Free Vibration Analysis of Multi-span Timoshenko Beams on Elastic Foundation Using Dynamic Stiffness Method

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Abstract

In this study, the exact first five natural frequencies of three-span Timoshenko beams on Winkler foundation are calculated using dynamic stiffness formulation. Different elastic foundation spring constants and different beam cross-sections are used to reflect their effects on natural frequencies. Moreover, the natural frequencies are also calculated via structural analysis software SAP2000 and tabulated with exact results. It is seen that the influence of elastic foundation spring stiffness in inner span is high in comparison with outer spans. The cross-section of the beam plays an important role on natural frequencies of multi-span Timoshenko beams on Winkler foundation.

Key words
Dynamic stiffness, free vibration, multi-span Timoshenko beam, Winkler foundation

1. INTRODUCTION

The calculation of exact natural frequencies of beams is of great interest to researchers for a long time. In the recent years, free vibration analyses of different types of beams with various loading conditions are performed using different methods [1-5]. The vibration problem of beams on elastic foundations can be encountered especially in civil engineering and mechanical engineering applications. The elastic foundations are modeled using elastic springs. The effects of elastic foundation can be important for beams or beam assembly structures. Thus, there are numerous studies about different foundation models carrying various types of beams and structures [6-10]. The dynamic stiffness method (DSM) is an effective method for calculating exact natural frequencies of beams or beam like structures as the method uses the exact shape functions. Vibration analyses of many types of beams and plates are performed by using DSM in recent years [11-17].

In this study, free vibration analysis of three-span simply supported beams on Winkler foundation is performed using DSM. Timoshenko Beam Theory (TBT) which considers shear deformation and rotational inertia is used. In the numerical analysis, different spring stiffness values for spans are selected to reveal the effect of elastic foundation on natural frequencies. To reflect the importance of beam geometry, several analyses are completed for different beam geometries. SAP2000 is a well known and widely used structural analysis software worldwide.
Thus, SAP2000 is used to obtain natural frequencies of beams on Winkler foundation and the results are compared with exact values.

2. MODEL AND FORMULATION

The mathematical model of three-span Timoshenko beam on Winkler foundation can be seen in Figure 1. Here, \( x \) and \( y \) represents the axes, \( k_{s1} \), \( k_{s2} \) and \( k_{s3} \) are Winkler foundation spring stiffnesses, \( L \) is the span length, \( b \) is width of the beam and \( h \) is the height of the beam.

![Figure 1. Three-span Timoshenko beam on Winkler foundation](image)

Let the denomination of span AB as 1, span BC as 2 and span CD as 3.

The assumptions listed below are considered to clarify and simplify the analysis procedure:

1) The beam is constructed by using an isotropic and homogenous material.
2) The cross-section of the beam is uniform.
3) The beam behaves linear and elastic.
4) The damping is neglected.
5) The foundation springs are linear and distributed along the beam length.

The governing equations of motion of vibrating Timoshenko beam resting on Winkler foundation are given as follows:

\[
\begin{align*}
\frac{AG}{k} \left( \frac{\partial^2 y_n(x,t)}{\partial x^2} - \frac{\partial \theta_n(x,t)}{\partial x} \right) - m \frac{\partial^2 y_n(x,t)}{\partial t^2} - k_{s1} y_n(x,t) &= 0 \\
\frac{EI}{kL} \frac{\partial^2 \theta_n(x,t)}{\partial x^2} - \frac{m}{kL} \frac{\partial^2 \theta_n(x,t)}{\partial t^2} - \frac{AG}{k} \left( \frac{\partial y_n(x,t)}{\partial x} - \theta_n(x,t) \right) &= 0
\end{align*}
\]

(1)

In Eq. (1), \( A \) is cross-sectional area, \( I \) is area moment of inertia, \( G \) is shear modulus, \( E \) is Young’s modulus, \( k \) is shear coefficient, \( m \) is mass per unit length. \( y_n(x,t) \) and \( \theta_n(x,t) \) are \( n \)th beam span’s deflection function and rotation function, respectively (\( n = 1, 2, 3 \)).

