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Research Article

On divided and regular divided rings

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Abstract

In this paper, we study the notion of divided and regular divided rings. Then we establish the transfer of these notions to trivial ring extension and amalgamated algebras along an ideal. These results provide examples of non-divided regular divided rings. The article includes a brief discussion of the scope and precision of our results.

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1. Introduction

All rings considered below are commutative with $1 \neq 0$; all subrings and ring homomorphisms are unital. If R is a ring, then $Spec(R)$ (resp., $Max(R)$; resp., $Min(R)$ denotes the set of all prime (resp., maximal; resp., minimal prime) ideals of R; $Z(R)$ the set of zero-divisors of R, $U(R)$ the set of units of R, $Reg(R) := R - Z(R)$ the set of regular elements of R, $Rad(R)$ the Jacobson radical of R, $Nil(B)$ the set of nilpotent elements of R; and $tq(R) = R_{R-Z(R)}$ the total quotient ring of R. A ring R is called a total ring of quotients if $R = tq(R)$, that is every element of R is invertible or zero-divisor.

Let R be a ring and P be a prime ideal of R. Recall from [2] that P is called a divided prime ideal of R if P is comparable (under inclusion) with each principal ideal of R. We say that P is a regular divided prime ideal of R if P is comparable with each ideal generated by a regular element (i.e., a non-zero-divisor) of R. Dobbs and Shapiro shows that if $tq(R)$ the quotient field of R is a von Neumann regular ring, then P is a regular divided prime ideal of R if and only if P is comparable under inclusion to each regular ideal of R (see [11, Proposition 2.1]).

Recall that A. Badawi in [2], say that a ring R is a divided ring if each of its prime ideals is a divided prime ideal; as in [10], a (commutative integral) domain that is a divided ring is called a divided domain. By [11], D.E. Dobbs and J. Shapiro say that a ring R is a regular divided ring if each $P \in Spec(R) - (Max(R) \cap Min(R))$ is comparable with each principal regular ideal of R. In [11, Corollary 3.4], it is shown that if R is a valuation ring [of $tq(R)$] and a ring whose total quotient ring is von Neumann regular, then R is a regular divided ring.

Remark that a divided ring is a regular divided ring and the converse is false (See for example Theorem 2.2(2)). See for instance [2, 3, 10, 11, 19].

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Let A be a ring, E be an A-module and $R := A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E. Recall that a prime (resp., maximal) ideal of R has always the form $M \propto E$, where M is a prime (resp., maximal) ideal of A [13, Theorem 25.1(3)].

Considerable work has been concerned with trivial ring extensions. Part of it has been summarized in Glaz's book [12] and Huckaba's book (where R is called the idealization of E by A) [13]. See for instance [4, 12–14, 16, 17].

The amalgamation algebras along an ideal, introduced and studied by D'Anna, Finocchiaro and Fontana in [7–9] and defined as follows:

Let A and B be two rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [5, 6]). See for instance [1, 7–9, 15–18].

This paper develops results of the transfer of divided and regular divided rings to trivial ring extensions and amalgamated algebras along an ideal in order to give us a wide class of regular divided rings and a class non-divided regular divided rings.

2. Main results

This paper develops a result of the transfer of divided and regular divided rings to trivial ring extensions, and amalgamated algebras along an ideal in order to give us a wide class of regular divided rings and examples of non-divided regular divided rings.

First, we will construct a class of regular divided rings.

Proposition 2.1. *Any total ring of quotients is a regular divided ring.*

Proof. It is clear since every element in a total ring is invertible or zero-divisor. □

The first main result establish necessary and sufficient conditions for the transfer of the divided and regular divided properties in special contexts of trivial ring extension of A by E, where E be an A-module. The result enriches the literature with original examples of regular divided rings and examples of non-divided regular divided rings.

Let $R \propto E$ be the trivial ring extension of a ring A by an A-module E. Remark by [13, Theorem 25.1(3)] that:

$$Spec(R) - (Max(R) \cap Min(R)) = \{P \propto E / P \in Spec(A) - (Max(A) \cap Min(A))\}.$$

Recall that $dim(R)$ means the Krull dimension of a ring R.

