CC_F(X)$ ($CS_c(X)$) is not prime in any subrings of $C(X)$ which contains the idempotents of $C(X)$ are determined. We investigate the primeness of $CC_F(X)$ in some subrings of $C(X)$.

2. Colocally socle

Let $O(X)$ be the set of open subsets of X and if $U \subseteq X$ is closed and open it is called clopen. The set of isolated point of X is denoted by $I(X)$ and $I_c(X)$ is the set of points $x \in X$ with countable open neighborhood. An element $x \in X$ is called λ-isolated point if x has a neighborhood with cardinality less than λ. The set of λ-isolated points of X is denoted by $I_\lambda(X)$. A space X is called λ-discrete space if $I_\lambda(X) = X$, see [11].

Definition 2.1. Let $f \in C(X)$ and S_f^λ be the union of all open subsets $U \subseteq X$ such that $|U\backslash Z(f)| < \lambda$, S_f^λ is called the local cozeroset of f. We denote the colocally socle of $C(X)$ by $C_\mu S_\lambda(X)$ and define it to be the set of all $f \in C(X)$ such that $|X\backslash S_f^\lambda| < \mu$. i.e.,

$$S_f^\lambda = \bigcup\{U : U \in O(X),\ |U\backslash Z(f)| < \lambda\},$$

$$C_\mu S_\lambda(X) = \{f \in C(X) : |X\backslash S_f^\lambda| < \mu\}.$$

In particular, we denote $C_{\aleph_0} S_{\aleph_0}(X) = CC_F(X)$, cofinite locally socle of $C(X)$, and $C_{\aleph_1} S_{\aleph_1}(X) = CS_c(X)$ is called cocountable locally socle of $C(X)$. i.e.,

$$S_f^F = S_f^{\aleph_0} = \bigcup\{U : U \in O(X),\ |U\backslash Z(f)| < \aleph_0\},$$

$$CC_F(X) = \{f \in C(X) : |X\backslash S_f^F| < \aleph_0\};$$

and

$$S_f^c = S_f^{\aleph_1} = \bigcup\{U : U \in O(X),\ |U\backslash Z(f)| < \aleph_1\},$$

$$CS_c(X) = \{f \in C(X) : |X\backslash S_f^c| < \aleph_1\}.$$

Definition 2.2. The set

$$LS_\lambda(X) = \{f \in C(X) : \overline{S_f^\lambda} = X\}$$

is called locally λ-super socle of $C(X)$. Let $LS_{\aleph_0}(X) = LC_F(X)$, and $LS_{\aleph_1}(X) = LS_c(X)$, see [15].

If $\lambda_1 < \lambda_2$, then $S_f^{\lambda_1} \subseteq S_f^{\lambda_2}$. Hence $C_\mu S_{\lambda_1}(X) \subseteq C_\mu S_{\lambda_2}(X)$ and $LS_{\lambda_1}(X) \subseteq LS_{\lambda_2}(X)$. If $\mu_1 < \mu_2$, we conclude that $C_{\mu_1} S_\lambda(X) \subseteq C_{\mu_2} S_\lambda(X)$.

Remark 2.3. Let C_f^λ be the union of all $U \in O(X)$ such that $|f(U)| < \lambda$, we define $L_\lambda(X) = \{f \in C(X) : \overline{C_f^\lambda} = X\}$. Let $L_{\aleph_0}(X) = L_F(X)$, $L_{\aleph_1}(X) = L_c(X)$, and $L_F(X)$ ($L_c(X)$) is called locally functionally finite (countable) subalgebra of $C(X)$, see [14]. Now, we define $L_{\mu\lambda}(X) = \{f \in C(X) : |X \backslash C_f^\lambda| < \mu\}$, let $L_{\aleph_0\aleph_0}(X) = L_{FF}(X)$ as cofinite locally functionally finite subalgebra of $C(X)$, and $L_{\aleph_1\aleph_0}(X) = L_{cF}(X)$ ($L_{\aleph_1\aleph_1}(X) = L_{cc}(X)$) are called cocountable locally functionally finite (countable) subalgebra of $C(X)$, see [12]. It is evident that, $L_{\mu\lambda}(X) \subseteq C_\mu S_\lambda(X)$.

