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# Asymptotic Behaviours for the Landau-Lifshitz-Bloch Equation

Kamel Hamdache[a], Djamila Hamroun[b]

[a] *Léonard de Vinci Pôle Universitaire, DVRC., 92 916 Paris La Défense Cedex. France.*

[b] *Laboratoire AMNEDP, Faculté de Mathématiques, Université USTHB, B. P. 32 El Alia, Bab Ezzouar, 16111, Alger, Algérie.*

**Abstract**

The Landau-Lifshitz-Bloch (LLB) equation is an interpolation between Bloch equation valid for high temperatures and Landau-Lifshitz equation valid for low temperatures. Conversely in this paper, we discuss the behaviours of the solutions of (LLB) equation both as the temperature goes to infinity or 0. Surprisingly in the first case, the behaviour depends also on the scaling of the damping parameter $\delta$ and the volume exchange parameter $a$. Three cases are considered and accordingly we get either a linear stationary equation, Bloch equation or Stokes equation. As for the small temperature behaviour, $\delta$ and $a$ being independent of the temperature, we show that the limit of (LLB) equation is Landau-Lifshitz-Gilbert equation.

*Keywords:* Landau-Lifshitz-Bloch equation; Landau-Lifshitz-Gilbert equation; Bloch equation.
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## 1. Introduction

A macroscopic description of the dynamics of magnetization $m$ of ferromagnets at low temperature as well as at elevated temperature is given by the Landau-Lifshitz-Bloch (LLB) equation coupled with the magnetostatic equation satisfied by the magnetic field $H$. (LLB) equation interpolates between the Landau-Lifshitz (LL) equation see [1, 3, 6] arising at temperatures $\theta$ below the Curie point $\theta_c$ and the Bloch equation when the temperatures exceed $\theta_c$ see [4]. (LLB) model involves the longitudinal variation of the magnetization so the saturation constraint $|m| = 1$ is not conserved as in case of (LL) equation. The (LLB) model first introduced in [4] has been discussed from the physical point of view in many recent papers see [10, 11] for

*Email addresses:* `kamel.hamdache@devinci.fr` (Kamel Hamdache), `djamroun@yahoo.fr` (Djamila Hamroun)

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example. The growing interest for this model is sparked by the many applications as the magnetic write head and the recording medium.

**The model equations.** We denote by $|\,|$, $\cdot$ and $\times$ respectively the Euclidean norm, the scalar and the cross products in $\mathbb{R}^3$ and we consider an open bounded domain $D \subset \mathbb{R}^3$ which is simply connected and regular with boundary $\Gamma$. We denote by $\nu$ the unit outward normal to $\Gamma$ and for $T > 0$ fixed, we set $D_T = (0,T) \times D$ and $\Gamma_T = (0,T) \times \Gamma$.
The (LLB) equation satisfied by the magnetization $m = (m_1, m_2, m_3)$ and the equation of the demagnetizing field $H = (H_1, H_2, H_3)$ take the form

$$\begin{aligned} &\partial_t m = -g\, m \times \mathcal{H} - \delta\, \alpha_{tr}\, \omega \times (\omega \times \mathcal{H}) + \delta\, \alpha_l\, (\omega \cdot \mathcal{H})\, \omega \quad \text{in } \ D_T, \\ &m(0) = m_0 \text{ in } D, \ \ (\nu \cdot \nabla) m = 0 \quad \text{on } \Gamma_T, \end{aligned} \tag{1}$$

$$\begin{aligned} &\operatorname{div}(H + m) = F, \ H = \nabla \varphi \quad \text{in } \ D_T, \\ &(H + m) \cdot \nu = 0 \text{ on } \Gamma_T, \end{aligned} \tag{2}$$

where $\omega := \omega(m) = \frac{m}{|m|}$ as long as $m \neq 0$, $\mathcal{H}$ is the effective magnetic field and $F$ is the applied magnetic field. Omitting the contribution of the anisotropic and the internal exchange fields, the effective magnetic field $\mathcal{H}$ is given, see [4] by

$$\mathcal{H} = a\, \Delta m + H. \tag{3}$$

The parameters $g, a > 0$ are respectively the gyromagnetic and the volume exchange coefficients, $\delta > 0$ is the damping parameter and $\alpha_{tr}$ and $\alpha_l$ are the transverse and longitudinal damping parameters given by means of the dimensionless temperature $\tau = \theta/\theta_c$, see [4] by

$$\begin{aligned} &\alpha_{tr} = 1 - \frac{\tau}{3}, \quad \alpha_l = \frac{2\tau}{3}, \qquad \text{if } 0 \le \tau < 1, \\ &\alpha_{tr} = \alpha_l = \frac{2}{3}\, \tau, \qquad \text{if } \tau \ge 1. \end{aligned}$$

Using the relation

$$\mathcal{H} = (\mathcal{H} \cdot \omega)\, \omega - \omega \times (\omega \times \mathcal{H}), \tag{4}$$

one can rewrite the magnetization equation (1) in the form

$$\begin{aligned} &\partial_t m = -g\, m \times \mathcal{H} + \delta\, \alpha_{tr}\, \mathcal{H} - \delta\, (\alpha_{tr} - \alpha_l)\, (\omega \cdot \mathcal{H})\, \omega \quad \text{in } \ D_T, \\ &m(0) = m_0 \text{ in } D, \ \ (\nu \cdot \nabla) m = 0 \quad \text{on } \Gamma_T, \end{aligned} \tag{5}$$

and in view of (4) we see that for $\tau \ge 1$ this equation simplifies into

$$\begin{aligned} &\partial_t m - \kappa(\tau)\, \mathcal{H} = -g\, m \times \mathcal{H}, \\ &m(0) = m_0 \text{ in } D, \ (\nu \cdot \nabla) m = 0 \text{ on } \Gamma_T, \end{aligned} \tag{6}$$

where we set

$$\kappa(\tau) = \frac{2}{3}\, \delta\, \tau. \tag{7}$$

**Asymptotic behaviours.** Our aim in this work is to discuss the behaviour of the system (1)-(2) first as the dimensionless temperature $\tau \to +\infty$ and then as $\tau \to 0$.

1. **High temperature limits.** Three different behaviors are discussed in section 3 when $\tau \to +\infty$, corresponding to different relationships of $\delta$ and $a$ among $\tau$.
   (1.1) In Theorem 2.2, we show that if $\lim_{\tau\to+\infty} \kappa(\tau) = +\infty$ and $a$ is fixed then at the limit the effective magnetic field $\mathcal{H}$ vanishes.
   (1.2) In Theorem 2.4, we prove that if $\lim_{\tau\to+\infty} \kappa(\tau) = 0$ and $a$ is fixed then at the limit the magnetization $m$ satisfies Bloch equation.
   (1.3) In Theorem 2.5, we consider the case when $\lim_{\tau\to+\infty} \kappa(\tau) = +\infty$, $a = \frac{1}{\kappa(\tau)}$ and the applied magnetic field is rescaled as $F = \frac{1}{\sqrt{\kappa(\tau)}} F_1$ then we show that at the limit, the magnetization $m$ satisfies Stokes equation and the magnetic field $H$ vanishes.
2. **Small temperature limit.** In section 4 we discuss the behavior as $\tau \to 0$ assuming $\delta$ and $a$ (independent of $\tau$) equal to 1 for simplicity. We prove that at the limit, the magnetization $m$ satisfies the classical Landau-Lifshitz-Gilbert (LLG) equation, see Theorem 2.7.

**Structure of the paper.** We present our results in the next section, starting by giving some notations to precise the functional framework and a reminder of the existence result for problem (1)-(2) available in [7]. We end section 2 by some properties related to the magnetostatic equations which will be useful later. In section 3, we prove Theorems 2.2, 2.4 and 2.5. First we provide some uniform estimates on the solutions $(m^\tau, H^\tau)$ of problem (1)-(2) allowing to pass to the limit as $\tau \to \infty$ in each case. Section 4 is devoted to the proof of Theorem 2.7 following globally the same steps as before in order to perform the limit as $\tau \to 0$. But regarding to the difficulty related to the possible canceling of the magnetization $m$, this case requires more technicalities.

## 2. Statement of the Main Results

Before stating our main results, let us precise some notations and the hypotheses under consideration then we will recall the existence result of solutions to problem (1)-(2).

**Notations.** Let $L^p(D)$, $W^{s,p}(D)$ and $H^s(D)$ be the usual Lebesgue and Sobolev spaces for scalar functions and let $\mathbb{L}^p(D), \mathbb{W}^{s,p}(D)$ and $\mathbb{H}^s(D)$ be the associated vectorial functional spaces, all equipped with the usual norms and we denote by $\|\cdot\|$ either the $L^2(D)$ or $\mathbb{L}^2(D)$ norm. We define the Hilbert space

$$\mathcal{M} = \{m \in \mathbb{H}^1(D),\ \operatorname{div} m = 0,\ m \cdot \nu = 0 \text{ on } \Gamma\}$$

equipped with the usual norm of $\mathbb{H}^1(D)$, see [2] for example and the Hilbert spaces

$$L^2_\sharp(D) = \{\phi \in L^2(D),\ \int_D \phi\, dx = 0\}, \quad H^1_\sharp(D) = H^1(D) \cap L^2_\sharp(D).$$

$L^2_\sharp(D)$ is equipped with the norm $\|\cdot\|$ of $L^2(D)$ and $H^1_\sharp(D)$ with the norm $\|\nabla\phi\|$ (which is equivalent to the $H^1$-norm thanks to Poincaré-Wirtinger inequality) and the vectorial functional spaces associated are denoted by $\mathbb{L}^2_\sharp(D)$ and $\mathbb{H}^1_\sharp(D)$.
For a general Banach space $V$ we denote the norm by $\|\cdot\|_V$ and the dual space by $V'$. A sequence $(v_n)_n \subset V$ is said to be strongly convergent to $v \in V$, if $\|v_n - v\|_V \to 0$ and weakly convergent if for all $f \in V'$, $f(v_n) \to f(v)$ and a sequence $(f_n)_n \subset V'$ converges weakly$-\star$ to $f \in V'$ if for all $v \in V$, $f_n(v) \to f(v)$.
The Bochner spaces associated with $V$ are denoted by $L^p(0,T;V)$ and we define as usual the spaces $H^s(0,T;V)$, $W^{s,p}(0,T;V)$ and $\mathcal{C}([0,T];V)$.