Theorem 2.2. *Let $R := A \propto E$ be the trivial ring extension of a ring A by an A-module E.*

(1) *Let D be an integral domain which is not a field, $K = qf(D)$, E be a K-vector space and $R := D \propto E$. Then the following are equivalent:*
  a) *R is a divided ring.*
  b) *R is a regular divided ring.*
  c) *D is a regular divided ring.*
  d) *D is a divided ring.*

(2) *Let (A, M) be a local ring (where M is a maximal ideal of A) such that $dim(A) \neq 0$, E be an A-module with $ME = 0$, and let $R := A \propto E$. Then:*

a) *R is a regular divided ring.*
b) *R is not a divided ring.*

Recall that an A-module E is said a torsion free if $ae = 0$ imply that $a = 0$ or $e = 0$ for every $a \in A$ and $e \in E$.

Before proving Theorem 2.2, we establish the following two Lemmas.

Lemma 2.3. *Let $R := A \propto E$ be the trivial ring extension of a ring A by an A-module E. Then:*

(1) *If R is a divided ring, then so is A.*
(2) *Assume that E is a torsion free A-module. Then, if R is a regular divided ring, then so is A.*

Proof. **1)** Let $P \in Spec(A)$ and $a \in A$. Then $P \propto E \subseteq R(a, e)$ or $R(a, e) \subseteq P \propto E$ for some $e \in E$. Therefore, $P \subseteq Aa$ or $Aa \subseteq P$, as desired.

We can have a second direct proof by using [2, Corollary 3] since $R/(0 \propto E) \cong A$ and R is a divided ring.

2) Assume that E is a torsion free A-module, let $P \in Spec(A) - (Max(A) \cap Min(A))$ be a prime ideal of A, and let $a \in A \setminus Z(A)$. Then, $P \propto E \in Spec(R) - (Max(R) \cap Min(R))$ $(a, e) \in R \setminus Z(R)$ for every $e \in E$. Then $P \propto E \subseteq R(a, e)$ or $R(a, e) \subseteq P \propto E$ since R is a regular divided ring. Therefore, $P \subseteq Aa$ or $Aa \subseteq P$, as desired. □

Lemma 2.4. *Let D be an integral domain which is not a field. Then $Min(D) = \{0\}$ and $Max(D) \cap Min(D) = \emptyset$.*
In particular, D is a divided regular ring if and only if D is a regular ring.

Proof. Clear since $\{0\}$ is a unique minimal prime ideal which is not maximal (since D is not a field). □

Proof of Theorem 2.2. **1)** Let D be a domain, $K = qf(D)$, E be a K-vector space and $R := D \propto E$.

Remark by above that in this context of trivial ring extension, we have $Min(R) = \{0 \propto E\}$ and so $Max(R) \cap Min(R) = \emptyset$ since D is an integral domain which is not a field.

a) $\Longrightarrow$ **b)**. Clear.

b) $\Longrightarrow$ **c)**. If R is a regular divided ring, then so is D by Lemma 2.3(2) since E is a torsion free D-module.

c) $\Longrightarrow$ **d)**. Clear by Lemma 2.4 since D is an integral domain.

d) $\Longrightarrow$ **a)**. Let $P \propto E$ be a prime ideal of R, where P is a prime ideal of D, and $(a, e) \in D \propto E$ be an element of R. Two cases are then possible:

Case 1: $a = 0$. Then $R(a, e) = R(0, e) = 0 \propto De \subseteq P \propto E$, as desired.

Case 2: $a \neq 0$. We have $P \subseteq Da$ or $Da \subseteq P$ since D is a divided ring. Note that $R(a, e) = Da \propto E$ since $aE = E$. Therefore, $R(a, e) = Da \propto E \subseteq P \propto E$ or $P \propto E \subseteq Da \propto E (= R(a, e))$, as desired.