Lemma 2.4. *If $f, g \in C(X)$, then the following statements hold.*

(1) $S_{f+g}^\lambda \supseteq S_f^\lambda \cap S_g^\lambda$.
(2) $S_{fg}^\lambda \supseteq S_f^\lambda \cup S_g^\lambda$.
(3) $S_{|f|}^\lambda = S_f^\lambda$.
(4) $S_f^\lambda \subseteq C_f^\lambda$.
(5) *If $f, g \in L_\lambda(X)$, then $\overline{C_f^\lambda \cap C_g^\lambda} = \overline{C_f^\lambda} = \overline{C_g^\lambda} = X$.*
(6) *If $f, g \in LS_\lambda(X)$, then $\overline{S_f^\lambda \cap S_g^\lambda} = \overline{S_f^\lambda} = \overline{S_g^\lambda} = X$.*

Proof. Let

$$S_f^\lambda = \bigcup\{U | U \in O(X), |U \backslash Z(f)| < \lambda\},$$

and

$$S_g^\lambda = \bigcup\{V | V \in O(X), |V \backslash Z(f)| < \lambda\}.$$

Hence

$$\begin{aligned} S_f^\lambda \cap S_g^\lambda &= \bigcup\{U \cap V | \, |U \backslash Z(f)| < \lambda, |V \backslash Z(g)| < \lambda\} \\ &= \bigcup\{W | W \in O(X), |W \backslash Z(f)| < \lambda, |W \backslash Z(g)| < \lambda\} \\ &\subseteq \bigcup\{W | W \in O(X), |W \backslash Z(f+g)| < \lambda\}. \end{aligned}$$

We have $U \backslash Z(f+g) \subseteq (U \backslash Z(f)) \cup (U \backslash Z(g))$, $U \backslash Z(fg) = (U \backslash Z(f)) \cap (U \backslash Z(g))$. Hence the proof of (1), (2) is obvious. Since $U \backslash Z(|f|) = U \backslash Z(f)$, we infer that (3) holds. For (4), let $f \in S_f^\lambda$, therefore $U \in O(X)$, $|U \backslash Z(f)| < \lambda$ which implies that $|f(U)| < \lambda$. We remind the reader that if $\overline{Y} = X$ and $G \in O(X)$, then $\overline{G \cap Y} = \overline{G}$. Since $C_f^\lambda, S_f^\lambda \in O(X)$ are dense, we infer that (5), (6). □

Proposition 2.5. *$C_\mu S_\lambda(X)$ is a z-ideal of $C(X)$.*

Proof. Let $f, g \in C_\mu S_\lambda(X)$, we show that $f + g \in C_\mu S_\lambda(X)$. By the previous lemma $S_{f+g}^\lambda \supseteq S_f^\lambda \cap S_g^\lambda$, so $\overline{S_{f+g}^\lambda} \supseteq \overline{S_f^\lambda \cap S_g^\lambda} = \overline{S_f^\lambda} = \overline{S_g^\lambda} = X$, hence $f + g \in C_\mu S_\lambda(X)$. Now, let $f \in C_\mu S_\lambda(X)$, $g \in C(X)$ we show that $fg \in C_\mu S_\lambda(X)$. By the previous lemma $S_{fg}^\lambda \supseteq S_f^\lambda \cup S_g^\lambda$, so $\overline{S_{fg}^\lambda} \supseteq \overline{S_f^\lambda \cup S_g^\lambda} = \overline{S_f^\lambda} \cup \overline{S_g^\lambda} = X$. Therefore, $C_\mu S_\lambda(X)$ is an ideal. Now, we prove that $C_\mu S_\lambda(X)$ is a z-ideal. For this mean let $f \in C_\mu S_\lambda(X)$ and $Z(f) \subseteq Z(g)$, we show that $g \in C_\mu S_\lambda(X)$. For each open subset $U \subseteq S_f$, we have $U \backslash Z(g) \subseteq U \backslash Z(f)$, so $S_f^\lambda \subseteq S_g^\lambda$. Hence $X = \overline{S_f^\lambda} \subseteq \overline{S_g^\lambda}$, therefore $g \in C_\mu S_\lambda(X)$. □

Similarly, the next fact is proved.