To end the notations, we point out that in the sequel $C > 0$ is a generic constant which depends only on the domain $D$ and not of the physical parameters appearing in the equations.

**Hypotheses.** In the sequel we make use of the following assumptions on the data

$$m_0 \in \mathbb{H}^1(D), \quad F \in H^1(0,T;L^2_\sharp(D)). \tag{8}$$

In section 4, $m_0$ is also assumed to satisfy the saturation condition

$$|m_0| = 1 \text{ a.e. in } D. \tag{9}$$

**Solutions to problem (1)-(2).** We consider the (LLB) system (1)-(2) and we recall the following existence result, see [7].

**Proposition 2.1.** *Let $\tau > 0$, $m_0 \in \mathbb{H}^1(D)$ and $F \in \mathcal{C}([0,T]; L^2_\sharp(D))$. Then problem (1)-(2) admits a global solution $(m^\tau, H^\tau)$ such that*

$$m^\tau \in \mathcal{C}([0,T]; \mathbb{H}^1(D)) \cap L^2(0,T; \mathbb{H}^2(D)) \cap H^1(0,T; \mathbb{L}^{3/2}(D)),$$

$$H^\tau \in \mathcal{C}([0,T]; \mathbb{H}^1(D)).$$

**Main theorems.** We shall prove that at high or low temperatures, the solutions $(m^\tau, H^\tau)$ of problem (1)-(2) provided by Proposition 2.1 behave according to the different cases already described as follows

**Theorem 2.2.** *Assume hypotheses (8) to be satisfied and $a$ is independent of $\tau$. As $\tau \to +\infty$, if $\kappa(\tau) \to +\infty$ then there exists a subsequence still labeled $(m^\tau, H^\tau)$ converging to a limit $(m, H)$ such that*

$$m \in L^\infty(0,T; \mathbb{H}^3(D)), \quad H \in L^\infty(0,T; \mathbb{H}^1_\sharp(D)), \tag{10}$$

*and $(m, H)$ satisfies the linear stationary problem*

$$\begin{gathered} \mathcal{H} = a\,\Delta m + H = 0 \text{ in } D_T, \\ \operatorname{div}(H + m) = F,\ H = \nabla\varphi \text{ in } D_T, \\ (\nu\cdot\nabla)m = 0, \quad (H+m)\cdot\nu = 0 \text{ on } \Gamma_T. \end{gathered} \tag{11}$$

**Remark 2.3.** *The results of Theorem 2.2 are far from the attempted Bloch equation for the magnetization. In order to capture this dynamic we shall use the second scaling of the parameters.*

**Theorem 2.4.** *Assume hypotheses (8) to be satisfied and $a$ is independent of $\tau$. As $\tau \to +\infty$, if $\kappa(\tau) \to 0$, then there exists a subsequence still labeled $(m^\tau, H^\tau)$ converging to a limit $(m, H) \in L^\infty(0,T; \mathbb{H}^1(D) \times \mathbb{H}^1_\sharp(D))$ such that $(m, H)$ satisfies Bloch equation coupled to the magnetostatic equation*

$$\begin{gathered} \partial_t m = -g\, m \times \mathcal{H} \text{ with } \mathcal{H} = a\,\Delta m + H \text{ in } D_T, \\ \operatorname{div}(H+m) = F,\ H = \nabla\varphi \text{ in } D_T, \\ m(0) = m_0 \text{ in } D,\ (m\times\nabla m)\cdot\nu = 0,\ (H+m)\cdot\nu = 0 \text{ on } \Gamma_T, \end{gathered} \tag{12}$$

*where $(m \times \nabla m)_i = m \times \partial_i m, i = 1,2,3.$*

**Theorem 2.5.** *Let $m_0 \in \mathcal{M}$, $a = \frac{1}{\kappa(\tau)}$ and $F = \frac{1}{\sqrt{\kappa(\tau)}} F_1$ where $F_1 \in H^1(0,T;\ L^2_\sharp(D))$. As $\tau \to +\infty$, if $\kappa(\tau) \to +\infty$ then there exists a subsequence $(m^\tau, H^\tau)$ converging to $(m, 0)$ with $m \in L^\infty(0,T; \mathcal{M})$ and there exists $\pi \in L^2(0,T; L^2(D))$ (which is unique up to a constant) such that $(m, \pi)$ satisfies Stokes equations with Navier's slip boundary conditions*

$$\begin{gathered} \operatorname{div} m = 0 \text{ in } D_T, \\ \partial_t m - \Delta m + \nabla\pi = 0 \text{ in } D_T, \\ m(0) = m_0 \text{ in } D, \quad m\cdot\nu = 0, \quad \nu\times(\nu\times(\nu\cdot\nabla)m) = 0 \text{ on } \Gamma_T. \end{gathered} \tag{13}$$

**Remark 2.6.** *This result is very surprising. To our knowledge, this is the first time that the Stokes equation is used to describe the dynamics of the magnetization. In our opinion, this is explained by the fact that the applied magnetic field is very small.*

**Theorem 2.7.** *Assume hypotheses (8) and (9) to be satisfied and $a = \delta = 1$. Then there exists a subsequence $(m^\tau, H^\tau)$ converging as $\tau \to 0$ to $(m, H)$ such that*

$$m \in L^\infty(0,T;\mathbb{H}^1(D)) \cap \mathbb{W}^{1,3/2}(D_T), \quad H \in L^\infty(0,T;\mathbb{H}^1_\sharp(D)),$$

*and $(m, H)$ is a global weak solution of the Landau-Lifshitz-Gilbert (LLG) equation coupled to the magnetostatic equations*

$$\begin{gathered} \partial_t m = -\frac{1+g^2}{g}\, m \times \mathcal{H} + \frac{1}{g}\, m \times \partial_t m \ \textit{in } D_T, \\ |m|^2 = 1 \ \ \textit{in } D_T, \ \ \mathcal{H} = \Delta m + H, \\ \operatorname{div}(H+m) = F, \quad H = \nabla\varphi \ \textit{in } D_T, \\ m(0) = m_0, \ |m_0|^2 = 1 \ \ \textit{in } D, \\ m \times (\nu \cdot \nabla) m = 0, \ (H+m)\cdot\nu = 0 \ \textit{on } \Gamma_T. \end{gathered} \tag{14}$$

The proofs of Theorems 2.2, 2.4 and 2.5 will be done in section 3 and the proof of Theorem 2.7 in section 4. To end this section, let us recall some useful results on the magnetostatic equations.

**The magnetostatic equations.** Let $m \in \mathcal{C}([0,T];\mathbb{L}^2(D))$ and $F \in \mathcal{C}([0,T];L^2_\sharp(D))$ be fixed and let $\varphi \in \mathcal{C}([0,T];H^1_\sharp(D))$ be the unique solution of the problem

$$\nabla\varphi = H, \ \operatorname{div}(H+m) = F \ \text{ in } D_T, \quad (H+m)\cdot\nu = 0 \ \text{ on } \Gamma_T. \tag{15}$$

Since

$$-\int_D H \cdot m\, dx = \|H\|^2 + \int_D F\,\varphi, \tag{16}$$

then by Poincaré-Wirtinger inequality we get the bound

$$|\int_D F\,\varphi\, dx| \le \frac{1}{2}\|H\|^2 + C\,\|F\|^2, \tag{17}$$

where $C > 0$ depends only on the domain $D$, which leads to the estimate

$$\|H(t)\| \le C\,(\|m(t)\| + \|F(t)\|), \quad t \in [0,T]. \tag{18}$$

In particular we have

**Lemma 2.8.** *The linear mapping $(m, F) \longmapsto H$ is continuous from $L^2(0,T;\mathbb{L}^2(D)\times L^2_\sharp(D))$ into $L^2(0,T;\mathbb{L}^2(D))$.*

Moreover if $m \in \mathcal{C}([0,T];\mathbb{H}^1(D))$, writing equation (15) in the form

$$\Delta\varphi = -\operatorname{div} m + F \ \text{ in } D_T, \quad \nabla\varphi\cdot\nu = -m\cdot\nu \ \text{ on } \Gamma_T, \tag{19}$$

where $\operatorname{div} m \in \mathcal{C}([0,T];L^2(D))$ and $m\cdot\nu \in \mathcal{C}([0,T];H^{1/2}(\Gamma))$, then applying elliptic regularity results we conclude that $\varphi \in \mathcal{C}([0,T];H^2(D))$ so $H \in \mathcal{C}([0,T];\mathbb{H}^1(D))$ and we have

$$\|H(t)\|_{\mathbb{H}^1(D)} \le C\,(\|m(t)\|_{\mathbb{H}^1(D)} + \|F(t)\|), \quad t \in [0,T], \tag{20}$$

where $C > 0$ depends only on the domain $D$. Next if $m \in H^1(0,T;\mathbb{L}^2(D))$ and $F \in H^1(0,T;L^2_\sharp(D))$ then we infer that

$$-\int_D H\cdot\partial_t m\, dx = \frac{1}{2}\frac{d}{dt}\|H\|^2 + \int_D \partial_t F\,\varphi\, dx, \tag{21}$$

with

$$|\int_D \partial_t F\,\varphi\, dx| \le \frac{1}{2}\|H\|^2 + C\,\|\partial_t F\|^2. \tag{22}$$

## 3. High Temperature Limits

We consider the (LLB) system (6)-(2) when $\tau \geq 1$. In this section we will prove the asymptotic behaviour results given in Theorems 2.2, 2.4 and 2.5.
To begin, let us give some energy estimates satisfied by the solutions $(m^\tau, H^\tau)$ of the problem provided by Proposition 2.1.