If the motion of the beam is harmonic and separation of variables method is applied, the following equation is obtained:

\[
\begin{align*}
\frac{AG}{kL} \frac{d^2 y_n(z)}{dz^2} - \frac{AG}{kL} \frac{d \theta_n(z)}{dz} + \frac{\bar{m} \omega^2 y_n(z)}{k} - k_{s1} y_n(z) &= 0 \\
\frac{EI}{E} \frac{d^2 \theta_n(z)}{dz^2} + \frac{AG}{kL} \frac{d y_n(z)}{dz} + \left( \frac{\bar{m} \omega^2}{k} - \frac{AG}{k} \right) \theta_n(z) &= 0
\end{align*}
\]

(2)

where \( z = x/L \) and \( \omega \) is natural frequency.

The solution is assumed as:

\[
\begin{align*}
y_n(z) &= \begin{bmatrix} C \end{bmatrix}_n e^{i\omega t} \\
\theta_n(z) &= \begin{bmatrix} D \end{bmatrix}_n e^{i\omega t}
\end{align*}
\]

(3)

Substituting Eq.(3) into Eq.(2), \( y_n(z) \) and \( \theta_n(z) \) functions are given in Eq.(4) and Eq. (5), respectively.

\[
y_n(z) = (\overline{C}_n e^{i\omega z} + \overline{D}_n e^{i\omega z} + \overline{C}_n e^{-i\omega z} + \overline{D}_n e^{-i\omega z})
\]

(4)
\[ \theta_i(z) = (K_{i0}C_{i0}e^{im\omega z} + K_{i1}C_{i1}e^{im\omega z} + K_{i2}C_{i2}e^{im\omega z} + K_{i3}C_{i3}e^{im\omega z}) \]  

(5)

where \( K_{im} = \frac{(AG)}{kL} + (m\omega^2) - k_n \); \( m=1,2,3,4; j=1,2,3,4 \)

The bending moment function and shear force function are defined in Eq. (6) and Eq. (7), respectively.

\[ M_i(z) = \frac{EI}{L} \frac{d\theta_i(z)}{dz} \]  

(6)

\[ Q_i(z) = \frac{AG}{kL} \frac{dy_i(z)}{dz} + \frac{AG}{k} \theta_i(z) \]  

(7)

3. DYNAMIC STIFFNESS METHOD (DSM) FOR CALCULATING NATURAL FREQUENCIES

DSM is a technique that can be used for calculating exact natural frequencies using exact mode shapes. First of all, the dynamic stiffness matrix should be obtained. The dynamic stiffness matrix can be constructed by using end displacements and end forces of beam. The vector of end displacements of beam and the vector of coefficients are given in Eqs. (8) and (9), respectively.

\[ \delta_e = \begin{bmatrix} y_{e0} & \theta_{e0} & y_{e1} & \theta_{e1} \end{bmatrix}^T \]  

(8)

\[ \tilde{C}_e = \begin{bmatrix} \tilde{C}_{e1} & \tilde{C}_{e2} & \tilde{C}_{e3} & \tilde{C}_{e4} \end{bmatrix}^T \]  

(9)

where

\[ y_{e0} = y_e(z=0) \]

\[ \theta_{e0} = \theta_e(z=0) \]

\[ y_{e1} = y_e(z=1) \]

\[ \theta_{e1} = \theta_e(z=1) \]

Eqs. (8) and (9) can be rewritten in the form below:

\[ \begin{bmatrix} y_{e0} \\ \theta_{e0} \\ y_{e1} \\ \theta_{e1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ K_{e1} & K_{e2} & K_{e3} & K_{e4} \\ e^{mu} & e^{nu} & e^{mu} & e^{nu} \\ K_{e1}e^{mu} & K_{e2}e^{nu} & K_{e3}e^{mu} & K_{e4}e^{nu} \end{bmatrix} \begin{bmatrix} \tilde{C}_{e1} \\ \tilde{C}_{e2} \\ \tilde{C}_{e3} \\ \tilde{C}_{e4} \end{bmatrix} \]  

(10)

The closed form of Eq. (10) is given in Eq. (11):

\[ \delta_e = A\tilde{C}_e \]  

(11)

where

\[ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ K_{e1} & K_{e2} & K_{e3} & K_{e4} \\ e^{mu} & e^{nu} & e^{mu} & e^{nu} \\ K_{e1}e^{mu} & K_{e2}e^{nu} & K_{e3}e^{mu} & K_{e4}e^{nu} \end{bmatrix} \]

