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Approximate Solution of Singular Integral Equations with Negative Index

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ABSTRACT

This paper is devoted to investigating a class of linear singular integral equations with a negative index on a closed, simple and smooth curve. In this paper, we propose the quadrature method for the approximate solution of the linear singular integral equations with negative index. Sufficient conditions are given for the convergence of this method in Hölder space.

Key Words: *Linear singular integral equation, Index of singular integral equation, Quadrature method.*

1. INTRODUCTION

It is well-known that singular integral equation theory has broad applications to theoretical and practical investigations in mathematics, mathematical physics, hydrodynamic and elasticity theory [1- 3]. This fact has not only given rise to multiple studies of singular integral equations (SIE), has also developed many effective approximate solution methods [4-24]. The doctoral thesis of V. N. Seychuk is about the approximate solutions of SIE with zero index [22]. In the studies of N. Mustafaev approximate solutions of SIE's with non-negative index were investigated by the collocation method [10], [11]. In the article of N. Mustafaev the quadrature method was applied to the solution of linear SIE (LSIE) with non-negative index [11].

The purpose of this study is to examine of the quadrature method to identify an approximate solution of the SIE's with negative index.

In this present study, we investigate the following type of LSIE:

$$K\varphi(t) = K^0\varphi(t) + \lambda \cdot k\varphi(t) = f(t),\ t \in \gamma\ . \quad (1)$$

Here,

$$K^0\varphi(t) = a(t)\varphi(t) + b(t)S\varphi(t),$$
$$S\varphi(t) = \frac{1}{\pi i}\int_\gamma \frac{\varphi(\tau)}{\tau - t}d\tau,$$
$$k\varphi(t) = \int_\gamma k(t,\tau)\varphi(\tau)d\tau,\ \ t \in \gamma,$$

where γ is a closed, simple and smooth curve in the complex plane, the functions), $a(t)$, $b(t)$, $f(t)$ and) , $k(t,\tau)$ are known functions in Hölder space, and $a^2(t) - b^2(t) \neq 0$

in γ, and λ is a complex parameter and $\varphi(t)$ is the unknown function.

We will denote the complex numbers by $\mathbb{C}$.

The present studies about the approximate solution of SIE's of type (1) are improved in two ways. In our paper the quadrature method is applied for the approximate solution of SIE's with negative index

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defined on an Alper curve. Consequently, it is shown that the idea which was given from B.I. Musaev in his studies [25-27] can be applicable to more broad class of SIE's. In the second section of the paper, we introduce some concepts needed to prove their main results.Then, in the third section, we show the convergence of the quadrature method applied to SIE (1).

2. BASIC ASSUMPTIONS AND AUXILIARY RESULTS

In this section of the paper, we will introduce some necessary information for proving the main results.

We consider a closed, simple and smooth curve γ with equation $t = t(s),\ 0 \le s \le \ell$ in the complex plane, where, s is the arc length calculated from a fixed point and $\ell = |\gamma|$ is the length of the curve γ. The interior and exterior of curve γ are denoted by γ^+ and γ^-, respectively. Let the origin $0 \in \gamma^+$.

Definition 2.1 [28]. Let the function θ (s) be the slope angle of the curve γ at the point $t(s)$ and let $\omega(x,\theta)$ be the continuity modulo of this function. If the condition

$$\int_0^{d_1} x^{-1}\omega(\theta, x)|\ln x|dx < \infty$$

is satisfied then the curve γ is called as Alper curve. We will denote the class of Alper curves with (A) . With d_1, d_2 we denote positive real numbers.

Let $C(\gamma)$ be the set of continuous functions, which are defined on the curve γ. For $0 < \alpha < 1$, let us take

$$H_\alpha(\gamma) = \left\{ \begin{array}{l} f \in C(\gamma) : H(f;\alpha) = \\ = \sup\left\{ \begin{array}{l} \dfrac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^\alpha} : \\ t_1, t_2 \in \gamma, t_1 \ne t_2 \end{array} \right\} < \infty \end{array} \right\}$$

as given above [1-3].

Let $H_\alpha^{(r)}(\gamma)$ be the set of r^{th} derivatives of functions from the $H_\alpha(\gamma)$. Here, r is a natural number.