Proposition 2.6. *$LS_\lambda(X)$ is a z-ideal of $C(X)$.*

Clearly, $C_\mu S_\lambda(X)$ is absolutely convex. In this paper we investigate more $C_{\aleph_0} S_{\aleph_0}(X) = CC_F(X)$, and $C_{\aleph_1} S_{\aleph_1}(X) = CS_c(X)$.

Proposition 2.7.

$$S_f^F = \bigcup\{U : U \in O(X)\,,\, |U \backslash Z(f)| < \aleph_0\} = \bigcup\{V : V \in O(X)\,,\, |V \backslash Z(f)| \leq 1\}.$$

Proof. $\bigcup\{V : V \in O(X), |V\backslash Z(f)| \leq 1\} \subseteq S_f^F$. Let $U\backslash Z(f) = \{x_1, x_2, \dots, x_n\}$, we define $V_i = U\backslash\{x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$. It is obvious that $V_i \in O(X)$ and $V_i\backslash Z(f) = \{x_i\}$. Now, put $U = \bigcup_{i=1}^n V_i$ and we are done. □

If U is an open subset in a Hausdorff space X and $x \in U$, then x is isolated, for $\{x\} = U\backslash(U\backslash\{x\})$. A point in a space X is isolated if and only if it has a finite neighborhood. Clearly, if $U \in O(X)$, then $\overline{\bigcup_{|U|<\aleph_0} U} = X$ if and only if $\overline{I(X)} = X$.

Remark 2.8. We note that if $U \in O(X)$ and $|U\backslash Z(f)| < \aleph_0$, then $U\backslash Z(f) \subseteq I(X)$ it means that $S_f^F \subseteq Z(f) \cup I(X)$. Whenever $|U\backslash Z(f)| \leq \aleph_0$, then $U\backslash Z(f) \subseteq I_c(X)$ and $|f(U)| \leq \aleph_0$ and $S_f^c \subseteq Z(f) \cup I_c(X)$.

Proposition 2.9. $S_\lambda(X) \subseteq C_\mu S_\lambda(X) \subseteq LS_\lambda(X) \subseteq L_\lambda(X)$.

Corollary 2.10. *For a topological space X the following statements hold.*

1. $C_F(X) \subseteq CC_F(X) \subseteq LC_F(X) \subseteq L_F(X)$.
2. $S_c(X) \subseteq CS_c(X) \subseteq LS_c(X) \subseteq L_c(X)$.

3. The coincidence of colocally socle with $C(X)$ and $C_F(X)$

If X is an uncountable scattered space, then $C_F(X) \subsetneq LC_F(X) = C(X)$ and if X is a connected space $(0) = C_F(X) = LC_F(X) \subsetneq C(X)$. Clearly, if X is a λ-discrete space, then $S_\lambda(X) \subseteq C_\mu S_\lambda(X) = LS_\lambda(X) = C(X)$. In particular, if X is discrete, then $CC_F(X) = LC_F(X) = C(X)$.

Theorem 3.1. $|X\backslash I_{\lambda(X)}| < \mu$ *if and only if* $C_\mu S_\lambda(X) = LS_\lambda(X) = C(X)$.