The end forces of the beam is given in vector form in Eq. (12):

\[ F = [Q_{e0} \quad M_{e0} \quad Q_{e1} \quad M_{e1}]^T \]  

(12)

where

\[ Q_{e0} = Q_e(z=0), M_{e0} = M_e(z=0), Q_{e1} = Q_e(z=1), M_{e1} = M_e(z=1) \]

Eqs. (12) and (9) can be written in the following form:
\[
\begin{bmatrix}
Q_{ei} \\
M_{ei} \\
Q_{si} \\
M_{si}
\end{bmatrix} =
\begin{bmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
A_1 & A_2 & A_3 & A_4 \\
-\lambda A e^{i \nu} & -\lambda A e^{i \nu} & -\lambda A e^{i \nu} & -\lambda A e^{i \nu} \\
-A e^{i \nu} & -A e^{i \nu} & -A e^{i \nu} & -A e^{i \nu}
\end{bmatrix}
\begin{bmatrix}
C_{ai} \\
C_{ei} \\
C_{si} \\
C_{se}
\end{bmatrix}
\]  
(13)

where \( \lambda_j = \frac{AG}{kL^2} \), \( A_j = \frac{EI}{kL} \), \( j = 1, 2, 3, 4 \).

The closed form of Eq. (13) can be written as:

\[ F_e = \kappa_e C_e \]  
(14)

where

\[ \kappa_e = \begin{bmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
A_1 & A_2 & A_3 & A_4 \\
-\lambda A e^{i \nu} & -\lambda A e^{i \nu} & -\lambda A e^{i \nu} & -\lambda A e^{i \nu} \\
-A e^{i \nu} & -A e^{i \nu} & -A e^{i \nu} & -A e^{i \nu}
\end{bmatrix} \]

Eqs.(11) and (14) are used to construct the dynamic stiffness matrix of the n th span of the Timoshenko beam on elastic foundation.

\[ F_e = \kappa_e A_e^{-1} C_e \]  
(15)

In Eq.(15), \( \kappa_e A_e^{-1} \) represents the dynamic stiffness matrix of the n th span. The natural frequencies of the beam are calculated by equating the determinant of assembly of \( \kappa_1 A_1^{-1} \), \( \kappa_2 A_2^{-1} \) and \( \kappa_3 A_3^{-1} \) to zero. It should be noted that the related rows and columns of dynamic stiffness matrix of the beam are erased according to boundary conditions.

4. NUMERICAL ANALYSIS AND DISCUSSION

A three-span Timoshenko beam on Winkler foundation is used for numerical analysis with the following properties: \( E = 2 \times 10^7 \) kN/m \(^2\), \( G = 7692308 \) kN/m \(^2\), \( k = 1.2 \), \( \rho = 25 \) kN/m \(^3\), \( L = 6 \) m, \( b = 1 \) m.

The boundary conditions are same for each span and given below:

\[ y_i(z = 0) = 0, M_i(z = 0) = 0, y_i(z = 1) = 0, M_i(z = 1) = 0 \]

The analyses are performed for constant \( k_{s2} \) and \( k_{s4} \) with varying \( k_{s1} \), constant \( k_{s1} \) and \( k_{s3} \) with varying \( k_{s2} \), constant spring stiffnesses with various beam height values. The first five natural frequencies of the beam are presented in Tables (1-3). It should be noted that SAP2000 results are obtained by dividing spans into 1 cm segments for accuracy.