Definition 2.2 [1-3]. The integer number $\nu = \frac{1}{2\pi}\left[\arg(D(t)/C(t))\right]_\gamma$

is called the index of the SIE (1) (or of the operator K).

Here, $D(t)=a(t)-b(t)$ and $C(t)=a(t)+b(t)$.

In the LSIE theory the following equation

$$K^0\varphi(t) = f(t),\ t \in \gamma \qquad (2)$$

is called as the characteristic equation of the SIE (1) [1-3]. The sufficient conditions for the solution of the SIE (1) can be derived from the existence of the solution of the characteristic equation (2).

Theorem 2.1 [1-3]. *Let* $a, b, f \in H_\alpha(\gamma)$ $0 < \alpha < 1$, *and* ν *is index of the SIE* (1).

1. *If* $0 > \nu$ *and the condition* ,

$$\int_\gamma \tau^{p-1}\varphi(\tau)d\tau = 0,\ \ p = 1,2,\cdots,\nu \quad (3)$$

is satisfied, then for every $f(t)$ *the characteristic equation* (2) *has the unique solution* $\varphi(t) = Rf(t)$ *in the space* $H_\alpha(\gamma), 0 < \alpha < 1$.

Here,

$$Rf(t) = \frac{Z(t)}{D(t)C(t)}\overline{K}^0(f(t)/Z(t)),$$

$$\overline{K}^0(f(t)/Z(t)) = a(t)\frac{f(t)}{Z(t)} -$$
$$-b(t)S(f(t)/Z(t)),$$
$$Z(t) = C(t)\psi^+(t) = t^{-\nu}D(t)\psi^-(t),$$
$$\psi\pm(t) = \exp\left[\Gamma^\pm(t)\right],$$
$$\Gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{\ln\left[\tau^{-\nu}G(\tau)\right]}{\tau - z}d\tau,\ z \notin \gamma,$$
$$G(t) = D(t)/C(t), \Gamma^\pm(t) =$$
$$= \Gamma(t) \pm \frac{1}{2}\ln\left[t^{-\nu}\cdot G(t)\right], t \in \gamma. \qquad (4)$$

2. *If* $0 = \nu$, *then characteristic equation* (2) *has the unique solution* $\varphi(t) = Rf(t)$ *in the space*

$H_\alpha(\gamma), 0 < \alpha < 1.$

3. *If* $0 < \nu$, *then existence of the unique solution* $\varphi(t) = Rf(t)$ *in the space* $H_\alpha(\gamma), 0 < \alpha < 1$ *depends upon meeting the following condition:*

$$\int_\gamma \frac{f(\tau)}{Z(\tau)}\tau^{p-1}d\tau = 0,\ p = 1,2,\cdots,-\nu. \quad (5)$$

Using the Theorem 2.1, can easily prove the following theorem.

Theorem 2.2. *Let* γ *be a closed, simple and smooth curve,*), $a, b, f \in H_\alpha(\gamma)$, $0 < \alpha < 1$ *and suppose that the index of the SIE* (1) *is* $\nu < 0$. *In this case, the following equation has the unique solution*

$$x(t) = (\varphi, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{-\nu})$$

$$K_\varepsilon^0 x(t) \equiv K^0\varphi((t) + \sum_{k=1}^{-\nu}\varepsilon_k h_k(t) = f(t),$$

$$t \in \gamma, \qquad (6)$$

in the space

Here, the functions $h_k(t) = b(t)\cdot t^{k-1}$, $k = 1,2,\cdots,-\nu$ *are linear independent solutions of the equation* $Rh(t) = 0$.

Now suppose that the index of the SIE (1) is $\nu < 0$ In this case, the SIE (1) together with the following conditions

$$L_p(f,k,\varphi) \equiv$$
$$\equiv \int_\gamma \frac{\tau^{p-1}}{Z(\tau)}\left[f(\tau) - \lambda\int_\gamma k(\tau,\xi)\varphi(\xi) \right] d\tau = 0,$$
$$p = 1,2,\cdots,-\nu \qquad (7)$$

is equivalent to the following Fredholm integral equation

$$F\varphi(t) \equiv \varphi(t) + \lambda\int_\gamma F(t,\tau)\varphi(\tau)d\tau = Rf(t), \qquad (8)$$

in the subspace

$$\overline{H}_\alpha(\gamma) = \left\{ \begin{array}{l} f \in H_\alpha(\gamma) : L_p(f,k,\varphi) = 0, \\ p = 1,2,\cdots,-\nu \end{array} \right\},$$
$0 < \alpha < 1$.

