On the Dynamics of the Recursive Sequence

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ABSTRACT

Our aim in this paper is to investigate the local stability of the positive solutions of the difference equation

\[ y_{n+1} = \frac{\alpha - y_n}{\beta y_{n-1}} - \gamma y_{n-1}, \quad n = 0, 1, 2, \ldots, \]

where the initial conditions \( y_{-1}, \ y_0 \) are arbitrary positive real numbers such that \( y_n \neq 0 \) for \( n = -1, 0, 1, \ldots \), \( \alpha, \beta, \gamma \in (0, \infty) \) and \( \alpha > \gamma \). Furthermore we investigate the periodic nature of the mentioned difference equation.

Key Words: Difference Equations, Local Stability, Period-two Solutions.

1. INTRODUCTION

Consider the difference equation

\[ y_{n+1} = \frac{\alpha - y_n}{\beta y_{n-1}} - \gamma y_{n-1}, \quad n = 0, 1, 2, \ldots, \] (1)

where the initial conditions \( y_{-1}, \ y_0 \) are arbitrary positive real numbers such that \( y_n \neq 0 \) for \( n = -1, 0, 1, \ldots \), \( \alpha, \beta, \gamma \in (0, \infty) \) and \( \alpha > \gamma \). Our aim in this paper is to investigate the local asymptotic stability of the positive equilibrium point of Eq. (1), furthermore the periodic nature of the solutions of Eq. (1) under specified conditions of the parameters \( \alpha, \beta \) and \( \gamma \).

Other nonlinear, rational difference equations were investigated in [1]-[6]. In [1], the global stability, the boundedness character and the periodic nature of the positive solutions of the recursive sequence

\[ x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \ldots, \] (2)

was investigated by Amleh et al.[1], where \( \alpha \in [0, \infty) \) and the initial conditions \( x_{-1} \) and \( x_0 \) are arbitrary positive real numbers. They showed that a necessary and sufficient condition which every positive solution of Eq. (2) be bounded is \( \alpha \geq 1 \). H. M. El-Owaidy et al. [2] studied the recursive sequence

\[ x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, 2, \ldots, \] (3)

where \( \alpha \in [1, \infty), \ k \in \{1, 2, \ldots\} \) and the initial conditions \( x_{-k}, \ldots, x_0 \) are arbitrary positive real numbers.

R. DeVault et al. [3] investigated the global stability and the periodic character of solutions of the equation

\[ y_{n+1} = (p+y_{n-k})/(q y_n + y_{n-k}) \]

where \( p \) and \( q \) are positive, \( k \in \{1, 2, \ldots\} \) where the initial conditions \( x_{-k}, \ldots, x_0 \) are arbitrary positive numbers. W. S. He and W. T. Li [4] studied the global stability and the periodic character of the positive solutions of the difference equation

\[ x_{n+1} = \frac{a - bx_{n-k}}{A + x_n}, \quad n = 0, 1, 2, \ldots, \] (4)

where \( a \geq 0, \ b, A > 0, \ k \in \{1, 2, \ldots\} \) and initial conditions \( x_{-k}, \ldots, x_0 \) are arbitrary real numbers. They
showed that the positive equilibrium of the equation is a global attractor with a basin that depends on certain conditions posed on the coefficients. Also, Yan et al. [5] investigated the global asymptotic stability, the global attractivity, the boundedness character, the periodic nature and chaotic behavior of solutions of the difference equation \( x_{n+1} = \alpha - \frac{x_{n-1}}{x_n} \), \( n = 0,1,2,... \),
where \( \alpha \) is a real number and the initial conditions \( x_{-1}, x_0 \) are arbitrary positive real numbers.

We first recall some results which will be useful in the sequel.

Let \( I \subset \mathbb{R} \) and let \( f : I \times I \rightarrow I \) be a continuous function. Consider the difference equation
\[
y_{n+1} = f(y_n, y_{n-1}), \quad n = 0,1,2,..., \tag{5}
\]
where the initial conditions \( y_{-1}, y_0 \in I \). We say that \( y \) is an equilibrium of Eq. (5) if
\[
y = f(y, y).
\]

Let
\[
s = \frac{\partial f}{\partial y}(y, y) \quad \text{and} \quad t = \frac{\partial f}{\partial y}(y, y)
\]
denote the partial derivatives of \( f(u,v) \) evaluated at an equilibrium \( y \) of Eq. (5). Then the equation
\[
x_{n+1} = sx_n + tx_{n-1}, \quad n = 0,1,2,...
\]
is called the linearized equation associated with Eq. (5) about the equilibrium point \( y \) [6].

