Gazi University Journal of Science
GU J Sci
24(1):35-39 (2011)

GUJS
www.gujs.org

The Riesz Core of a Sequence

Celal ÇAKAN[1,♠], Abdullah M. ALOTAIBI[2]

[1]İnönü University, Faculty of Education, 44280, Malatya, Turkey
[2]School of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Received: 03/11/2010 Revised: 05/11/2010 Accepted: 08/11/2010

ABSTRACT

The Riesz sequence space r_c^q including the space c has recently been defined in [14] and its some properties have been investigated. In the present paper, we introduce a new type core, K_q-core, of a complex valued sequence and also determine the required conditions for a matrix B for which K_q-*core* $(Bx) \subseteq K$-*core* (x), K_q-*core* $(Bx) \subseteq st_A$-*core* (x) and K_q-*core* $(Bx) \subseteq K_q$-*core* (x) hold for all $x \in \ell_\infty$.

Keywords: *Matrix transformations, core of a sequence, statistical convergence*

1. INTRODUCTION

Let E be a subset of N={0,1,2,...}. Natural density δ of E is defined by

$$\delta(E) = \lim_n \frac{1}{n} |\{k \leq n : k \in E\}|,$$

where the vertical bars indicate the number of elements in the enclosed set. A sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for every ε, $\delta\{k: |x_k - \ell| \geq \varepsilon\} = 0$, [9]. By st and st_0, we denote the sets of statistically convergent and statistically null sequences.

For a given nonnegative regular matrix $A=(a_{nk})$, the number $\delta_A(F)$ is defined by

$$\delta_A(F) = \lim_n \sum_{k \in F} a_{nk}$$

and it is said to be the A-density of $F \subseteq N$, [10]. A sequence $x=(x_k)$ is said to be A-statistically convergent to a number s if for every $\varepsilon > 0$ the set $\delta\{k: |x_k - s| \geq \varepsilon\}$ has A-density zero, [4].

In this case, we write st_A-$lim\ x$ = s. By $st(A)$ and $st(A)_0$, we respectively denote the sets of all A-statistically convergent and A-statistically null sequences.

Let $x=(x_k)$ be a sequence in C, the set of all complex numbers, and R_k be the least convex closed region of complex plane containing x_k, x_{k+1}, x_{k+2},.... The Knopp Core (or K-*core*) of x is defined by the intersection of all R_k (k=1,2,...), [3, p.137]. In [15], it is shown that

$$K\text{-}core(x) = \bigcap_{z \in C} B_x(z)$$

for any bounded sequence $x=(x_k)$, where $B_x(z) = \{w \in C: |w-z| \leq limsup_k |x_k - z|\}$.

In [8], the notion of the statistical core of a complex valued sequence introduced by Fridy and Orhan [11] has been extended to the A-statistical core (or st_A-*core*) and it is shown for a A-statistically bounded seqeunce x that

$$st_A\text{-}core(x) = \bigcap_{z \in C} C_x(z),$$

where $C_x(z) = \{w \in C: |w-z| \leq st_A\text{-}limsup_k |x_k - z|\}$.

The inequalities related to the core of a sequence have been studied by many authors. For instance, see [1, 5, 6,

♠Corresponding author, e-mail: ccakan@inonu.edu.tr

7, 8, 11, 15] and the others. The matrix $R=(r_{nk})$ defined by

$$r_{nk}=\begin{cases} q_k/Q_n, & k\le n \\ 0, & k>n \end{cases}$$

is called Riesz matrix and denoted by (R, q_k) or shortly R, where (q_k) is a sequence of non-negative numbers which are not all zero and $Q_n = q_1+q_2+\ldots+q_n$, $n\in N$; $q_1 > 0$. It is well-known that R is regular if and only if $\lim_n Q_n = \infty$, [14].

Using the convergence domain of the Riesz matrix, the new sequence spaces r_c^q and r_0^q respectively including the spaces c and c_0 have been constructed by Malkowsky & Rakòević in [13] and Altay & Başar in [2] and their some properties have been investigated, where c and c_0 are the spaces of all convergent and null sequences, respectively.

Let B be an infinite matrix of complex entries b_{nk} and $x=(x_k)$ be a sequence of complex numbers. Then $Bx=\{(Bx)_n\}$ is called the B transform of x, if $(Bx)_n = \sum_k b_{nk}x_k$ converges for each n. For two sequence spaces X and Y we say that $B=(b_{nk}) \in (X, Y)$ if $Bx\in Y$ for each $x=(x_k) \in X$. If X and Y are equipped with the limits X-lim and Y-lim, respectively, $B=(b_{nk}) \in (X, Y)$ and $Y\text{-}\lim_n (Bx)_n = X\text{-}\lim_k x_k$ for all $x=(x_k) \in X$, then we say B regularly transforms X into Y and write $B=(b_{nk}) \in (X, Y)_{reg}$.