Proof. Let $|X\backslash I_\lambda(X)| < \mu$ and $f \in C(X)$. Since $X\backslash S_f^\lambda \subseteq X\backslash I_\lambda(X)$, we infer that $f \in C_\mu S_\lambda(X)$. Conversely, let $0 \neq r \in C(X) = C_\mu S_\lambda(X)$ and $Z(r) = \emptyset$. Hence $S_r^\lambda = \bigcup\{U | U \in O(X), |U| < \lambda\}$, so $U \subseteq I_\lambda(X)$. Therefore $S_r^\lambda = I_\lambda(X)$ and since $r \in C_\mu S_\lambda(X)$, we conclude that $|X\backslash I_\lambda(X)| = |X\backslash S_r^\lambda| < \mu$. □

Corollary 3.2. *Let X be a topological space, then we have*

1. $|X\backslash I(X)| < \aleph_0$ *if and only if* $CC_F(X) = LC_F(X) = C(X)$.
2. $|X\backslash I_c(X)| < \aleph_1$ *if and only if* $CS_c(X) = LS_c(X) = C(X)$.

Definition 3.3. A topological space X is called almost λ-discete whenever the set of λ-isolated points of X is dense, i.e., $\overline{I_\lambda(X)} = X$.

Theorem 3.4. *X is an almost λ-discrete space if and only if* $LS_\lambda(X) = C(X)$.

Proof. Let $\overline{I_\lambda(X)} = X$ and $f \in C(X)$. Since $I_\lambda(X) \subseteq S_f^\lambda$, we infer that $\overline{S_f^\lambda} = X$, i.e., $f \in LS_\lambda(X)$. Conversely, let r be a nonzero constant function, hence $0 \neq r \in C(X) = LS_\lambda(X)$ and $Z(r) = \emptyset$, therefore $\overline{S_r^\lambda} = X$. Now, we suppose that $G \in O(X)$, so there exists an open subset U in X such that $|U\backslash Z(r)| < \lambda$ and $U \cap G \neq \emptyset$. Hence $U \subseteq I_\lambda(X)$ and $\emptyset \neq U \cap G \subseteq I_\lambda(X)$. This means that $\overline{I_\lambda(X)} = X$. □

Corollary 3.5. *Let X be any topological space, then*

1. *X is an almost discrete space (i.e., $\overline{I(X)} = X$) if and only if* $LC_F(X) = C(X)$.
2. *X is an almost countably discrete space (i.e., $\overline{I_c(X)} = X$) if and only if* $LC_F(X) = C(X)$.

Proposition 3.6. $|I(X)| < \aleph_0$ *if and only if* $C_F(X) = LC_F(X)$.

Proof. Let $C_F(X) = LC_F(X)$. If $|I(X)| > \aleph_0$, there exists an infinite countable subset $A = \{x_1,\ x_2,\ \dots,\ x_n, \dots\} \subseteq I(X)$. We define a function f where for each $x_n \in A$, $f(x_n) = \frac{1}{n}$ and otherwise $f(x_n) = 0$. In this case if $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $\frac{1}{n} < \varepsilon$. Hence $G = X \setminus \{x_1, x_2, ..., x_k\}$ is a clopen subset and for each $x \in G$, $|f(x)| < \varepsilon$. Therefore $f \in C(X)$. But $X \setminus Z(f) = A$ is infinite, hence $f \notin C_F(X)$. Now, we show that $f \in LC_F(X)$. Let $G \subseteq X$ be an arbitrary open set, we must find an open $U \subseteq X$ where $|U \cap Coz(f)| < \aleph_0$ and $U \subseteq G$. We consider two cases: if $x \in G \cap I(X) \neq \emptyset$, it is sufficient put $U = \{x\}$. If $G \cap I(X) = \emptyset$, then $G \subseteq X \setminus I(X) \subseteq X \setminus A$. Hence $G \cap Coz(f) = G \cap A \subseteq (X \setminus A) \cap A = \emptyset$, so we put $U = G$. Therefore $f \in LC_F(X) \setminus C_F(X)$ and it is a contradiction. Conversely, let $|I(X)| < \aleph_0$, we prove that $C_F(X) = LC_F(X)$. Let $f \in LC_F(X)$, hence $\overline{S_f} = X$, and by Proposition 2.7 we have $S_f = \bigcup\{U | U \subseteq X, |U \setminus Z(f)| < 1\}$. i.e., for each open $G \subseteq X$, there exists an open subset $U \subseteq X$ where $U \cap Coz(f) \subseteq \{x\}$ and $U \subseteq G$. Therefore

$$X \setminus Z(f) = \overline{S_f} \setminus \overline{Z(f)} \subseteq \overline{S_f \setminus Z(f)} = \overline{(\cup U) \setminus Z(f)} = \overline{\cup (U \setminus Z(f))} \subseteq \overline{I(X)} = I(X).$$