2. LOCAL ASYMPOTOTIC STABILITY AND PERIOD-TWO SOLUTIONS

In this section, we discuss the local stability of the positive equilibrium point \( y \) and period-two solutions of Eq. (1). The equilibrium points of Eq. (1) are solutions of the equation
\[
\beta y^2 - (\alpha - \gamma) = 0. \tag{6}
\]
So the equilibrium points of Eq. (1) are
\[
y = \frac{\pm \sqrt{\alpha - \gamma}}{\beta}, \tag{7}
\]
where \( \alpha > \gamma \). The linearized equation associated with Eq. (1) about the equilibrium point \( y \) is
\[
x_{n+1} - \frac{\gamma - 2y}{\alpha - \gamma} x_n + \frac{\alpha - 2y}{\alpha - \gamma} x_{n-1} = 0, \quad n = 0,1,2,... \tag{8}
\]
Hence, its characteristic equation is
\[
\lambda^2 - \frac{\gamma - 2y}{\alpha - \gamma} \lambda + \frac{\alpha - 2y}{\alpha - \gamma} = 0. \tag{9}
\]

Theorem 2.1. Suppose that \( \alpha, \beta, \gamma \in (0, \infty) \) and \( \alpha > \gamma \). Then the following statements are true.

(i) The positive equilibrium point \( y \) of Eq. (1) is locally asymptotically stable if
\[
4\frac{(\alpha - \gamma)}{\alpha} < \beta < 4\frac{(\alpha - \gamma)}{\gamma^2}. \tag{10}
\]

(ii) The positive equilibrium point \( y \) of Eq. (1) is unstable (furthermore is a repeller point) if
\[
4\frac{(\alpha - \gamma)}{\alpha^2} < 4\frac{(\alpha - \gamma)}{\gamma^2} < \beta. \tag{11}
\]

Proof. (i) Since
\[
\beta < 4\frac{(\alpha - \gamma)}{\gamma^2}, \quad \text{then}
\]
\[
\gamma^2 < 4\frac{(\alpha - \gamma)}{\beta} \tag{12}
\]
and
\[
\gamma < \frac{\sqrt{\alpha - \gamma}}{\beta}. \tag{13}
\]

Using positive equilibrium point of Eq. (1) in (7) and (14), we have
\[
\gamma - 2y < 0
\]
and
\[
2\alpha - \gamma - 2y < 2\alpha - 2\gamma. \tag{15}
\]

Hence from (15), we get
\[
1 + \frac{\alpha - 2y}{\alpha - \gamma} < 2. \tag{16}
\]

On the other hand, using positive equilibrium point of Eq. (1) in (7) and
\[
4\frac{(\alpha - \gamma)}{\alpha^2} < \beta,
\]
we have
\[
2y < \alpha
\]
and
\[
\frac{\gamma - 2y}{\alpha - \gamma} < 1 + \frac{\alpha - 2y}{\alpha - \gamma}. \tag{17}
\]

Since
\[
\gamma - 2y < 0
\]
we can write
\[
\frac{\gamma - 2\gamma}{\alpha - \gamma} < 1 + \frac{\alpha - 2\gamma}{\alpha - \gamma}.
\] (18)

Then, one can see from (16) and (18) that, the positive equilibrium point \( \bar{y} \) of Eq. (1) is locally asymptotically stable. This completes the proof of the first part in the theorem.