**Proposition 3.1.** *Under hypotheses (8), the following estimates hold for all $t \in [0,T]$,*

$$\begin{gathered} \|m^\tau(t)\|^2 + 2\kappa(\tau) \int_0^t (a\|\nabla m^\tau(s)\|^2 + \frac{1}{2}\|H^\tau(s)\|^2)\, ds \leq \\ \|m_0\|^2 + C\,\kappa(\tau) \int_0^t \|F(s)\|^2\, ds, \end{gathered} \tag{23}$$

$$\begin{gathered} a\|\nabla m^\tau(t)\|^2 + \|H^\tau(t)\|^2 + 2\kappa(\tau) \int_0^t \|a\Delta m^\tau(s) + H^\tau(s)\|^2\, ds \leq \\ \mathcal{E}_0 + \frac{1}{\kappa(\tau)}\|m_0\|^2 + C\|F\|^2_{H^1(0,T;L^2(D))}, \end{gathered} \tag{24}$$

$$\begin{gathered} a\|\nabla m^\tau(t)\|^2 + \|H^\tau(t)\|^2 + 2\kappa(\tau) \int_0^t \|a\Delta m^\tau(s) + H^\tau(s)\|^2\, ds \leq \\ (\mathcal{E}_0 + C\|\partial_t F\|^2_{L^2(D_T)})e^T, \end{gathered} \tag{25}$$

*where $C > 0$ depends only on the domain $D$, $\mathcal{E}_0 = a\|\nabla m_0\|^2 + \|H_0\|^2$, $H_0$ being the demagnetizing field associated to $m_0$ and the source term $F(0)$ that is $H_0$ solves the problem*

$$\begin{gathered} div\,(H_0 + m_0) = F(0),\ H_0 = \nabla\varphi_0 \quad in\ D, \\ (H_0 + m_0)\cdot\nu = 0\ on\ \Gamma. \end{gathered} \tag{26}$$

*Proof.* For simplicity we drop the index $\tau$ in $(m^\tau, H^\tau)$.
We multiply the magnetization equation by $m$ then by $\mathcal{H} \in L^2(0,T;\mathbb{L}^2(D))$ and integrate by parts to get

$$\frac{1}{2}\frac{d}{dt}\|m\|^2 - \kappa(\tau)\int_D (a\Delta m + H)\cdot m\, dx = 0, \tag{27}$$

and

$$-\int_D \partial_t m \cdot (a\Delta m + H)\, dx + \kappa(\tau)\,\|\mathcal{H}\|^2 = 0. \tag{28}$$

Hence (27) leads straightforwardly to estimate (23) using relations (16) and (17). Now using relations (28), (21) and (22) and setting

$$\mathcal{E} = \mathcal{E}(m,H) = a\|\nabla m\|^2 + \|H\|^2, \tag{29}$$

we get

$$\mathcal{E}(t) + 2\kappa(\tau)\int_0^t \|\mathcal{H}(s)\|^2\, ds \leq \mathcal{E}_0 + \int_0^t (\|H(s)\|^2 + C\|\partial_t F(s)\|^2)\, ds, \tag{30}$$

which leads to estimate (24) by using (23). To prove (25), we rewrite (30) as

$$\mathcal{E}(t) + 2\kappa(\tau)\int_0^t \|\mathcal{H}(s)\|^2\, ds \leq \mathcal{E}_0 + C\int_0^t \|\partial_t F(s)\|^2\, ds + \int_0^t \mathcal{E}(s)\, ds, \tag{31}$$

and use Gronwall lemma which leads to

$$\mathcal{E}(t) \leq (\mathcal{E}_0 + C\int_0^t \|\partial_t F(s)\|^2\, ds)e^t. \tag{32}$$

□

*3.1. The stationary limit.*

We will prove Theorem 2.2, so uniform bounds of $m^\tau$ and $H^\tau$ are needed. First the results of Proposition 3.1 allow to deduce that $(m^\tau, H^\tau)$ satisfy the estimates below.

**Corollary 3.2.** *We have the estimates*

$$\begin{gathered} a\|\nabla m^\tau\|^2_{L^2(D_T)} + \frac{1}{2}\|H^\tau\|^2_{L^2(D_T)} \le \frac{1}{\kappa(\tau)}\,\|m_0\|^2 + C\|F\|^2_{L^2(D_T)}, \\ a\|\nabla m^\tau\|^2_{L^\infty(0,T;L^2(D))} + \|H^\tau\|^2_{L^\infty(0,T;L^2(D))} \le \mathcal{F}^\tau_0, \\ \|a\Delta m^\tau + H^\tau\|^2_{L^2(D_T)} \le \frac{1}{2\kappa(\tau)}\,\mathcal{F}^\tau_0, \end{gathered} \tag{33}$$

*where* $\mathcal{F}^\tau_0 = \mathcal{E}_0 + \frac{1}{\kappa(\tau)}\|m_0\|^2 + C\|F\|^2_{H^1(0,T;L^2(D))}$ *and* $C > 0$ *is independent of* $\tau$.

Next we will prove the following $L^\infty(0,T;\mathbb{H}^1(D))$ uniform estimates.

**Lemma 3.3.** *The solutions* $(m^\tau, H^\tau)$ *satisfy the uniform bounds*

$$\begin{aligned} \|m^\tau\|^2_{L^\infty(0,T;\mathbb{H}^1(D))} &\le C\big(\mathcal{E}_0 + \frac{1}{\kappa(\tau)}\,\|m_0\|^2 + \|F\|^2_{H^1(0,T;L^2(D))}\big), \\ \|H^\tau\|^2_{L^\infty(0,T;\mathbb{H}^1(D)} &\le C\,\big(\mathcal{E}_0 + \frac{1}{\kappa(\tau)}\,\|m_0\|^2 + \|F\|^2_{H^1(0,T;L^2(D))}\big), \end{aligned} \tag{34}$$

*where* $C > 0$ *is independent of* $\tau$.

*Proof.* We need to apply Poincaré-Wirtinger inequality for $m^\tau$ so we shall estimate its mean value on the domain $D$. To this purpose we introduce the notation $\langle f\rangle = |D|^{-1}\int_D f(x)\,dx$ for a scalar or a vectorial function $f$, where $|D|$ is the Lebesgue measure of $D$.
We multiply the magnetostatic equation by $x_i$ for $i = 1, 2, 3$ and integrate by parts to obtain the relation

$$\langle m^\tau(t)\rangle = -\langle H^\tau(t)\rangle - \langle x\,F(t)\rangle, \tag{35}$$

for all $t \in [0, T]$. Therefore the inequality $|\langle x\,F(t)\rangle| \le C\,\|F(t)\|$ implies that

$$\|\langle m^\tau\rangle\|_{L^\infty(0,T)} \le C\,(\|H^\tau\|_{L^\infty(0,T;\mathbb{L}^2(D))} + \|F\|_{L^\infty(0,T;L^2(D))}), \tag{36}$$

where throughout this demonstration $C > 0$ denotes different constants depending only on the domain $D$. By using Poincaré-Wirtinger inequality $\|m^\tau - \langle m^\tau\rangle\| \le C\|\nabla m^\tau\|$, we deduce that

$$\begin{aligned} \|m^\tau\|_{L^\infty(0,T;\mathbb{L}^2(D))} \le C\big(\|H^\tau\|_{L^\infty(0,T;\mathbb{L}^2(D))} &+ \|F\|_{L^\infty(0,T,L^2(D))} + \\ &\|\nabla m^\tau\|_{L^\infty(0,T;\mathbb{L}^2(D))}\big). \end{aligned} \tag{37}$$

Hence (33) leads to the first estimate of the lemma and the second one follows by using (20). □