**Table.1. First five natural frequencies (h=0.75 m, k_{s2}=10000 kN/m, k_{s3}=10000 kN/m)**

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd Mode</td>
<td>34.2849</td>
<td>34.1590</td>
<td>34.7388</td>
<td>34.6111</td>
<td>35.2293</td>
<td>35.1000</td>
<td>35.7462</td>
<td>35.6153</td>
<td>36.2791</td>
<td>36.1467</td>
</tr>
<tr>
<td>3rd Mode</td>
<td>47.7167</td>
<td>47.5440</td>
<td>47.8737</td>
<td>47.6664</td>
<td>47.9689</td>
<td>48.1097</td>
<td>47.7954</td>
<td>48.2607</td>
<td>48.0889</td>
<td></td>
</tr>
<tr>
<td>4th Mode</td>
<td>97.9198</td>
<td>98.4130</td>
<td>98.0322</td>
<td>98.9265</td>
<td>98.1410</td>
<td>99.0360</td>
<td>98.2463</td>
<td>99.1424</td>
<td>98.3481</td>
<td>99.2454</td>
</tr>
</tbody>
</table>
It is seen from Table 1 that, the natural frequencies are increased with increasing spring stiffness of an outer span of three-span beam. Table 2 shows that there is also an augmentation in natural frequencies when the spring stiffness of middle span is increased. There is no significant difference between the particular increment of spring stiffness of middle span and an outer span on natural frequencies. Table 3 reveals that the natural frequencies are increased due to increasing beam height and higher modes are more sensitive to this effect.

Table 2. First five natural frequencies ($h=0.75$ m, $k_{s1}=10000$ kN/m, $k_{s3}=10000$ kN/m)

<table>
<thead>
<tr>
<th>Natural Frequency (Hz)</th>
<th>500</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd Mode</td>
<td>34.7134</td>
<td>34.5983</td>
<td>34.7388</td>
<td>34.6111</td>
<td>34.7642</td>
</tr>
<tr>
<td>3rd Mode</td>
<td>47.3841</td>
<td>47.2243</td>
<td>47.6373</td>
<td>47.6665</td>
<td>48.2066</td>
</tr>
<tr>
<td>4th Mode</td>
<td>97.9209</td>
<td>98.8141</td>
<td>98.0322</td>
<td>98.9265</td>
<td>98.1421</td>
</tr>
</tbody>
</table>

Table 3. First five natural frequencies ($k_{s1} = k_{s2} = k_{s3} = 10000$ kN/m)

<table>
<thead>
<tr>
<th></th>
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<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd Mode</td>
<td>27.9915</td>
<td>27.8159</td>
<td>31.2701</td>
<td>31.1127</td>
<td>34.7388</td>
<td>34.6111</td>
<td>38.2623</td>
<td>38.1773</td>
<td>41.7629</td>
<td>41.7336</td>
</tr>
<tr>
<td>3rd Mode</td>
<td>37.6691</td>
<td>37.4618</td>
<td>42.8127</td>
<td>42.5751</td>
<td>47.8373</td>
<td>47.6645</td>
<td>52.7395</td>
<td>52.6141</td>
<td>57.4279</td>
<td>57.3397</td>
</tr>
<tr>
<td>4th Mode</td>
<td>75.5556</td>
<td>75.6747</td>
<td>87.0604</td>
<td>87.5351</td>
<td>98.0322</td>
<td>98.9265</td>
<td>108.3847</td>
<td>109.7625</td>
<td>118.095</td>
<td>119.9983</td>
</tr>
<tr>
<td>5th Mode</td>
<td>84.3307</td>
<td>84.4696</td>
<td>96.7702</td>
<td>97.2515</td>
<td>108.3922</td>
<td>109.2864</td>
<td>119.1587</td>
<td>120.5124</td>
<td>129.078</td>
<td>130.9151</td>
</tr>
</tbody>
</table>

Figure 2. Fundamental frequencies for different $k_{s1}$ and $k_{s2}$ values

Figure 2 implies that fundamental frequency of three-span Timoshenko beam on Winkler foundation is more sensitive to spring stiffness of middle span in comparison with outer span especially for high stiffness values. Figure 3 represents the variation of first three natural frequencies with different beam height values.
Figure 3. First three natural frequencies for different h values (k_{s1}=k_{s2}=k_{s3}=10000 kN/m)

6. CONCLUSIONS

The first five exact natural frequencies of three-span Timoshenko beams on Winkler foundation are obtained using dynamic stiffness approach. The spring stiffnesses of middle span is more effective than outer span. Different beam height values are used in the numerical analysis and effects on natural frequencies are observed. SAP2000 provides fairly well results when segment number increased sufficiently. The DSM can be used for calculating exact natural frequencies of multi-span Timoshenko beams on elastic foundation with different support conditions and foundation models.

REFERENCES