(ii) Since

\[
\frac{4(\alpha - \gamma)}{\beta} < \gamma^2,
\] (19)

then

\[
\frac{4(\alpha - \gamma)}{\beta} < \gamma^2,
\] (20)

and

\[
\gamma > 2\left\frac{\alpha - \gamma}{\beta}\right\gamma.
\] (21)

Using positive equilibrium point of Eq. (1) in (7) and (21), we have

\[
\gamma - 2\gamma > 0
\]

and

\[
\alpha - 2\gamma > \alpha - \gamma.
\] (22)

Thus we get

\[
\frac{\alpha - 2\gamma}{\alpha - \gamma} > 1.
\] (23)

On the other hand, using positive equilibrium point of Eq. (1) in (7) and

\[
\gamma < \alpha
\]

we have

\[
\gamma - 2\gamma < \alpha - \gamma + \alpha - 2\gamma
\]

and

\[
\gamma - 2\gamma < 1 + \frac{\alpha - 2\gamma}{\alpha - \gamma}
\] (24)

Since

\[
\gamma - 2\gamma > 0,
\]

we can write

\[
\frac{\gamma - 2\gamma}{\alpha - \gamma} < \frac{\alpha - 2\gamma}{\alpha - \gamma}.
\] (25)

Then, one can see from (23) and (25) that, the positive equilibrium point \( \bar{y} \) of Eq. (1) is unstable (furthermore a repeller point). This completes the proof.

**Theorem 2.2** Suppose that \( \alpha, \beta, \gamma \in (0, \infty) \), \( \alpha > \gamma \) and

\[
\beta = \frac{4(\alpha - \gamma)}{\gamma^2} < \frac{4(\alpha - \gamma)}{\gamma^2}.
\] (26)

Then the positive equilibrium point \( \bar{y} \) of Eq. (1) is non-hyperbolic point.

**Proof.** Since

\[
\beta = \frac{4(\alpha - \gamma)}{\gamma^2},
\] (27)

then

\[
\alpha = 2\sqrt{\frac{\alpha - \gamma}{\beta}}.
\] (28)

Using positive equilibrium point of Eq. (1) in (7) and (28), we have

\[
\alpha - 2\gamma = 0
\]

and

\[
\gamma - 2\gamma = \gamma - \alpha.
\] (29)

Hence from (29), we can write

\[
\gamma - 2\gamma = |\gamma - \alpha| = |\alpha - \gamma|.
\] (30)

On the other hand, since

\[
\alpha - 2\gamma = 0,
\]

we have

\[
|\gamma - 2\gamma| = |\alpha - \gamma + \alpha - 2\gamma|
\] (31)

and

\[
|\gamma - 2\gamma| = |1 + \frac{\alpha - 2\gamma}{\alpha - \gamma}|
\] (32)

Thus, we get \(|\bar{s}| = |1 - t|\) where \( s = \frac{\gamma - 2\gamma}{\alpha - \gamma} \) and \( t = \frac{\alpha - 2\gamma}{\alpha - \gamma} \). Then the positive equilibrium point \( \bar{y} \) of Eq. (1) is non-hyperbolic point.

**Theorem 2.3** Suppose that \( \alpha, \beta, \gamma \in (0, \infty) \), \( \alpha > \gamma \) and

\[
\beta = \frac{4(\alpha - \gamma)}{\gamma^2} > \frac{4(\alpha - \gamma)}{\gamma^2}.
\] (33)

Then the positive equilibrium point \( \bar{y} \) of Eq. (1) is non-hyperbolic point.

**Proof.** Since

\[
\beta = \frac{4(\alpha - \gamma)}{\gamma^2},
\] (34)

then
\( \gamma = 2 \sqrt{\frac{(\alpha - \gamma)}{\beta}}. \) \hspace{1cm} (35)

Using positive equilibrium point of Eq. (1) in (7) and (35), we have

\( \gamma - 2\gamma = 0 \)

and

\( \alpha - 2\gamma = \alpha - \gamma. \) \hspace{1cm} (36)

Hence from (36), we can write

\( \frac{\alpha - 2\gamma}{\alpha - \gamma} = 1. \) \hspace{1cm} (37)

On the other hand, we have

\( t = \frac{\alpha - 2\gamma}{\alpha - \gamma} = -1. \)

Furthermore, since \( \gamma - 2\gamma = 0 \) we get

\[ |T| = \left| \frac{\gamma - 2\gamma}{\alpha - \gamma} \right| = 0 < 2. \] \hspace{1cm} (38)

Thus, the positive equilibrium point \( \overline{\gamma} \) of Eq. (1) is non-hyperbolic point under the condition (33).