Hence, R is a divided ring and this completes the proof of 1).

2) Let (A, M) be a local ring such that $dim(A) \neq 0$, E be an A-module with $ME = 0$, and let $R := A \propto E$.

a) By Proposition 2.1 since R is a total ring (by [14, Proof of Theorem 2.6]).

b) Let P be a non maximal prime ideal of R since $dim(R) \neq 0$ and let $a \in M - P$. It's clear that $R(a, e) \nsubseteq P \propto E$.

Also, we claim that $P \propto E \nsubseteq R(a, e)$. Deny. Then $P \propto E \subseteq R(a, e)$ and so for every $u \in E - \{0\}$, we have $(0, u) \in P \propto E \subseteq R(a, e)$ and so $(0, u) = (a, e)(b, f) = (ab, be)$ for some $(b, f) \in R$ (since $a \in M - P$ and $ME = 0$). We claim that $b \in M$.

Assume that $b \notin M$. Then $(b, f) \notin M \propto E$ and so (b, f) is invertible in R (since R is a local ring with maximal ideal $M \propto E$). Therefore, $(a, e) = (b, f)^{-1}(0, u) \in P \propto E$ and so $a \in P$, a contradiction. Hence, $b \in M$.

Therefore, $(0, u) = (ab, be) = (ab, 0)$ (since $b \in M$) and so $u = 0$, a contradiction.

Hence, R is not a divided ring, and this completes the proof of Theorem 2.2. □

We know that in an integral domain, the two notions of divided domains and regular divided domains collapse (see Lemma 2.4). Now, by Theorem 2.2, we can construct a class of rings with zerodivisors such that the two above notions collapse.

Corollary 2.5. *Let D be a domain which is not a field, $K = qf(D)$, E be a K-vector space and $R := D \propto E$. Then R is a divided ring if and only if R is a regular divided ring.*

By Theorem 2.2, we have the following Examples.

Example 2.6. Let (V, M) be a valuation domain with maximal ideal M, $K := qf(V)$ and $R := V \propto K$ be the trivial ring extension of V by K. Then R is a (regular) divided ring.

Example 2.7. Let (V, M) be a valuation domain with maximal ideal M and $R := V \propto (V/M)$ be the trivial ring extension of V by V/M. Then R is a non-divided regular divided ring.

In our second main result, we study the transfer of divided property between a ring R and his amalgamated algebras along some ideals of R. The result enriches the literature with original examples of regular divided rings and examples of non-divided regular divided rings.

Let $f : A \to B$ be a ring homomorphism, J an ideal of B such that $J \subseteq Rad(B)$ and set $R = A \bowtie^f J$. Remark by [8, Proposition 2.6] and since $J \subseteq Rad(B)$ that:

$$Spec(R) - (Max(R) \cap Min(R)) = \{P \bowtie^f J / P \in Spec(A) - (Max(A) \cap Min(A))\}.$$

Theorem 2.8. *Let $f : A \to B$ be a ring homomorphism, J an ideal of B and let $R = A \bowtie^f J$. Then:*

(1) **a)** *A is divided provided so is R.*
b) *Assume that $dim(A) \neq 0$ and $f(a) \in J$ for each $a \in A - U(A)$. If R is divided then so is A and $J^2 = J$.*
c) *Assume that $f(Reg(A)) \subseteq Reg(B)$. Then A is regular divided provided so is R.*

(2) *Assume that $J \subseteq Nil(B)$ and $f(a)$ is invertible for each $a \in A - \{0\}$. Then:*
a) *R is divided if and only if so is A.*
b) *R is regular divided if and only if so is A.*

(3) *Let A be a total ring and assume that $f(Z(A)) \subseteq J$ and $J^2 = 0$. Then, R is regular divided.*

Proof. **1)a)** Let $a \in A$ and $P \in Spec(A)$. Hence, $R(a, f(a))$ and $Q = P \bowtie^f J$ ($\in Spec(R)$) are comparable since R is divided. Therefore, Aa and P are comparable, as desired.