Here

$$F(t,\tau) = \frac{a(t)k(t,\tau)}{D(t)C(t)} - \frac{b(t)Z(t)}{D(t)C(t)} \cdot \frac{1}{\pi i}\int_\gamma \frac{k(\xi,\tau)}{Z(\xi)}\frac{d\xi}{\xi - t}.$$

Really, if (1) has a solution φ then, due to (5) this solution automatically satisfies (7). Then, by (4) it follows that φ is a solution of (8). Conversely, if $\varphi \in \overline{H}_\alpha(\gamma)$ then φ is a solution of (1). Hence, problem (1) is equivalent to problem (7)-(8).

Moreover, the

$$K_\varepsilon x(t) \equiv K\varphi(t) + \sum_{k=1}^{-\nu} \varepsilon_k h_k(t) = f(t), \qquad (9)$$

is equivalent to the Fredholm integral equation (8) together with the following conditions

$$\sum_{k=1}^{-\nu} \varepsilon_k \int_\gamma \frac{h_k(\tau)}{Z(\tau)}\tau^{p-1}d\tau = L_p(f,k,\varphi),$$
$$p = 1,2,\cdots,-\nu \qquad (10)$$

in the subspace

$$\overline{X} = \left\{ \begin{array}{l} x \in X : L_p(f,k,\varphi) = \\ = \sum_{k=1}^{-\nu} \varepsilon_k \cdot \int_\gamma \frac{h_k(\tau)}{Z(\tau)}\tau^{p-1}d\tau, p = 1,2,\cdots - \nu \end{array} \right\}.$$

Indeed, if $x(t) = (\varphi, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{-\nu})$ is a solution of (9), then (10) is automatically satisfied.

Conversely, if $x(t) = (\varphi, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{-\nu})$ is a solution of (8), then φ is the solution of (9).

Note 2.1. We call equation (9) as the "regularization" of the SIE (1). I also want to indicate that previously in the studies of V.V. Ivanov [8] the idea of presenting the unknowns $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{-\nu}$ (regularized parameter) was given.Afterwards this idea was used by B.I. Musaev [25-27].

We now provide certain pertinent information regarding the Fredholm integral equation theory. Let us take the following homogeneous Fredholm integral equation in the space

$H_\beta(\gamma), 0 < \beta < \alpha < 1$

$$F\varphi(t) \equiv \varphi(t) + \lambda\int_\gamma F(t,\tau)\varphi(\tau)d\tau = 0. \quad (11)$$

Definition 2.3 [1, 2]. If, for a value of the λ parameter, there exist a non-zero solution of the homogeneous Fredholm integral equation (11), then we will call this value of the parameter an eigenvalue of the kernel $F(t,\tau)$.

Theorem 2.3 [1, 2]. *If the parameter* λ *is not an eigenvalue of the kernel*) , $F(t,\tau)$*, then the non-homogeneous Fredholm integral equation* (8) *has only one solution for every*

$f,k \in H_\alpha(\gamma), 0 < \alpha < 1.$

In this study, we assume that the homogeneous equation (11) has only the zero solution. In this case, according to Theorem 2.3 and Theorem 2.2, equation (9) has the unique solution

$x(t) = (\varphi, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{-\nu})$

for every

$f \in H_\alpha(\gamma), \ 0 < \alpha < 1.$

Here, the function

$$\varphi(t) = Rf(t) - \int_\gamma \overline{F}(t,\tau)Rf(\tau)d\tau$$

is the solution of the non-homogeneous Frdeholm integral equation (8). $\overline{F}(t,\tau)$ is the resolvent kernel of the equation (8) and can be clearly expressed with the help of the function) , $F(t,\tau)$ [1] . The components , $\varepsilon_k, \ k = 1,2,\cdots,-\nu$ are found from the equation (10).