**Example 2.1.** Let \( \alpha = 5, \beta = 4, \gamma = 1, y_{-1} = 1 \) and \( y_0 = 0.5 \). The values of \( \alpha, \beta \) and \( \gamma \) verify (10). The graph of the first 100 iterations of Eq. (1) is given in Figure 1. The graph suggests that the solution of Eq. (1) is converging to a stable equilibrium value of about \( \overline{\gamma} = 1 \). If this converge continues in the form of a limit as \( n \to \infty \), future values of \( y_n \) can be predicted to be at or extremely near the equilibrium value (furthermore see Table 1).

**Example 2.2.** Let \( \alpha = 20, \beta = 2.8, \gamma = 5, y_{-1} = 0.2 \) and \( y_0 = 0.5 \). Note that the values of \( \alpha, \beta \) and \( \gamma \) verify (11). The graph of the first 50 iterations of Eq. (1) is given in Figure 2. It does not contain any of the predictable patterns of solutions. Absent of any pattern or repetition, the general long-term behavior and specific values of \( y_n \) for large \( n \) are impossible to predict from the graph (furthermore see Figure 2 and Table 2).

**Theorem 2.4.** Suppose that \( \{y_n\}_{n=-1}^{\infty} \) is a solution of Eq. (1) and \( \alpha, \beta, \gamma \in (0, \infty) \). Then the following statements are true.

(i) If \( y_{-1} = \sqrt{\frac{\alpha + \gamma}{\beta}} \) and \( y_0 = -\sqrt{\frac{\alpha + \gamma}{\beta}} \) (or \( y_{-1} = -\sqrt{\frac{\alpha + \gamma}{\beta}} \) and \( y_0 = \sqrt{\frac{\alpha + \gamma}{\beta}} \)), then Eq. (1) has period-two solutions.

(ii) Assume that \( A = \frac{\alpha^2 - 4(\alpha - \gamma)}{\beta} > 0 \). If \( y_{-1} = \frac{\alpha + \sqrt{A}}{2} \) and \( y_0 = \frac{\alpha - \sqrt{A}}{2} \) (or \( y_{-1} = \frac{\alpha - \sqrt{A}}{2} \) and \( y_0 = \frac{\alpha + \sqrt{A}}{2} \)), then Eq. (1) has period-two solutions.

**Proof.** Let

\[ \ldots, \phi, \psi, \phi, \psi, \ldots \]

be a period-two solutions of Eq. (1). Then,

\[ \phi = \frac{\alpha - \psi}{\beta \phi} = \frac{\alpha \psi - \psi^2 - \gamma \phi + \phi^2}{\beta \psi}, \quad (39) \]

\[ \psi = \frac{\alpha - \phi}{\beta \psi} = \frac{\alpha \phi - \phi^2 - \gamma \psi + \psi^2}{\beta \phi}, \quad (40) \]

Subtracting (40) from (39), we have

\[ (\phi - \psi)(\beta \psi + \alpha + \gamma - 2(\phi + \psi)) = 0. \quad (41) \]

Since \( \phi \) and \( \psi \) period-two solutions of Eq. (1), then \( \psi \neq \phi \). Hence from (41), we have

\[ \beta \psi + \alpha + \gamma - 2(\phi + \psi) = 0. \quad (42) \]

Thus period-two solutions of Eq. (1) are also the solutions of (42). From (39) and (40), we get

\[ \frac{\phi}{\psi} = \frac{\alpha \psi - \psi^2 - \gamma \phi + \phi^2}{\alpha \phi - \phi^2 - \gamma \psi + \psi^2} \]

and

\[ (\phi - \psi)(\phi + \psi)(\alpha - \phi - \psi) = 0. \quad (43) \]

Firstly, assume that \( \phi + \psi = 0 \) in (43). From this, we get \( \psi = -\phi \). Using \( \psi = -\phi \) in (42), we obtain

\[ \phi_{1,2} = \pm \sqrt{\frac{\alpha + \gamma}{\beta}}. \quad (44) \]

So the period-two solutions must be of the form

\[ \ldots, \sqrt{\frac{\alpha + \gamma}{\beta}}, \sqrt{\frac{\alpha + \gamma}{\beta}}, \ldots \quad (45) \]
Table 1. Values of the iteration solutions of Eq. (1) for $\alpha = 5$, $\beta = 4$, $\gamma = 1$, $y_{-1} = 1$ and $y_0 = 0.5$.