In the present paper, we firstly introduce a new type core, K_q-*core*, of a complex valued sequence and also determine the necessary and sufficient conditions on a matrix B for which $K_q\text{-core}(Bx) \subseteq K\text{-core}(x)$, $K_q\text{-core}(Bx) \subseteq st_A\text{-core}(x)$ and $K_q\text{-core}(Bx) \subseteq K_q\text{-core}(x)$ for all $x \in \ell_\infty$, where ℓ_∞ is the space of all bounded complex sequences. To do these, we need to characterize the classes $(c, r_c^q)_{reg}$, $(r_c^q, r_c^q)_{reg}$ and $(st(A)\cap \ell_\infty, r_c^q)_{reg}$.

2. LEMMAS

In this section, we prove some lemmas which will be useful to our main results. For brevity, in what follows we write $\tilde{b}_{nk}$ in place of

$$\frac{1}{Q_n}\sum_{k=0}^{n} q_k b_{nk} \; ; (n,k \in N).$$

Lemma 2.1. $B \in (\ell_\infty, r_c^q)$ if and only if

$$\|B\|_r = \sup_n \sum_k |\tilde{b}_{nk}| < \infty, \tag{2.1}$$

$$\lim_n \tilde{b}_{nk} = \alpha_k \quad \text{for each } k, \tag{2.2}$$

$$\lim_n \sum_k |\tilde{b}_{nk} - \alpha_k| = 0. \tag{2.3}$$

Proof. Let $x \in \ell_\infty$ and consider the equality

$$\frac{1}{Q_n}\sum_{j=0}^{n} q_k \sum_{k=0}^{m} b_{nk}x_k = \sum_{k=0}^{m}\frac{1}{Q_n}\sum_{j=0}^{n} q_k b_{jk}x_k \; ; (m, n)\in N)$$

which yields as $m \to \infty$ that

$$\frac{1}{Q_n}\sum_{j=0}^{n} q_k (Bx)_j = (Dx)_n; \; (n \in N), \tag{2.4}$$

where $D = (d_{nk})$ defined by

$$d_{nk} = \begin{cases} \frac{1}{Q_n}\sum_{j=0}^{n} q_k b_{jk}, & 0\le k\le n \\ 0, & k>n. \end{cases}$$

Therefore, one can easily see that $B \in (\ell_\infty, r_c^q)$ if and only if $D\in(\ell_\infty, c)$ (see [13]) and this completes the proof.

Lemma 2.2. $B \in (c, r_c^q)_{reg}$ if and only if the conditions (2.1) and (2.2) of the Lemma 2.1 hold with $\alpha_k = 0$ for all $k \in N$ and

$$\lim_n \sum_k \tilde{b}_{nk} = 1. \tag{2.5}$$

Since the proof is easy we omit it.

Lemma 2.3. $B \in (st(A) \cap \ell_\infty, r_c^q)_{reg}$ if and only if $B \in (c, r_c^q)_{reg}$ and

$$\lim_n \sum_{k\in E} |\tilde{b}_{nk}| = 0 \tag{2.6}$$

for every $E \subset N$ with $\delta_A(E) = 0$.

Proof (Necessity). Because of $c \subset st(A) \cap \ell_\infty$, $B \in (c, r_c^q)_{reg}$. Now, for any $x \in \ell_\infty$ and a set $E \subset N$ with $\delta_A(E) = 0$, let us define the sequence $z = (z_k)$ by

$$z_k = \begin{cases} x_k, & k\in E \\ 0, & k\notin E. \end{cases}$$

Then, since $z \in st(A)_0$, $Az \in r_0^q$, where r_0^q is the space of sequences consisting the Riesz transforms of them in c_0. Also, since

$$\sum_k \tilde{b}_{nk} z_k = \sum_{k\in E} \tilde{b}_{nk} x_k \, ,$$

the matrix $D = (d_{nk})$ defined by $d_{nk} = \tilde{b}_{nk}$ $(k\in E)$, $= 0$ $(k\notin E)$ is in the class (ℓ_∞, r_c^q). Hence, the necessity of (2.6) follows from Lemma 2.1.