**Passing to the limit.** Using (33) and (34) we infer that

**Corollary 3.4.** *There exist a subsequence still denoted* $(m^\tau, H^\tau = \nabla\varphi^\tau)$ *and* $(m, H = \nabla\varphi)$ *such that as* $\tau \to +\infty$*, if* $\kappa(\tau) \to +\infty$ *then*

$$\begin{gathered} m^\tau \rightharpoonup m \textit{ weakly} - \star \textit{ in } L^\infty(0,T;\mathbb{H}^1(D)), \\ H^\tau \rightharpoonup H \textit{ weakly} - \star \textit{ in } L^\infty(0,T;\mathbb{H}^1(D)) \;\; \textit{and weakly in } \mathbb{L}^2(D_T), \\ \mathcal{H}^\tau = a\Delta m^\tau + H^\tau \to 0 \textit{ strongly in } L^2(0,T;\mathbb{L}^2(D)), \end{gathered} \tag{38}$$

$$m^\tau \times \mathcal{H}^\tau \to 0 \textit{ strongly in } L^2(0,T;\mathbb{L}^{3/2}(D)), \tag{39}$$

$$\Delta m^\tau \rightharpoonup \Delta m \textit{ weakly in } L^2(0,T;\mathbb{L}^2(D)), \tag{40}$$

*and*

$$a\Delta m + H = 0 \quad \textit{in } D_T.$$

*Proof.* We have only to prove the two last convergences. To get (39), we use embedding $H^1(D) \subset L^6(D)$ and inequality

$$\|m^\tau \times \mathcal{H}^\tau\|_{L^2(0,T;\mathbb{L}^{3/2}(D))} \leq \|m^\tau\|_{L^\infty(0,T;\mathbb{L}^6(D))}\|\mathcal{H}^\tau\|_{L^2(0,T;\mathbb{L}^2(D))},$$

and to prove (40) we write

$$a\Delta m^\tau = \mathcal{H}^\tau - H^\tau \rightharpoonup -H \text{ weakly in } L^2(D_T),$$

and since $a\Delta m^\tau \to a\Delta m$ in the sense of distributions, then the result follows. □

Now we introduce the weak formulation of problem (6)-(2). Let $\Phi \in (\mathcal{D}([0,T[\times\overline{D}))^3$ and $\phi \in \mathcal{D}([0,T]\times\overline{D})$ be two test functions. The weak formulation of the magnetization equation takes the form

$$\begin{aligned} -\frac{1}{\kappa(\tau)}\int_{D_T} m^\tau \cdot \partial_t\Phi \, dxdt + a\int_{D_T} \nabla m^\tau \cdot \nabla\Phi \, dxdt \\ -\int_{D_T} H^\tau \cdot \Phi \, dxdt = \frac{1}{\kappa(\tau)}\int_D m_0 \cdot \Phi(0)\, dx \\ -\frac{g}{\kappa(\tau)}\int_{D_T} m^\tau \times \mathcal{H}^\tau \cdot \Phi \, dxdt, \end{aligned} \tag{41}$$

and the weak formulation of the magnetostatic equation writes as

$$\int_{D_T} (H^\tau + m^\tau)\cdot\nabla\phi \, dxdt = -\int_{D_T} F\,\phi\, dxdt. \tag{42}$$

The results of Corollary 3.4 allow to pass to the limit as $\tau \to \infty$ in (41) and (42), since $\kappa(\tau) \to \infty$ we get

$$a\int_{D_T} \nabla m \cdot \nabla\Phi\, dxdt - \int_{D_T} H \cdot \Phi \, dxdt = 0, \tag{43}$$

$$\int_{D_T} (H + m)\cdot\nabla\phi\, dxdt = -\int_{D_T} F\,\phi\, dxdt, \tag{44}$$

for all $\Phi \in (\mathcal{D}([0,T[\times\overline{D}))^3$ and $\phi \in \mathcal{D}([0,T]\times\overline{D})$ so integrating by parts we see that $(m,H)$ solves the problem

$$\begin{gathered} a\,\Delta m + H = 0 \text{ in } D_T, \\ (\nu\cdot\nabla)m = 0 \text{ on } \Gamma_T, \quad \langle H\rangle = 0 \text{ in } \ (0,T), \\ \operatorname{div}(H+m) = F, \ H = \nabla\varphi \text{ in } D_T, \\ (H+m)\cdot\nu = 0 \text{ on } \Gamma_T, \end{gathered} \tag{45}$$

so $\Delta m \in L^\infty(0,T;\mathbb{H}^1(D))$ which ends the proof of Theorem 2.2.

*3.2. The Bloch limit.*

In this paragraph we assume that $\lim_{\tau\to+\infty}\kappa(\tau)=0$ and $a$ is independent of $\tau$. We aim to prove the results of Theorem 2.4. First since the solutions $(m^\tau, H^\tau)$ of (6)-(2) satisfy the uniform estimates (23) and (25) given in Proposition 3.1, we infer that

**Corollary 3.5.** *$m^\tau$ and $H^\tau$ are uniformly bounded with respect to $\tau$ in $L^\infty(0,T;\mathbb{H}^1(D))$ and $\sqrt{\kappa(\tau)}\,(a\Delta m^\tau + H^\tau)$ is bounded in $L^2(0,T;\mathbb{L}^2(D))$.*

The uniform estimate of $H^\tau$ in $L^\infty(0,T;\mathbb{H}^1(D))$ is derived by the same bound of $m^\tau$, using inequality (20). Now we look for a bound of the time derivative of $m^\tau$.

**Lemma 3.6.** *$\partial_t m^\tau$ is uniformly bounded with respect to $\tau$ in $L^2(0,T;(\mathbb{H}^2(D))')$.*

*Proof.* We write $\partial_t m^\tau = \kappa(\tau)\,\mathcal{H}^\tau - g\,m^\tau\times H^\tau - a\,g\,m^\tau\times\Delta m^\tau$ and use the results of Corollary 3.5 and the embedding $H^1(D)\subset L^6(D)$ to see that the two first terms are uniformly bounded in $\mathbb{L}^2(D_T)$ and $L^\infty(0,T;\mathbb{L}^3(D))$ respectively. Next we write $m^\tau\times\Delta m^\tau = \operatorname{div}(m^\tau\times\nabla m^\tau)$ then since $m^\tau\times\nabla m^\tau$ is bounded in $L^\infty(0,T;\mathbb{L}^{3/2}(D))$ we deduce that $m^\tau\times\Delta m^\tau$ is bounded in $L^\infty(0,T;(\mathbb{H}^2(D))')$. □

**Passing to the limit.** As previously we will pass to the limit in the weak formulation of (6)-(2) as $\tau\to+\infty$, assuming that $a$ is independent of $\tau$ and $\kappa(\tau)\to 0$. By using Corollary 3.5 and Lemma 3.6, we deduce using Aubin's compactness lemma see [8, 9] and the compact embedding $H^1(D)\subset L^p(D)$ for all $1\le p<6$, the following convergence results.

**Corollary 3.7.** *There exists a subsequence $(m^\tau, H^\tau)$ and $(m, H=\nabla\varphi)$ such that*

$$\begin{aligned}
&m^\tau \rightharpoonup m \ \textit{weakly}-\star \ \textit{in}\ L^\infty(0,T;\mathbb{H}^1(D)),\\
&m^\tau \to m \ \textit{strongly in}\ L^2(0,T;\mathbb{L}^p(D)),\ 1\le p<6,\\
&H^\tau \rightharpoonup H \ \textit{weakly}-\star \ \textit{in}\ L^\infty(0,T;\mathbb{H}^1(D)),\\
&\kappa(\tau)\,\mathcal{H}^\tau \to 0 \ \textit{strongly in}\ L^2(0,T;\mathbb{L}^2(D)).
\end{aligned}\tag{46}$$

*Moreover, we have*

$$\begin{aligned}
&H^\tau \to H \ \textit{strongly in}\ L^2(0,T;\mathbb{L}^p(D)),\ 1\le p<6,\\
&m^\tau\times H^\tau \rightharpoonup m\times H \ \textit{weakly in}\ L^2(0,T;\mathbb{L}^2(D)),\\
&m^\tau\times\nabla m^\tau \rightharpoonup m\times\nabla m \ \textit{weakly in}\ L^2(0,T;\mathbb{L}^{3/2}(D)).
\end{aligned}\tag{47}$$

*Proof.* It remains to prove the convergences given in (47). By Lemma 2.8, we see that $H^\tau\to H$ strongly in $L^2(0,T;\mathbb{L}^2(D))$ but since $H^\tau$ is bounded in $L^2(0,T;\mathbb{L}^6(D))$, the strong convergence of $H^\tau$ is true in $L^2(0,T;\mathbb{L}^p(D))$ with $1\le p<6$. Next the sequence $m^\tau\times H^\tau$ is uniformly bounded in $L^2(0,T;\mathbb{L}^2(D))$ and the strong convergence of $m^\tau$ and $H^\tau$ implies that $m^\tau\times H^\tau\to m\times H$ strongly in $L^1(0,T;\mathbb{L}^2(D))$ which leads to the desired result. Similarly the sequence $m^\tau\times\nabla m^\tau$ is bounded in $L^2(0,T;\mathbb{L}^{3/2}(D))$ then by the weak-strong convergence principle we deduce the stated convergence. □

These results enable us to end the proof of Theorem 2.4. First using Lemma 2.8, we infer that the magnetostatic equation is satisfied. Let $\Phi\in(\mathcal{D}([0,T[\times\overline{D}))^3$, we write the weak formulation of the magnetization equation

$$\begin{aligned}
-\int_{D_T} m^\tau\cdot\partial_t\Phi\,dxdt + \kappa(\tau)\int_{D_T}\mathcal{H}^\tau\cdot\Phi\,dxdt = \int_D m_0\cdot\Phi(0)\,dx+\\
a\,g\int_{D_T} m^\tau\times\nabla m^\tau\cdot\nabla\Phi\,dxdt - g\int_{D_T} m^\tau\times H^\tau\cdot\Phi\,dxdt,
\end{aligned}\tag{48}$$

and we pass to the limit by using the convergence results given in Corollary 3.7 to get

$$
\begin{gathered}
-\int_{D_T} m\cdot\partial_t\Phi\,dxdt=\int_D m_0\cdot\Phi(0)\,dx+\\
a\,g\int_{D_T} m\times\nabla m\cdot\nabla\Phi\,dxdt-g\int_{D_T} m\times H\cdot\Phi\,dxdt,
\end{gathered}
\tag{49}
$$

for all $\Phi\in(\mathcal{D}([0,T[\times\overline{D}))^3$. From there, it is easy to conclude that $(m,H)$ satisfies the system of equations (12). This ends proof of Theorem 2.4.