Hence by topological definition of $C_F(X)$, we conclude that $f \in C_F(X)$. □

Corollary 3.7. *If X is a connected space, then $C_F(X) = LC_F(X) = (0)$.*

Definition 3.8. A space (X, τ) is called λ-open if any subset A of X with $|A| > \lambda$ has nonempty interior.

Proposition 3.9. *The following statements are equivalent.*

(1) *(X, τ) is a λ-open space.*
(2) *If $clA = X$, then $|X \setminus A| < \lambda$.*
(3) *If $A \subseteq X$, then $|A \setminus intA| < \lambda$.*
(4) *If $A \subseteq X$, then $|clA \setminus A| < \lambda$.*
(5) *If $intA = \emptyset$, then $|A| < \lambda$.*

Proof. (1) → (2). Let (X, τ) be λ-open and $clA = X$, then $int(X \setminus A) = \emptyset$. So by (1), $|X \setminus A| < \lambda$.
(2) → (3). $int(A \setminus intA) = \emptyset$, hence $cl(X \setminus (A \setminus intA)) = X$. Therefore by hypothesis $|A \setminus intA| < \lambda$.
(3) → (4). $clA \setminus A = (X \setminus A) \setminus int(X \setminus A)$, so by (3), $|clA \setminus A| < \lambda$.
(4) → (5). Let $intA = \emptyset$, hence by (4), $|A| = |cl(X \setminus A) \setminus (X \setminus A)| < \lambda$.
(5) → (1). By definition it is clear. □

We remind the reader that, if $|X \setminus I_X| < \aleph_1$ then X is $\aleph_1$-open. Conversely, let $A = X \setminus I_X$ and suppose that $|A| > \aleph_1$. Then A can be represented as a disjoint union $A = \{A_n : n \in \mathbb{N}\}$ where $|A_n| > \aleph_0$. Since X is $\aleph_0$-open there is a point $a_n \in intA_n$ for each $n \in \mathbb{N}$. If $B = \{a_n : n \in \mathbb{N}\}$, then $|B| > \aleph_0$ and hence there exists an $m \in \mathbb{N}$ such that $a_m \in intB$. Clearly $intB \cap intA_m = \{a_m\}$ so that $a_m \in I_X$, contradicting the fact that $a_m \in A$. Hence X is $\aleph_1$-open if and only if the set of nonisolated points of X is countable.

Proposition 3.10. *Let X be μ-open, then*

(1) $L_\lambda(X) = L_{\lambda\mu}(X)$.
(2) $LS_\lambda(X) = C_\mu S_\lambda(X)$.

Corollary 3.11. *Let X be $\aleph_0$-open, then*

(1) $L_F(X) = L_{FF}(X)$.
(2) $LC_F(X) = CC_F(X)$.

Corollary 3.12. *Let X be $\aleph_1$-open, then*

(1) $L_c(X) = L_{cc}(X)$.
(2) $LS_c(X) = CS_c(X)$.