We can have a second direct proof by using [2, Corollary 3] since $R/(0 \bowtie^f J) \cong A$ and R is a divided ring.

b) Assume that $dim(A) \neq 0$ and $f(a) \in J$ for each $a \in A - U(A)$.

Let $a \in A - U(A)$ such that $a \notin P$ for some $P \in Spec(A)$ (since $dim(A) \neq 0$). Hence, $P \bowtie^f J \subseteq R(a, f(a))$ since R is divided and $a \notin P$. We claim that $J^2 = J$.
We have $J^2 \subseteq J$. Conversely, let $k \in J$. Then $(0, k) \in P \bowtie^f J \subseteq R(a, f(a))$ and so $(0, k) = (a, f(a))(b, f(b) + j)$ for some $b \in A$ and $j \in J$. Thus, $ab = 0$ and $k = f(a)(f(b) + j) = f(ab) + f(a)j = f(a)j \in j^2$ since $f(a) \in J$ (since $a \in A - U(A)$), as

desired.

c) Let a be a regular element of A and $P \in Spec(A) - (Max(A) \cap Min(A))$. Then, $P \bowtie^f J \in Spec(R) - (Max(R) \cap Min(R)$. We claim that $(a, f(a))$ is a regular element of R.
Indeed, let $(b, f(b) + j) \in R$ such that $(a, f(a))(b, f(b) + j) = (0, 0)$, where $b \in A$ and $j \in J$. Then, $(0, 0) = (ab, f(ab) + f(a)j)$ and hence $ab = 0$ and so $b = 0$ since a is a regular element of A. Therefore, $f(a)j = 0$ and so $j = 0$ since $f(a)$ is a regular element of B (since $f(Reg(A)) \subseteq Reg(B)$), as desired.

We have $P \bowtie^f J \subseteq R(a, f(a))$ or $R(a, f(a)) \subseteq P \bowtie^f J$ since R is regular divided. Therefore, $P \subseteq Ra$ or $Ra \subseteq P$ and so A is regular divided.

2) Assume that $J \subseteq Nil(B)$ and $f(a)$ is invertible in B for each $a \in A - \{0\}$. Remark that $Spec(A \bowtie^f J) = \{P \bowtie^f J / P \in Spec(A)\}$ since $J \subseteq Nil(B) \subseteq Rad(B)$.

a) By 1)a), it remains to show that R is a divided ring provided so is A.

Conversely, let $Q = P \bowtie^f J \in Spec(A \bowtie^f J)$ for some $P \in Spec(A)$ and let $(a, f(a) + j) \in R$. Two cases are then possible, $a \in P$ or $P \subseteq Aa$ (since A is a divided ring).

Case 1: $a \in P$. Then $(a, f(a) + j) \in Q$, as desired.
Case 2: $P \subseteq Aa$ and $a \neq 0$. In this case, we claim that $Q = P \bowtie^f J \subseteq (A \bowtie^f J)(a, f(a) + j)$. Indeed, let $(c, f(c) + k) \in Q = P \bowtie^f J$ where $c \in P$ and $k \in J$. Hence, there exists $b \in A$ such that $c = ba$ (since $c \in P \subseteq Aa$).

On the other hand, let $l \in J$ such that $l(f(a) + j) = k - j(f(b))$ as $f(a) + j \in U(B)$ (since $f(a) \in U(B)$ and $j \in J \subseteq Nil(B)$). Therefore, $(c, f(c) + k) = (ab, f(ab) + l(f(a) + j) + jf(b)) = (a, f(a) + j)(b, f(b) + l) \in (A \bowtie^f J)(a, f(a) + j)$, as desired.

b) By 1)c), it remains to show that R is a regular divided ring provided so is A since $f(Reg(A)) \subseteq Reg(B)$ (since $f(a)$ is invertible for each $a \in A - \{0\}$).