*3.3. The Stokes limit.*

Now we shall discuss the behavior of the problem (6)-(2) under hypotheses of Theorem 2.5 so $a=\frac{1}{\kappa(\tau)}$ and $F=\frac{1}{\sqrt{\kappa(\tau)}}F_1$ and we investigate the case when $\tau\to+\infty$ and $\kappa(\tau)\to+\infty$. With this scaling of $F$, the right hand sides of estimates (23) and (24) given in Proposition 3.1 are bounded with respect to $\tau$. Indeed these inequalities are written as

$$
\begin{gathered}
\|m^\tau(t)\|^2+2\int_0^t(\|\nabla m^\tau(s)\|^2+\frac{\kappa(\tau)}{2}\|H^\tau(s)\|^2)\,ds\le\\
\|m_0\|^2+C\int_0^t\|F_1(s)\|^2\,ds,
\end{gathered}
\tag{50}
$$

$$
\begin{gathered}
\frac{1}{\kappa(\tau)}\|\nabla m^\tau(t)\|^2+\|H^\tau(t)\|^2+2\kappa(\tau)\int_0^t\|\frac{\Delta m^\tau(s)}{\kappa(\tau)}+H^\tau(s)\|^2\,ds\le\\
\mathcal{E}_0^\tau+\frac{1}{\kappa(\tau)}\,(\|m_0\|^2+C\|F_1\|^2_{H^1(0,T;L^2(D))}),
\end{gathered}
\tag{51}
$$

where $C>0$ depends only on the domain $D$ and $\mathcal{E}_0^\tau=\frac{1}{\kappa(\tau)}\|\nabla m_0\|^2+\|H_0^\tau\|^2$. Here $H_0^\tau$ is the solution of the problem

$$
\operatorname{div}H_0^\tau=\frac{1}{\sqrt{\kappa(\tau)}}F_1(0),\quad H_0^\tau=\nabla\varphi_0^\tau\text{ in }D,\qquad H_0^\tau\cdot\nu=0\text{ on }\Gamma,
\tag{52}
$$

because $m_0\in\mathcal{M}$, so $\|H_0^\tau\|\le\frac{C}{\sqrt{\kappa(\tau)}}\|F_1(0)\|$ where $C>0$ depends only on the domain $D$. Therefore

$$
\mathcal{E}_0^\tau\le\frac{1}{\kappa(\tau)}(\|\nabla m_0\|^2+C_T\|F_1\|^2_{H^1(0,T;L^2(D))}),
\tag{53}
$$

where $C_T>0$ depends only on the domain $D$ and $T$ involving the inequality

$$
\begin{gathered}
\frac{1}{\kappa(\tau)}\|\nabla m^\tau(t)\|^2+\|H^\tau(t)\|^2+2\kappa(\tau)\int_0^t\|\frac{\Delta m^\tau(s)}{\kappa(\tau)}+H^\tau(s)\|^2\,ds\le\\
\frac{1}{\kappa(\tau)}\big(\|m_0\|^2_{\mathbb{H}^1(D)}+C_T\|F_1\|^2_{H^1(0,T;L^2(D))}\big).
\end{gathered}
\tag{54}
$$

Hence from estimates (50) and (54), we infer that

**Corollary 3.8.** *Under hypotheses of Theorem 2.5, there exists $C,C_T>0$ independent of $\tau$ such that for all*

$t \in [0,T]$ *we have*

$$
\begin{aligned}
&\|m^\tau(t)\|^2 + 2\int_0^t \|\nabla m^\tau(s)\|^2\,ds \le \|m_0\|^2 + C\|F_1\|^2_{L^2(0,T;L^2(D))},\\
&\|\nabla m^\tau(t)\|^2 \le \|m_0\|^2_{\mathbb{H}^1(D)} + C_T\|F_1\|^2_{H^1(0,T;L^2(D))},\\
&\|H^\tau(t)\|^2 \le \frac{1}{\kappa(\tau)}\,(\|m_0\|^2_{\mathbb{H}^1(D)} + C_T\|F_1\|^2_{H^1(0,T;L^2(D))}),\\
&\int_0^t \|\frac{1}{\kappa(\tau)}\,\Delta m^\tau(s) + H^\tau(s)\|^2\,ds \le\\
&\tfrac{1}{2\kappa^2(\tau)}\,(\|m_0\|^2_{\mathbb{H}^1(D)} + C_T\|F_1\|^2_{H^1(0,T;L^2(D))}).
\end{aligned}
\tag{55}
$$

**Passing to the limit.** We use the previous bounds to deduce that

**Corollary 3.9.** *Under hypotheses of Theorem 2.5, there exist a subsequence $(m^\tau, H^\tau = \nabla\varphi^\tau)$ and $m$ such that the following convergences hold when $\tau \to \infty$*

$$
\begin{aligned}
&m^\tau \rightharpoonup m \textit{ weakly} - \star \textit{ in } L^\infty(0,T;\mathbb{H}^1(D)),\\
&H^\tau \to 0 \textit{ strongly in } L^\infty(0,T;\mathbb{L}^2(D)),\\
&\mathcal{H}^\tau = \tfrac{1}{\kappa(\tau)}\Delta m^\tau + H^\tau \to 0 \textit{ strongly in } L^2(0,T;\mathbb{L}^2(D)),\\
&m^\tau \times \mathcal{H}^\tau \to 0 \textit{ strongly in } L^2(0,T;\mathbb{L}^{3/2}(D)).
\end{aligned}
\tag{56}
$$

*Proof.* It remains only to prove the last convergence result and for this one, we use the inequality

$$
\|m^\tau \times \mathcal{H}^\tau\|_{L^2(0,T;\mathbb{L}^{3/2}(D))} \le \|m^\tau\|_{L^\infty(0,T;\mathbb{L}^6(D))}\,\|\mathcal{H}^\tau\|_{L^2(0,T;\mathbb{L}^2(D))}.
$$

□

We consider the following weak formulation of the problem. Let $\Phi \in \mathcal{M}$ and $\xi \in \mathcal{D}([0,T[)$ and let $\phi \in \mathcal{D}([0,T]\times\overline{D})$, since $H^\tau = \nabla\varphi^\tau$ then $\int_D H^\tau \cdot \Phi\,dx = 0$ and the weak formulation of problem (6)-(2) writes as

$$
\begin{aligned}
&-\int_{D_T} \xi'(t)\,m^\tau \cdot \Phi\,dxdt + \int_{D_T} \xi(t)\,\nabla m^\tau \cdot \nabla\Phi\,dxdt\\
&= \int_D \xi(0)\,m_0 \cdot \Phi\,dx - g\int_{D_T} \xi(t)\,m^\tau \times \mathcal{H}^\tau \cdot \Phi\,dxdt,
\end{aligned}
\tag{57}
$$

$$
-\int_{D_T} (H^\tau + m^\tau)\cdot\nabla\phi\,dxdt = \frac{1}{\sqrt{\kappa(\tau)}}\int_{D_T} F_1\phi\,dxdt.
\tag{58}
$$

Passing to the limit by using the convergence results given in (56), we get

$$
\begin{aligned}
&-\int_{D_T} \xi'(t)\,m \cdot \Phi\,dxdt + \int_{D_T} \xi(t)\,\nabla m \cdot \nabla\Phi\,dxdt =\\
&\qquad\int_D \xi(0)\,m_0 \cdot \Phi\,dx,
\end{aligned}
\tag{59}
$$

$$
-\int_{D_T} m \cdot \nabla\phi\,dxdt = 0,
\tag{60}
$$

for all $\Phi \in \mathcal{M}$, $\xi \in \mathcal{D}([0,T[)$ and $\phi \in \mathcal{D}([0,T] \times \overline{D})$. Hence we deduce that $m$ satisfies

$$\operatorname{div} m = 0 \text{ in } D_T,\ m \cdot \nu = 0 \text{ on } \Gamma_T, \tag{61}$$

so that $m \in L^\infty(0,T;\mathcal{M})$. Finally integrating by parts in (59) we deduce that for all $\Phi \in \mathcal{M}$

$$\frac{d}{dt}\int_D m.\Phi\, dx + \int_D \nabla m.\nabla\Phi\, dx = 0 \ \text{ in } \mathcal{D}'(0,T),$$