4. The primeness of $CC_F(X)$ in some subrings of $C(X)$

Theorem 4.1. *Let X have at least two infinite (uncountable) components with no finite (countable) subset. Then $CC_F(X)$ $(CS_c(X))$ is never prime in any subring A of $C(X)$ which contains the idempotents of $C(X)$.*

Proof. We suppose that X_1, X_2 are infinite (uncountable). We define

$$f(x) = \begin{cases} 1 \quad , \quad x \in X\backslash X_1 \\ \\ 0 \quad , \quad x \in X_1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 \quad , \quad x \in X_1 \\ \\ 0 \quad , \quad x \in X\backslash X_1 \end{cases}$$

So $f,\ g \in C(X)$ and $fg = 0 \in CC_F(X)$ $(CS_c(X))$. But $f,\ g \notin CC_F(X)$ $(CS_c(X))$. For $X_1, X_2 \subseteq X$ are open and $|X_1|, |X_2|$ are infinite (uncountable). $X_1 \subseteq X\backslash S_g$ and $X_2 \subseteq X\backslash S_f$. We note that if $U \subseteq X_1$, $U \subseteq X_2$ be open and $|U \cap Coz(g)|$, $|U \cap Coz(f)|$ be finite (countable), then it is a contradiction with hypothesis. i.e., $f,\ g \notin CC_F(X)$ $(CS_c(X))$. □

Corollary 4.2. *Let X have finite components and at least two of them are infinite. Then $CC_F(X)$ is never prime in any subring A of $C(X)$ which contains the idempotents of $C(X)$.*

We recall that $C_1(X\backslash I(X)) = \{f \in C(X) : |f(X\backslash I(X))| = 1\}$. The next theorem characterized the primeness of $CC_F(X)$ in some subrings of $C(X)$.

Theorem 4.3. *Let $|I(X)| < \aleph_0$ and $R \subseteq C(X)$. If $R \subseteq C_1(X\backslash I(X))$, then $CC_F(X)$ is prime in R. Conversely, if $CC_F(X)$ is prime in R and R contains the idempotents of $C(X)$, then $X\backslash I(X)$ is connected.*

Proof. Let $fg = 0 \in CC_F(X)$ and $f,\ g \in R$. Since $R \subseteq C_1(X\backslash I(X))$ we infer that $f(X\backslash I(X)) = 0$ or $g(X\backslash I(X)) = 0$. Hence $S_f^F = X$ or $S_g^F = X$. Therefore $f \in CC_F(X)$ or $g \in CC_F(X)$, i.e., $CC_F(X)$ is prime in R. Conversely, let $Y = X\backslash I(X) = A \cup B$ where A, B are two nonempty clopen subsets in Y and get a contradiction. Since Y in X is clopen we infer that A, B in X are clopen. It is evident that $X = I(X) \cup A \cup B$. Now, we define $f, g \in R \subseteq C(X)$ such that $f(A \cup I(X)) = 1$, $f(B) = 0$ and $g(B \cup I(X)) = 1$, $g(A) = 0$. It is obvious that $fg = 0 \in CC_F(X)$. Since $B \subseteq X\backslash S_f^F$ and $A \subseteq X\backslash S_g^F$ we infer that $f, g \notin CC_F(X)$ i.e., $CC_F(X)$ is not prime in R that is a contradiction. □

Corollary 4.4. *If $|I(X)| < \aleph_0$ and $X\backslash I(X)$ is connected, then $CC_F(X)$ is prime in $C_c(X)$ and $C^F(X)$.*

Corollary 4.5. *If $|I(X)| < \aleph_0$ and $X\backslash I(X)$ is disconected, then $CC_F(X)$ is never prime in any subring R of $C(X)$ which contains the idempotents of $C(X)$.*

The previous facts also hold for $LC_F(X)$, see [15].

Theorem 4.6. *Let C be a module and $A \leq C$, then A is an intersection of essential submodules of C if and only if* Soc$(C) \leq A$.

Proof. See [10]. □

The next fact gives a simple poof for this theorem.

Proposition 4.7. *$CC_F(X)$ $(CS_c(X))$ is an intersection of essential ideals.*

Proof. $CC_F(X)$ $(CS_c(X))$ is a z-ideal, hence it is an intersection of prime ideals, see [2]. Since every z-ideal which contains $C_F(X)$ is essential, we infer that $CC_F(X)$ $(CS_c(X))$ is an intersection of essential ideals. □

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