Conversely, let $Q = P \bowtie^f J \in\in Spec(R) - (Max(R) \cap Min(R)$ for some $P \in Spec(A) - (Max(A) \cap Min(A)$ and let $(a, f(a) + j)$ be a regular element of R. We claim that a is a regular element of A.

Indeed, we first claim that $a \neq 0$. Deny. Then, $j \neq 0$ since $(0, j)$ is a regular element of R. Let n be a non negative integer such that $j^n = 0$ and $j^{n-1} \neq 0$ since $j \in J \subseteq Nil(B)$. Hence, $(0, j)(0, j^{n-1}) = (0, 0)$, a desired contradiction since $(0, j)$ is regular and $(0, j^{n-1}) \neq (0, 0)$.

Now, let $b \in A$ such that $ab = 0$ and set $k := -(f(a))^{-1} f(b)j \in J$ (since $f(a)$ is invertible in B (since $a \neq 0$). Then, $(a, f(a) + j)(b, f(b) + k) = (ab, f(ab) + f(a)k + f(b)j) = (0, 0)$ and so $(b, f(b) + k) = (0, 0)$ (since $(a, f(a) + j)$ is a regular element of R), then $b = 0$.

Hence, a is a regular element of A. We finish the proof by the same argue as in the proof of 2)a) above.

3) Let A be a total ring and assume that $f(Z(A)) \subseteq J$ and $J^2 = 0$. To show that R is regular divided, it suffices to show that R is a total ring by Proposition 2.1.

Let $(a, f(a) + j) \in R$. Two cases are then possible:

Case 1: a is invertible. Then $(a, f(a) + j)(a^{-1}, f(a^{-1}) - f(a^{-2})j) = (1, 1)$ and so $(a, f(a) + j)$ is invertible in R.

Case 2: a is zero-divisor. Then there exists a nonzero element b of A such that $ab = 0$ (since A is total). Two cases are then possible:
If $a = 0$, then, $(0, j) \in Z(R)$ since $j \in J \subseteq Z(B)$.
If $a \neq 0$, then b is non invertible (since $ab = 0$) and so $f(b) \in J$. Hence, $(a, f(a) +$

$j)(b, f(b)) = (ab, f(ab) + jf(b)) = (0, 0)$ since $J^2 = 0$, as desired and this completes the proof of Theorem 2.8. □

By the above Theorem 2.8, we have the following Corollaries:

Corollary 2.9. *Let (A, M) be a local ring such that $dim(A) \neq 0$, $f : A \to B$ be an homomorphism of rings, J an ideal of B such that $f(M) \subseteq J$ and $J^2 \neq J$. Then $R = A \bowtie^f J$ is never a divided ring.*

Corollary 2.10. *Let A be an integral domain, $K = qf(A)$, B a K-algebra, J an ideal of B such that $J \subseteq Nil(B)$. Then:*

(1) *$A \bowtie^f J$ is a divided ring if and only if so is A.*
(2) *$A \bowtie^f J$ is a regular divided ring if and only if so is A.*

Corollary 2.11. *Let M be an ideal of a ring A. Then:*

(1) *A is regular divided provided so is $A \bowtie M$.*
(2) *Assume that (A, M) be a local total ring such that $M^2 = 0$. Then:*
   a) *$A \bowtie M$ is regular divided.*
   b) *Assume that $dim(A) \neq 0$. Then $A \bowtie M$ is not divided.*

By the above Theorem 2.8, we have the following examples:

Example 2.12. Let (A, M) be a local domain such that $M^2 \neq M$. Then $A \bowtie M$ is never a divided ring by Corollary 2.9.

Example 2.13. Let A be a divided (resp., regular divided) domain, $K = qf(A)$, $B = \frac{K[X]}{(X^n)}$, $J = XB$ and $f : A \to B$ be the injectif homomorphism ring. Then $A \bowtie^f J$ is a divided (resp., regular divided) ring by Corollary 2.10.

Example 2.14. Let (D, M) be local domain which is not a field and $A := D/M^2$. Then $A \bowtie (M/M^2)$ is a regular divided ring by Corollary 2.11(2.a).

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