so $\dfrac{d}{dt}\displaystyle\int_D m.\Phi\, dx \in L^\infty(0,T)$ and $\partial_t m \in L^\infty(0,T;\mathcal{M}')$. Therefore $m \in \mathcal{C}([0,T];\mathcal{M}')$ and the initial condition $m(0) = m_0$ is satisfied. Using the De Rham theorem and the classical results for Stokes equation, we deduce that there exists $\pi \in L^\infty(0,T;L^2(D))$ such that $(m,\pi)$ satisfies the equation $\partial_t m - \Delta m + \nabla\pi = 0$ in $D_T$. Testing again this equation by $\xi(t)\Phi$ with $\Phi \in \mathcal{M}$ and $\xi \in \mathcal{D}(0,T)$ we deduce that $\displaystyle\int_0^T \xi(t)\langle(\nu\cdot\nabla)m, \Phi\rangle\, dt = 0$, where the symbol $\langle .\,,\,.\ \rangle$ is the dual bracket between $\mathbb{H}^{-1/2}(\Gamma)$ and $\mathbb{H}^{1/2}(\Gamma)$ . Therefore $\langle(\nu\cdot\nabla)m, \Phi\rangle = 0$ for a.e. $t \in (0,T)$ and all $\Phi \in \mathcal{M}$. Writing $\Phi = -\nu \times (\nu \times \Phi)$, we see that this condition is equivalent to $\langle \nu \times (\nu \times (\nu\cdot\nabla)m), \Phi\rangle = 0$ for a.e. $t \in (0,T)$ and all $\Phi \in \mathcal{M}$ and we retrieve the Navier's slip boundary condition

$$\nu \times (\nu \times (\nu\cdot\nabla)m) = 0 \text{ on } \ \Gamma_T. \tag{62}$$

This ends the proof of Theorem 2.5.

## 4. Small Temperature Limit

This section deals with the asymptotic behaviour of (LLB) system (1)-(2) when $\tau \to 0$, $\delta$ and $a$ being independent of $\tau$ so without loss of generality we take them equal to 1. As we expect that the limit equation of the magnetization is (LLG) equation, we assume that the initial data satisfies the saturation constraint $|m_0| = 1$ a.e. in $D$.
From now on, we see that $\kappa(\tau) = \alpha_l = \frac{2}{3}\tau$ and we set $\gamma(\tau) := \alpha_{tr} = 1 - \frac{1}{2}\kappa(\tau)$. We recall that for $0 < \tau < 1$, the existence result of Proposition 2.1 holds true, but regarding to the indetermination contained in the equation (due to the fact that $m$ can cancel so $\omega(m)$ is not defined), the magnetization equation is rewritten in the following form, see [7]

$$\begin{aligned}
|m^\tau|^2\partial_t m^\tau = -g\,|m^\tau|^2\, m^\tau \times \mathcal{H}^\tau - \gamma(\tau)\, m^\tau \times (m^\tau \times \mathcal{H}^\tau)\\
+\,\kappa(\tau)\,(m^\tau\cdot\mathcal{H}^\tau)\, m^\tau \text{ in } D_T,
\end{aligned} \tag{63}$$

$$|m^\tau(0)|^2 m^\tau(0) = |m_0|^2 m_0 \text{ in } D,\ |m^\tau|^2(\nu\cdot\nabla)m^\tau = 0 \text{ on } \Gamma_T,$$

where $\mathcal{H}^\tau = \Delta m^\tau + H^\tau$. To be complete, we give again the magnetostatic equation satisfied by $H^\tau$

$$\begin{aligned}
\operatorname{div}(H^\tau + m^\tau) = F,\ H^\tau = \nabla\varphi^\tau \quad \text{in } \ D_T,\\
(H^\tau + m^\tau)\cdot\nu = 0 \text{ on } \Gamma_T,
\end{aligned} \tag{64}$$

We aim to prove the results stated in Theorem 2.7 so the first step is to establish uniform bounds of the solutions with respect to the small parameter $\tau$.

### *4.1. Uniform estimates*

We introduce the following notations. For $t > 0$ we set $D_t = (0,t) \times D$ and we define the function $\chi$ by $\chi(m) = 1$ if $m \neq 0$ and $\chi(m) = 0$ elsewhere. Our first estimates are given below

**Proposition 4.1.** *Let $(m^\tau, H^\tau)$ be the solution of (63)-(64) provided by Proposition 2.1. Under hypotheses (8), there exists $C > 0$ depending only on the domain $D$ such that for all $t \in [0, T]$ it holds*

$$\|m^\tau(t)\|^2 + 2\kappa(\tau) \int_0^t (\|\nabla m^\tau\|^2 + \frac{1}{2}\|H^\tau\|^2)\, ds \leq \|m_0\|^2 + C\,\kappa(\tau)\,\|F\|^2_{L^2(D_T)}, \tag{65}$$

$$\|\nabla m^\tau(t)\|^2 + \|H^\tau(t)\|^2 + 2\kappa(\tau) \int_{D_t} \chi(m^\tau)|\mathcal{H}^\tau|^2\, dxds \leq e^T(\mathcal{E}_0 + C\|\partial_t F\|^2_{L^2(0,T;L^2(D))}), \tag{66}$$

*where $\mathcal{E}_0 = \|\nabla m_0\|^2 + \|H_0\|^2$ and $H_0 = \nabla\varphi_0$ is the demagnetizing field associated to the magnetization $m_0$ and the source term $F(0)$.*

*Proof.* We drop the index $\tau$ for simplicity. We multiply the magnetization equation (63) by $m$ to obtain

$$|m|^2\partial_t m \cdot m = \kappa(\tau)\,|m|^2(m \cdot \mathcal{H}), \tag{67}$$

or equivalently

$$\partial_t m \cdot m = \kappa(\tau)\,(m \cdot \mathcal{H}). \tag{68}$$

Therefore integrating over $D_t$ leads to

$$\frac{1}{2}(\|m(t)\|^2 - \|m_0\|^2) + \kappa(\tau) \int_0^t \|\nabla m\|^2 ds = \kappa(\tau) \int_{D_t} (H \cdot m)\, dxds, \tag{69}$$

and we get estimate (65) using (16) and (17). Now we multiply the magnetization equation (63) by $-\mathcal{H} \in L^2(0, T; \mathbb{L}^2(D))$ to get the equality

$$-\,|m|^2\partial_t m \cdot \mathcal{H} = (1-\tau)\,(m \cdot \mathcal{H})^2 - \gamma(\tau)\,|m|^2|\mathcal{H}|^2, \tag{70}$$

which is equivalent to

$$-\,\chi(m)\partial_t m \cdot \mathcal{H} = (1-\tau)\chi(m)\,(\omega \cdot \mathcal{H})^2 - \gamma(\tau)\chi(m)|\mathcal{H}|^2, \tag{71}$$

where $\omega = \omega(m) = \frac{m}{|m|}$ if $m \neq 0$ and we set $\omega(0) = u$, $u$ being any unit vector. Integrating over $D_t$ and using the inequality $(\omega \cdot \mathcal{H})^2 \leq |\mathcal{H}|^2$ we obtain

$$-\int_{D_t} \chi(m)\partial_t m \cdot \mathcal{H}\, dxds \leq -\kappa(\tau) \int_{D_t} \chi(m)|\mathcal{H}|^2\, dxds. \tag{72}$$

Since $\partial_t m = 0$ on the subset $\{(s,x) \in D_t; m(s,x) = 0\}$ see [5], we have

$$\int_{D_t} \chi(m)\partial_t m \cdot \mathcal{H}\, dxds = \int_{\{(s,x)\in D_t; m(s,x)\neq 0\}} \partial_t m \cdot \mathcal{H}\, dxds = \int_{D_t} \partial_t m \cdot \mathcal{H}\, dxds,$$

then from the equality

$$-\int_{D_t} \partial_t m \cdot \mathcal{H}\, dx = \frac{1}{2}(\|\nabla m(t)\|^2 - \|\nabla m_0\|^2) + \frac{1}{2}(\|H(t)\|^2 - \|H_0\|^2) + \int_{D_t} \partial_t F\,\varphi\, dxds, \tag{73}$$

and inequality (22) we deduce

$$\begin{aligned}
\kappa(\tau)\int_{D_t}\chi(m)|\mathcal{H}|^2\,dxds + \frac{1}{2}(\|\nabla m(t)\|^2+\|H(t)\|^2) \leq \\
\frac{1}{2}(\|\nabla m_0\|^2+\|H_0\|^2) + \frac{1}{2}\int_0^t\|H\|^2 ds + C\int_0^t\|\partial_t F\|^2 ds,
\end{aligned}\tag{74}$$

where $C > 0$ depends only on $D$. Setting $\mathcal{E}(t) = \|\nabla m(t)\|^2 + \|H(t)\|^2$, we see that

$$\mathcal{E}(t) + 2\kappa(\tau)\int_{D_t}\chi(m)|\mathcal{H}|^2\,dxds \leq \mathcal{E}_0 + \int_0^t \mathcal{E}(s)\,ds + C\int_0^t\|\partial_t F\|^2\,ds, \tag{75}$$

and by using Gronwall inequality, we get estimate (66). □

This result allows us to prove that

**Proposition 4.2.** *Under hypotheses (8) and (9), it holds for all* $t \in [0, T]$

$$\|\,|m^\tau(t)|^2 - 1\|^2 \leq C_T\sqrt{\kappa(\tau)}, \tag{76}$$

*where* $C_T > 0$ *depends on* $T$, $\mathcal{E}_0$ *and* $F$ *but not of* $\tau$. *Therefore as* $\tau \to 0$,

$$|m^\tau|^2 \to 1 \quad \textit{strongly in } L^\infty(0, T; L^2(D)). \tag{77}$$