<table>
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Figure 1. Graph of the iteration solutions of Eq. (1) for $\alpha = 5$, $\beta = 4$, $\gamma = 1$, $y_{-1} = 1$ and $y_0 = 0.5$.

Table 2. Values of the iteration solutions of Eq. (1) for $\alpha = 20$, $\beta = 2.8$, $\gamma = 5$, $y_{-1} = 0.2$ and $y_0 = 0.5$.

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Figure 2. Graph of the iteration solutions of Eq. (1) for $\alpha = 20$, $\beta = 2.8$, $\gamma = 5$, $y_{-1} = 0.2$ and $y_0 = 0.5$.

Table 3. Values of the iteration solutions of Eq. (1) for $\alpha = 6$, $\beta = 0.2$, $\gamma = 5$, $y_{-1} = 5$, $y_0 = 1$.

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Figure 3. Graph of the period-two solutions of Eq. (1) for $\alpha = 6$, $\beta = 0.2$, $\gamma = 5$, $y_{-1} = 5$, $y_0 = 1$. 
Secondly, assume that \((\alpha - \phi - \psi) = 0\). By using \(\psi = \alpha - \phi\) in (42) we obtain
\[
\phi_{3,4} = \frac{\alpha \pm \sqrt{\Delta}}{2},
\]
(46)

(i) Let \(B = \sqrt{\frac{\alpha + \gamma}{\beta}}\), \(y_{-1} = B\) and \(y_0 = -B\). Then from Eq. (1), we have
\[
y_1 = \frac{\alpha - y_0 - \gamma - y_{-1}}{\beta y_{-1}} = \frac{\alpha + B + \gamma - B}{\beta B} = B, \\
y_2 = \frac{\alpha - y_1 - \gamma - y_0}{\beta y_0} = \frac{\alpha - B + \gamma + B}{\beta B} = -B, \\
y_3 = \frac{\alpha - y_2 - \gamma - y_1}{\beta y_1} = \frac{\alpha + B - \gamma - B}{\beta B} = B, \\
y_4 = \frac{\alpha - y_3 - \gamma - y_2}{\beta y_2} = \frac{\alpha - B + \gamma - B}{\beta B} = -B, \ldots
\]
The case \(y_{-1} = -B\) and \(y_0 = B\) is similar and will be omitted.

(ii) Using \(y_1 = \frac{\alpha + \sqrt{\Delta}}{2}\) and \(y_0 = \frac{\alpha - \sqrt{\Delta}}{2}\) in Eq. (1), we have
\[
y_1 = \frac{\alpha - y_0 - \gamma - y_{-1}}{\beta y_{-1}} = \frac{\alpha + B + \gamma - B}{\beta B} = B, \\
y_2 = \frac{\alpha - y_1 - \gamma - y_0}{\beta y_0} = \frac{\alpha - B + \gamma + B}{\beta B} = -B, \\
y_3 = \frac{\alpha - y_2 - \gamma - y_1}{\beta y_1} = \frac{\alpha + B - \gamma - B}{\beta B} = B, \\
y_4 = \frac{\alpha - y_3 - \gamma - y_2}{\beta y_2} = \frac{\alpha - B + \gamma - B}{\beta B} = -B, \ldots
\]
The case \(y_{-1} = -B\) and \(y_0 = B\) is similar and will be omitted. This completes the proof.

Example 2.3. Let \(\alpha = 6\), \(\beta = 0.2\), \(\gamma = 5\). Using the initial conditions \(y_{-1} = 5\) and \(y_0 = 1\), we have the the graph of the first 100 iterations of Eq. (1) and this graph is shown in Figure 3. The graphs show that, the solution of Eq. (1) oscillates between two values of 5 and 1. It is easy to predict future values of \(y_n\) (furthermore see Figure 3 and Table 3).

REFERENCES


[3] DeVault, R., Kosmala, W., Ladas, G., Schultz, S.W., “Global Behavior of \(y_{n+1} = \frac{p + y_{n-k}}{q y_n + y_{n-k}}\), Nonlinear Analysis, 47: 4743-4751 (2001).