(Sufficiency). Let $x \in st(A) \cap \ell_\infty$ with $st_A\text{-}\lim x = \ell$. Then, the set E defined by $E = \{k: |x_k - \ell| \ge \varepsilon\}$ has A-density zero and $|x_k - \ell| \le \varepsilon$ if $k\notin E$. Now, we can write

$$\sum_k \tilde{b}_{nk} x_k = \sum_k \tilde{b}_{nk}(x_k - l) + k\sum_k \tilde{b}_{nk} . \qquad (2.7)$$

Since

$$|\sum_k \tilde{b}_{nk}(x_k - l)| \leq \| x \| \sum_{k\in E} \tilde{b}_{nk} + \varepsilon \| B \|,$$

letting $n\to\infty$ in (2.7) with (2.6), we have

$$\lim_n \sum_k \tilde{b}_{nk} x_k = \ell.$$

This implies that $B \in (\mathrm{st}(A) \cap \ell_\infty, r_c^q)_{reg}$ and the proof is completed. When B is chosen as the Cesáro matrix in Lemma 2.3, we have the following corollary.

Corollary 2.4. $B \in (\mathrm{st} \cap \ell_\infty, r_c^q)_{reg}$ if and only if $B \in (c, r_c^q)_{reg}$ and

$$\lim_n \sum_{k\in E} | \tilde{b}_{nk} | = 0$$

for every $E \subset N$ with $\delta(E) = 0$.

Lemma 2.5. $B \in (r_c^q, r_c^q)_{reg}$ if and only if

$(b_{nk}) \in cs$ (2.8)

holds and $C \in (c, r_c^q)$, where $C = (c_{nk})$ is defined by

$$c_{nk} = \Delta\left(\frac{b_{nk}}{q_k}\right)Q_k$$

for all $n,k \in N$ and cs is the space of all convergent series.

Proof. (Sufficiency). Take $x\in r_c^q$. Then, the sequence $\{b_{nk}\}_{k\in N} \in [r_c^q]^\beta$ for all $n\in N$ and thisimplies the existence of the B-transform of x.

Let us now consider the following equality derived by using the relation,

$$y_k = \sum_{i=0}^{k} \frac{q_i}{Q_k} x_i$$

from the m^{th} partial sum of the series $\sum_k b_{nk} x_k$,

$$\sum_{k=0}^{m} b_{nk} x_k = \sum_{k=0}^{m-1} \Delta\left(\frac{b_{nk}}{q_k}\right) Q_k y_k + \frac{b_{nm}}{q_m} Q_m y_m \; (m,n \in N). \qquad (2.9)$$

Then, using (2.1), we obtain from (2.9) as $m\to\infty$ that

$$\sum_k b_{nk} x_k = \sum_k \Delta\left(\frac{b_{nk}}{q_k}\right) Q_k y_k , \qquad (2.10)$$

i.e. $Bx = Cy$. Since $x\in r_c^q$ if and only if $y\in c$, (2.2) implies that $B\in (r_c^q, r_c^q)$.

(Necessity). Conversely, let $B\in (r_c^q, r_c^q)$. Then, since $\{b_{nk}\}_{k\in N}\in [r_c^q]^\beta$ for all $n\in N$, the necessity of (2.1) is immediate. On the other hand, (2.2) follows from (2.4).

3. K_q-CORE

Let us write

$$t_n^q(x) = A^r(x) = \frac{1}{Q_n}\sum_{k=0}^{n} q_k x_k .$$

Then, we can define K_q-*core* of a complex sequence as follows.

Definition 3.1. Let H_n be the least closed convex hull containing t_n^q, t_{n+1}^q, t_{n+2}^q, ... Then, K_q-*core* of x is the intersection of all H_n, i.e.,

$$K_q\text{-}core(x) = \bigcap_{n=1}^{\infty} H_n .$$

Note that, actually, we define K_q-*core* of x by the *K-core* of the sequence (t_n^q). Hence, we can construct the following theorem which is an analogue of *K-core*, (see [16]).

Theorem 3.2. For any $z\in C$, let

$$G_x(z) = \{w\in C: |w-z| \leq \limsup_n |t_n^q - z|\}.$$

Then, for any $x\in \ell_\infty$,

$$K_q\text{-}core = \bigcap_{z\in C} G_x(z) .$$

Note that in the case $q_n=1$ for all n, the Riesz core is reduced to the Cesáro core.

Now, we may give some inclusion theorems.