*In particular for small values of* $\tau$, *we have*

$$m^\tau \neq 0 \quad \textit{a.e. in} \quad D_T. \tag{78}$$

*Proof.* We will drop again the index $\tau$ and use the proof of Proposition 4.1. We multiply equality (68) by $|m|^2 - 1$ and integrate over $D_t$ to obtain

$$\int_{D_t}\partial_t(|m|^2-1)^2\,dxds = 4\kappa(\tau)\int_{D_t}(m\cdot\mathcal{H})(|m|^2-1)\,dxds. \tag{79}$$

Since $|m_0| = 1$ we get by means of Cauchy-Schwarz inequality

$$\begin{aligned}
\|\,|m^\tau(t)|^2-1\|^2 = 4\kappa(\tau)\int_{D_t}\chi(m)\,(m\cdot\mathcal{H})(|m|^2-1)\,dxds \\
\leq \sqrt{\kappa(\tau)}\Big(4\kappa(\tau)\int_{D_t}\chi(m)|\mathcal{H}|^2\,dxds + \int_{D_t}|m|^2(|m|^2-1)^2\,dxds\Big),
\end{aligned}$$

where $|m|^2(|m|^2-1)^2 \leq (|m|^2+1)(|m|^2-1)^2 \leq C(|m|^6+1)$ then using embedding $H^1(D) \subset L^6(D)$ together to estimates (65) and (66), we conclude the proof of the proposition. □

According to the previous results and using the same notations, we infer that

**Corollary 4.3.** *For* $\tau > 0$ *small, the solutions* $(m^\tau, H^\tau)$ *satisfy the following uniform bound for all* $t \in [0, T]$

$$\begin{aligned}
\|\nabla m^\tau(t)\|^2 + \|H^\tau(t)\|^2 + 2\kappa(\tau)\int_0^t\int_D|\mathcal{H}^\tau|^2\,dxds \leq \\
e^T(\mathcal{E}_0 + C\|\partial_t F\|^2_{L^2(0,T;L^2(D))}).
\end{aligned}\tag{80}$$

*Proof.* Estimate (80) derives from (66) since by Proposition 4.2, $\chi(m^\tau) = 1$ a.e. in $D_T$.

□

In order to pass to the limit in the problem when $\tau \to 0$, a uniform bound on the time derivative of $m^\tau$ is needed. To begin, since for $\tau > 0$ small enough, $m^\tau \neq 0$ a.e. in $D_T$, we can rewrite equation (63) of $m^\tau$ in its first form (1) that is

$$\begin{aligned} \partial_t m^\tau = -g\, m^\tau \times \mathcal{H}^\tau - \gamma(\tau)\, \omega^\tau \times (\omega^\tau \times \mathcal{H}^\tau) + \\ \kappa(\tau)\, (\omega^\tau \cdot \mathcal{H}^\tau)\omega^\tau \quad \text{in } D_T, \\ (\nu \cdot \nabla)m^\tau = 0 \quad \text{on } \Gamma_T, \quad m^\tau(0) = m_0 \quad \text{in } D, \end{aligned} \tag{81}$$

where $\omega^\tau = \omega(m^\tau)$ and we used the property $|m_0|^2 = 1$ which implies that $|m^\tau(0)|^2 = 1$.
Below we will prove the following result.

**Lemma 4.4.** *Under hypotheses (8) and (9), $\partial_t m^\tau$ is uniformly bounded in $\mathbb{L}^{3/2}(D_T)$ with respect to the small parameter $\tau$.*

*Proof.* From equation (81) we get the equality

$$\partial_t m^\tau = \kappa(\tau)\, (\omega^\tau \cdot \mathcal{H}^\tau)\, \omega^\tau - \omega^\tau \times \big(g|m^\tau|\mathcal{H}^\tau + \gamma(\tau)\, \omega^\tau \times \mathcal{H}^\tau\big), \tag{82}$$

the terms of the right hand side being orthogonal, therefore

$$\frac{|\partial_t m^\tau|^2}{g^2|m^\tau|^2 + \gamma^2(\tau)} + (\omega^\tau \cdot \mathcal{H}^\tau)^2 - |\mathcal{H}^\tau|^2 = \frac{\kappa^2(\tau)}{g^2|m^\tau|^2 + \gamma^2(\tau)} (\omega^\tau \cdot \mathcal{H}^\tau)^2.$$

On another side multiplying equation (81) by $(-\mathcal{H}^\tau)$ and using relation (4) we see that

$$-\gamma(\tau)|\mathcal{H}^\tau|^2 = -\partial_t m^\tau \cdot \mathcal{H}^\tau - (\gamma(\tau) - \kappa(\tau))(\omega^\tau \cdot \mathcal{H}^\tau)^2,$$

which leads to

$$\frac{\gamma(\tau)|\partial_t m^\tau|^2}{g^2|m^\tau|^2 + \gamma^2(\tau)} + \kappa(\tau)(\omega^\tau \cdot \mathcal{H}^\tau)^2 - \partial_t m^\tau \cdot \mathcal{H}^\tau \le \frac{\kappa^2(\tau)}{\gamma(\tau)} (\omega^\tau \cdot \mathcal{H}^\tau)^2. \tag{83}$$

Therefore taking $0 < \tau \le 3/5$, we get for a.e. in $D_T$ the inequality

$$\gamma(\tau)\, \frac{|\partial_t m^\tau|^2}{g^2|m^\tau|^2 + \gamma^2(\tau)} + \frac{\kappa(\tau)}{2} (\omega^\tau \cdot \mathcal{H}^\tau)^2 - \partial_t m^\tau \cdot \mathcal{H}^\tau \le 0. \tag{84}$$

Let $\mathcal{E}^\tau(t) = \|\nabla m^\tau(t)\|^2 + \|H^\tau(t)\|^2$, since

$$-\int_D \partial_t m^\tau \cdot \mathcal{H}^\tau\, dx = \frac{1}{2}\frac{d}{dt}\, \mathcal{E}^\tau + \int_D \partial_t F \varphi\, dx,$$

using inequality (22) and estimate (80), we get for all $t \in [0, T]$

$$\begin{aligned} \mathcal{E}^\tau(t) + \kappa(\tau) \int_0^t \int_D (\omega^\tau \cdot \mathcal{H}^\tau)^2\, dxds + 2\gamma(\tau) \int_0^t \int_D \frac{|\partial_t m^\tau|^2}{g^2|m^\tau|^2 + \gamma^2(\tau)}\, dxds \le \\ \mathcal{E}_0 + \int_0^t (\|H^\tau\|^2 + C\|\partial_t F\|^2)\, ds \le \mathcal{E}_0 + C_T(\mathcal{E}_0 + \|\partial_t F\|^2_{L^2(0,T,L^2(D)}). \end{aligned}$$

Therefore as $2\gamma(\tau) > 1$ for $\tau > 0$ small enough, we deduce that for all $t \in [0, T]$ and $\tau > 0$ small enough it holds

$$\begin{aligned} \int_0^t \int_D \frac{|\partial_t m^\tau|^2}{g^2|m^\tau|^2 + \gamma^2(\tau)}\, dxds + \mathcal{E}^\tau(t) + \kappa(\tau) \int_0^t \int_D (\mathcal{H}^\tau \cdot \omega^\tau)^2\, dxds \\ \le C_T(\mathcal{E}_0 + \|F\|^2_{H^1(0,T,L^2(D)}), \end{aligned} \tag{85}$$

where $C_T > 0$ is independent of $\tau$. Now we write

$$\int_{D_T} |\partial_t m^\tau|^{3/2}\, dxdt = \int_{D_T} \frac{|\partial_t m^\tau|^{3/2}}{(g^2|m^\tau|^2 + \gamma^2(\tau))^{3/4}}\, (g^2|m^\tau|^2 + \gamma^2(\tau))^{3/4}\, dxdt$$
$$\leq C(\int_{D_T} (|m^\tau|^2 + 1)^3 dxdt)^{1/4}\, (\int_{D_T} \frac{|\partial_t m^\tau|^2}{g^2|m^\tau|^2 + \gamma^2(\tau)} dxdt)^{3/4},$$

and use the uniform boundedness of $m^\tau$ in $L^\infty(0, T; \mathbb{H}^1(D))$ and the embedding $H^1(D) \subset L^6(D)$ to get the bound of $\partial_t m^\tau$. □

*4.2. The LLG limit.*

By using the previous bounds and Aubin's compactness lemma we deduce the following convergence results

**Corollary 4.5.** *There exists a subsequence still denoted $(m^\tau, H^\tau)$ and $(m, H = \nabla\varphi)$ such that as $\tau \to 0$*

$$m^\tau \rightharpoonup m \ \textit{weakly} - \star \ \textit{in}\ L^\infty(0, T; \mathbb{H}^1(D)),$$
$$\partial_t m^\tau \rightharpoonup \partial_t m \ \textit{weakly in}\ \mathbb{L}^{3/2}(D_T),$$
$$m^\tau \to m \ \textit{strongly in}\ L^p(0, T; \mathbb{L}^q(D)),\ 1 < p < \infty,\ \ \tfrac{3}{2} \leq q < 6 \tag{86}$$
$$H^\tau \rightharpoonup H \ \textit{weakly} - \star \ \textit{in}\ L^\infty(0, T; \mathbb{H}^1(D)),$$
$$H^\tau \to H \ \textit{strongly in}\ L^2(0, T; \mathbb{L}^2(D)).$$