Theorem 3.3. Let $B\in (c, r_c^q)_{reg}$. Then, K_q-*core* $(Bx) \subseteq$ *K-core (x)* for all $x\in \ell_\infty$ if and only if

$$\lim_n \sum_k | \tilde{b}_{nk} | = 1. \qquad (3.1)$$

Proof (Necessity). Let us define a sequence $x = x^{(k)} = \{x^{(k)}_n\}$ by

$$x^{(k)}_n = sgn\; \tilde{b}_{nk}$$

for all $n\in N$. Then, since *limsup* $x^{(k)} = 1$ for all $n\in N$, *K-core(x)* $\subseteq B_1(0)$. Therefore, by hypothesis,

$$\left\{ w\in C : |w| \leq \limsup_n \sum_k | \tilde{b}_{nk} | \right\} \subseteq B_1(0)$$

which gives the necessity of (3.1).

(Sufficiency). Let $w\in K_q$-*core(Bx)*. Then, for any given $z\in C$, we can write

$$|w-z| \leq \limsup_n | t_n^q (Bx) - z| \qquad (3.2)$$

$$= \limsup_n |z - \sum_k \tilde{b}_{nk} x_k |$$

$$\leq \limsup_n | \sum_k \tilde{b}_{nk}(z - x_k) | + \limsup_n |z||1 - \sum_k \tilde{b}_{nk} |$$

$= \limsup_n |\sum_k \tilde{b}_{nk}(z-x_k)|.$

Now, let $limsup_k |x_k\text{-}z| = l$. Then, for any ε > 0, $|x_k\text{-}z| \leq \ell + \varepsilon$ whenever $k \geq k_0$. Hence, one can write that

$$\sum_k \tilde{b}_{nk}(z-x_k) =$$

$$|\sum_{k<k_0} \tilde{b}_{nk}(z-x_k) + \sum_{k\geq k_0} \tilde{b}_{nk}(z-x_k)| \qquad (3.3)$$

$$\leq \sup_k |z\text{-}x_k| \sum_{k<k_0} |\tilde{b}_{nk}| + (\ell+\varepsilon) \sum_{k\geq k_0} |\tilde{b}_{nk}|$$

$$\leq \sup_k |z\text{-}x_k| \sum_{k<k_0} |\tilde{b}_{nk}| + (\ell+\varepsilon) \sum_{k} |\tilde{b}_{nk}|.$$

Therefore, applying $limsup_n$ under the light of the hypothesis and combining (3.2) with (3.3), we have

$$|w\text{-}z| \leq \limsup_n |\sum_k \tilde{b}_{nk}(z-x_k)| \leq \ell + \varepsilon$$

which means that $w \in$ *K-core(x)*. This completes the proof.

Theorem 3.4. Let $B \in (\mathrm{st}(A) \cap \ell_\infty, r_c^q)_{\mathrm{reg}}$. Then, K_q-*core* (*Bx*) $\subseteq$ st_A-*core (x)* for all $x \in \ell_\infty$ if and only if (3.1) holds.

Proof.(Necessity). Since st_A-*core (x)* $\subseteq$ *K-core (x)* for any sequence x [9], the necessity of the condition (3.1) follows from Theorem 3.3.

(Sufficiency). Take $w \in K_q$-*core (Bx)*. Then, we can write again (3.2). Now; if $\mathrm{st_A}$-limsup $|x_k\text{-}z| = s$, then for any ε > 0, the set E defined by $E = \{k: |x_k\text{-}z| > s + \varepsilon\}$ has A-density zero, (see [9]). Now, we can write

$$|\sum_k \tilde{b}_{nk}(z-x_k)| = |\sum_{k\in E} \tilde{b}_{nk}(z-x_k) +$$

$$\sum_{k\notin E} \tilde{b}_{nk}(z-x_k)|$$

$$\leq \sup_k |z\text{-}x_k| \sum_{k\in E} |\tilde{b}_{nk}| + (s+\varepsilon) \sum_{k\notin E} |\tilde{b}_{nk}|$$

$$\leq \sup_k |z\text{-}x_k| \sum_{k\in E} |\tilde{b}_{nk}| + (s+\varepsilon) \sum_{k} |\tilde{b}_{nk}|.$$

Thus, applying the operator $limsup_n$ and using the condition (3.1) with (2.6), we get that

$$\limsup_n |\sum_k \tilde{b}_{nk}(z-x_k)| \leq s + \varepsilon. \qquad (3.4)$$

Finally, combining (3.2) with (3.4), we have $|w\text{-}z| \leq st_A\text{-}limsup_k |x_k\text{-}z|$ which means that $w \in st_A$-*core(x)* and the proof is completed. As a consequence of Theorem 3.4, we have

Theorem 3.5. Let $B \in (\mathrm{st} \cap \ell_\infty, r_c^q)_{\mathrm{reg}}$. Then, K_q-*core* (*Bx*) $\subseteq$ *st -core (x)* for all $x \in \ell_\infty$ if and only if (3.1) holds.