*Moreover $m$ satisfies the length constraint*

$$|m|^2 = 1 \quad a.e.\ \ in \ \ D_T. \tag{87}$$

*Proof.* The strong convergence of $H^\tau$ is a consequence of Lemma 2.8 and we deduce that $H = \nabla\varphi$ satisfies the magnetostatic equation (64) while property (87) is a direct consequence of the strong convergence of $m^\tau$ and (77). □

As a first step towards (LLG) equation, let us rewrite the magnetization equation (81) in a new form. We observe that if we take the cross product of (81) by $m^\tau$ we get

$$-\, g\, m^\tau \times (m^\tau \times \mathcal{H}^\tau) = m^\tau \times \partial_t m^\tau - \gamma(\tau)\, m^\tau \times \mathcal{H}^\tau, \tag{88}$$

and inserting this expression in (81) we obtain the new formulation

$$\begin{aligned}\frac{g|m^\tau|^2}{g^2|m^\tau|^2 + \gamma^2(\tau)}\, \partial_t m^\tau = -\, m^\tau \times \mathcal{H}^\tau + \frac{\gamma(\tau)}{g^2|m^\tau|^2 + \gamma^2(\tau)}\, m^\tau \times \partial_t m^\tau \\ + \frac{g\kappa(\tau)}{g^2|m^\tau|^2 + \gamma^2(\tau)}\, (m^\tau \cdot \mathcal{H}^\tau)\, m^\tau,\end{aligned} \tag{89}$$

with $\gamma(\tau) = 1 - \frac{\kappa(\tau)}{2} \to 1$ as $\tau \to 0$.

We consider now the weak formulation of equation (89) given by

$$
\begin{aligned}
&- \int_{D_T} \frac{g|m^\tau|^2}{g^2|m^\tau|^2+\gamma^2(\tau)}\, m^\tau \cdot \partial_t \Phi \, dxdt \\
&-2g\gamma^2(\tau) \int_{D_T} \frac{m^\tau \cdot \partial_t m^\tau}{(g^2|m^\tau|^2+\gamma^2(\tau))^2}\, m^\tau \cdot \Phi \, dxdt \\
&- \int_{D_T} m^\tau \times \nabla m^\tau \cdot \nabla \Phi \, dxdt + \int_{D_T} m^\tau \times H^\tau \cdot \Phi \, dxdt \\
&= \frac{g}{g^2+\gamma^2(\tau)} \int_D m_0 \cdot \Phi(0)\, dx + \\
&\int_{D_T} \frac{\gamma(\tau)}{g^2|m^\tau|^2+\gamma^2(\tau)}\, m^\tau \times \partial_t m^\tau \cdot \Phi \, dxdt \\
&+ \int_{D_T} \frac{g\kappa(\tau)}{g^2|m^\tau|^2+\gamma^2(\tau)}\, (m^\tau \cdot \mathcal{H}^\tau)\, m^\tau \cdot \Phi \, dxdt,
\end{aligned}
\tag{90}
$$

for test functions $\Phi \in (\mathcal{D}([0,T[\times \overline{D}))^3$. We will examine the convergence of the nonlinear terms appearing in (90). We have

**Lemma 4.6.** *Letting $\tau \to 0$, we have (at least) the following convergences*

$$m^\tau \times H^\tau \to m \times H \text{ strongly in } \mathbb{L}^{4/3}(D_T), \tag{91}$$

$$m^\tau \times \nabla m^\tau \rightharpoonup m \times \nabla m \text{ weakly in } L^2(0,T;\mathbb{L}^{3/2}(D)), \tag{92}$$

$$\frac{|m^\tau|^2}{g^2|m^\tau|^2+\gamma^2(\tau)}\, m^\tau \to \frac{1}{g^2+1}\, m \text{ strongly in } \mathbb{L}^2(D_T), \tag{93}$$

$$\gamma^2(\tau) \frac{m^\tau \cdot \partial_t m^\tau}{(g^2|m^\tau|^2+\gamma^2(\tau))^2}\, m^\tau \rightharpoonup 0 \quad \text{weakly in } \mathbb{L}^{3/2}(D_T), \tag{94}$$

$$\frac{\gamma(\tau)}{g^2|m^\tau|^2+\gamma^2(\tau)}\, m^\tau \times \partial_t m^\tau \rightharpoonup \frac{1}{g^2+1}\, m \times \partial_t m \text{ weakly in } \mathbb{L}^{6/5}(D_T), \tag{95}$$

$$\frac{\kappa(\tau)}{g^2|m^\tau|^2+\gamma^2(\tau)} (m^\tau \cdot \mathcal{H}^\tau)\, m^\tau \to 0 \text{ strongly in } L^2(0,T;\mathbb{L}^2(D)). \tag{96}$$

*Proof.* The convergence (91) follows directly from the strong convergence of $m^\tau$ and $H^\tau$ stated in Corollary 4.5. Next the strong convergence of $m^\tau$ and the weak-$\star$ convergence of $\nabla m^\tau$ in $L^\infty(0,T;\mathbb{L}^2(D))$ imply the weak convergence

$$m^\tau \times \nabla m^\tau \rightharpoonup m \times \nabla m \quad \text{in } L^2(0,T;\mathbb{L}^r(D)),$$

for $1 \le r < 3/2$ and leads to the result stated in (92) because the sequence is uniformly bounded in $L^2(0,T;\mathbb{L}^{3/2}(D))$. To proceed with the remaining convergences, we see first that as a consequence of (77) and Lebesgue dominated convergence theorem, we have

$$\frac{|m^\tau|^2}{g^2|m^\tau|^2+\gamma^2(\tau)} \to \frac{1}{g^2+1} \text{ strongly in } L^p(0,T;L^q(D))\ 1 \le p,q < \infty. \tag{97}$$

Therefore taking $p = q = 4$ and using the strong convergence of $m^\tau$ in $\mathbb{L}^4(D_T)$, we deduce convergence (93). Similarly we see that for $1 \le i,j \le 3$

$$\frac{m_i^\tau m_j^\tau}{(g^2|m^\tau|^2+\gamma^2(\tau))^2} \to \frac{m_i m_j}{(g^2+1)^2} \text{ strongly in } L^p(0,T;L^q(D))\ 1 \le p,q < \infty,$$

so taking $p = q = 6$ and using the weak convergence of $\partial_t m^\tau$ in $\mathbb{L}^{3/2}(D_T)$, we deduce the convergence

$$\frac{m^\tau \cdot \partial_t m^\tau}{(g^2|m^\tau|^2 + \gamma^2(\tau))^2}\, m^\tau \rightharpoonup \frac{1}{(g^2+1)^2}(m \cdot \partial_t m)m = 0,$$

weakly first in $\mathbb{L}^{6/5}(D_T)$ then in $\mathbb{L}^{3/2}(D_T)$ since the sequence is uniformly bounded in this space. As previously we see that

$$\frac{\gamma(\tau)}{g^2|m^\tau|^2 + \gamma^2(\tau)}\, \partial_t m^\tau \rightharpoonup \frac{1}{g^2+1}\, \partial_t m \text{ weakly in } \mathbb{L}^{3/2}(D_T),$$

so the strong convergence of $m^\tau$ implies the weak convergence of $\dfrac{\gamma(\tau) m^\tau \times \partial_t m^\tau}{g^2|m^\tau|^2 + \gamma^2(\tau)}$ in a space $\mathbb{L}^r(D_T)$ with $r < 6/5$ then since this sequence is bounded in $\mathbb{L}^{6/5}(D_T)$ we get the convergence result (95). To obtain the last convergence of the lemma, it is enough to see that

$$\| \frac{\kappa(\tau)}{g^2|m^\tau|^2 + \gamma^2(\tau)}(m^\tau \cdot \mathcal{H}^\tau)\, m^\tau \|_{\mathbb{L}^2(D_T)} \leq \frac{\sqrt{\kappa(\tau)}}{g^2}\, \| \sqrt{\kappa(\tau)} \mathcal{H}^\tau \|_{\mathbb{L}^2(D_T)}, \tag{98}$$

and use the estimate (80) which implies a uniform bound of $\sqrt{\kappa(\tau)}\mathcal{H}^\tau$ in $\mathbb{L}^2(D_T)$. □

Now we come back to the weak formulation (90) of the problem and pass to the limit as $\tau \to 0$ to get by using the results of the previous lemma

$$\begin{aligned} -\frac{g}{g^2+1} \int_{D_T} m \cdot \partial_t \Phi \, dxdt - \int_{D_T} m \times \nabla m \cdot \nabla \Phi \, dxdt \\ - \int_{D_T} m \times H \cdot \Phi \, dxdt = \frac{g}{g^2+1} \int_D m_0 \cdot \Phi(0)\, dx + \\ \frac{1}{g^2+1} \int_{D_T} m \times \partial_t m \cdot \Phi \, dxdt, \end{aligned} \tag{99}$$

for all $\Phi \in (\mathcal{D}([0,T[\times D))^3$. From here it is easy to deduce that $m$ satisfies the (LLG) equation with the initial and boundary conditions stated in problem (14). Theorem 2.7 is then proved.

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