Theorem 3.5. Let $B \in (r_c^q, r_c^q)_{\mathrm{reg}}$. Then, K_q-*core* (*Bx*) $\subseteq$ K_q *-core (x)* for all $x \in \ell_\infty$ if and only if (3.1) holds.

Proof. (Necessity). Since K_q *-core (x)* $\subseteq$ *K-core (x)* for all $x \in \ell_\infty$, the necessity of the condition (3.1) follows from Theorem 3.3.

(Sufficiency). Let $w \in$ *Kq-core (Bx)*. Then, we can write (3.2). Now; if $\mathrm{limsup_k} |t_k^q(x)\text{-}z| = v$, then for any ε > 0, $|t_k^q(x)\text{-}z| \leq v + \varepsilon$ whenever $k \geq k_0$. Hence, we can write

$$\sum_k \tilde{b}_{nk}(x_k - z)| = |\sum_{k<k_0} c_{nk}(t_k^q(x)-z) +$$

$$\sum_{k\geq k_0} c_{nk}(t_k^q(x)-z)| \qquad (3.5)$$

$$\leq \sup_k |t_k^q(x)-z| \sum_{k<k_0} |c_{nk}| + (v+\varepsilon) \sum_{k\geq k_0} |c_{nk}|$$

$$\leq \sup_k |t_k^q(x)-z| \sum_{k<k_0} |c_{nk}| + (v+\varepsilon) \sum_{k} |c_{nk}|,$$

where c_{nk} is defined as in Lemma 2.5.

Therefore, considering the operator $limsup_n$ in (3.5) and using the hypothesis, we get that $|w\text{-}z| \leq v + \varepsilon$. This means that $w \in K_q$-*core (x)* and the proof is completed.

ACKNOWLEDGEMENT

We are grateful to the referees for their valuable suggestions which are improved the paper considerably.

REFERENCES

[1] Abdullah M. Alotaibi, "Cesáro statistical core of complex number sequences", *Inter. J. Math. Math. Sci.,* Article ID 29869 (2007).

[2] B. Altay, F. Başar, "Some paranormed Riesz sequence spaces of non-absolute type", *Southeast Asian Bull. Math*. 30(5): 591-608 (2006).

[3] F. Başar, "A note on the triangle limitation methods", *Fırat Univ. Fen & Müh. Bil. Dergisi*, 5(1): 113-117 (1993).

[4] R. G. Cooke, "Infinite matrices and sequence spaces", *Macmillan*, New York (1950).

[5] J. Connor, "On strong matrix summability with respect to a modulus and statistical convergence", *Canad. Math. Bull*. 32: 194-198 (1989).

[6] C. Çakan, H. Çoşkun, "Some new inequalities related to the invariant means and uniformly bounded function sequences", *Applied Math. Lett.* 20(6): 605-609 (2007).

[7] H. Çoşkun, C. Çakan, "A class of statistical and σ-conservative matrices", *Czechoslovak Math. J.* 55(3): 791-801 (2005).

[8] H. Çoşkun, C. Çakan, Mursaleen, "On the statistical and σ –cores", *Studia Math.* 154(1):(2003).

[9] K. Demirci, "A-statistical core of a sequence", *Demonstratio Math.*, 33: 43-51 (2000).

[10] H. Fast, "Sur la convergence statisque", *Colloq. Math.*, 2: 241-244 (1951).

[11] A. R. Freedman, J. J. Sember, "Densities and summability", *Pasific J. Math.*, 95:293-305 (1981).

[12] J. A. Fridy, C. Orhan, "Statistical core theorems", *J. Math. Anal. Appl.*, 208: 520-527 (1997).

[13] I. J. Maddox, "Elements of Functional Analysis", *Cambridge University Press*, Cambridge (1970).

[14] E. Malkowsky, V. Rakoćević, "Measure of noncompactness of linear operators between spaces of sequences that are ($\overline{N}$, q) summable or bounded", *Czechoslovac Math. J.*, 51(126): 505-522 (2001).

[15] G. M. Petersen, "Regular matrix transformations", *McGraw-Hill*, (1966).

[16] A. A. Shcherbakov, "Kernels of sequences of complex numbers and their regular transformations", *Math. Notes*, 22: 948-953 (1977).