We will denote set of natural numbers using $\mathbb{N}$. For every function) $f \in C(\gamma)$and $n \in \mathbb{N}$ let us define the Lagrange interpolation polynomial using the following formula [29],

$$U_n f(t) \equiv U_n(f,t) = \sum_{j=1}^{2n} f(t_j)l_j(t),$$
$$t \in \gamma. \qquad (12)$$

Here,

$$l_j(t) = \prod_{k=0, k\neq j}^{2n} \frac{t-t_k}{t_j-t_k}\left(\frac{t_j}{t}\right)^n, \; t\in\gamma,$$

$$t_j = \phi(w_j), \; w_j = \exp\left[\frac{2\pi i}{2n+1}(j-k)\right],$$

$$i^2=-1, \; j=0,1,\cdots,2n \qquad (13)$$

and $z=\phi(w)$ is a conform transformation which satisfies the conditions $\phi(\infty)=\infty, \; \phi'(\infty)>0$ and transforms the region outside the unit circle centred at the origin to the region γ^-. As it can be seen from the formula (13),

we can write

$$l_j(t) = \sum_{k=-n}^{n} \Lambda_k^{(j)} t^k, \; j=0,1,\cdots,2n,$$

$$t\in\gamma. \qquad (14)$$

Here the coefficients

$$\Lambda_k^{(j)} = t_j^k \Big/ \prod_{m=0,m\neq j}^{2k}(t_j-t_m),$$

$$k=-n,\cdots,0,1,\cdots,n; \; j=0,1,\cdots,2n$$

can be clearly expressed by the points t_j.The points t_j are defined by formula (13). Therefore, we can write formula (12) as it given below

$$U_n f(t) \equiv U_n(f,t) = \sum_{k=-n}^{n} U_k^{(n)} t^k. \qquad (15)$$

Here,

$$U_k^{(n)} = \sum_{j=0}^{2n} \Lambda_k^{(j)} f(t_j), k=-n,\cdots,0,1,\cdots,n.$$

Note 2.2. The formula (15) is called as the "quadrature" formula for the function $f\in C(\gamma)$

Lemma 2.1. (see [11, Corollary 1.2.1]) *Let* $\gamma\in(A)$, *and* $f\in H_\alpha^{(r)}(\gamma), \; 0<\beta<\alpha<1$ *(0 ≤ r-is an integer). In this case, for every* $n\in\mathbb{N}$ *we have*

$$\|f-U_nf\| \le \le (d_3+d_4\cdot\ln n)\cdot H(f,\alpha)\cdot n^{\beta-\alpha-r}. \qquad (16)$$

3. QUADRATURE METHOD FOR THE SIE (1)

Let us take

$$X=\{x=(\varphi,\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{-\nu}) : \varphi\in H_\beta(\gamma), \varepsilon_k\in \mathbb{C}, \; k=1,2,\cdots,-\nu\},$$

and define the norm in X with

$$\|x\|_X = \|\varphi\|_\beta + \sum_{k=1}^{-\nu}|\varepsilon_k|, 0<\beta<\alpha<1.$$

Here, $\nu < 0$ is the index of the SIE (1). We can write equation (9) in the space X, as following linear operator equation 1

$$C(t)\cdot P\varphi(t) + D(t)\cdot Q\varphi(t) + +\lambda\cdot k\varphi(t) + \sum_{k=1}^{-\nu}\varepsilon_k h_k(t) = f(t), \qquad (17)$$

where $P=\frac{1}{2}(I+S), \quad Q=\frac{1}{2}(I-S)$ are projection operators, I is the identity operator on $H_\alpha(\gamma)$ and S is the linear singular integral operator with Cauchy kernel that is defined by (1).

We will seek the approximate solution of equation (9) in the form

$$x_{n-\nu} = (\varphi_{n-\nu}, \varepsilon_{1,n}, \varepsilon_{2,n}, \cdots, \varepsilon_{-\nu,n})$$

Here,

$$\varphi_{n-\nu} = \varphi_{n-\nu}^+ - \varphi_n^-, \varphi_{n-\nu}^+(t) = \sum_{k=0}^{n-\nu}\alpha_k t^k,$$

$$\varphi_n^-(t) = -\sum_{k=-n}^{-1}\alpha_k t^k.$$

We will find the $\alpha_{-n},\cdots,\alpha_{n-\nu},\varepsilon_{1,n},\cdots,\varepsilon_{-\nu,n}$ unknown values from the following linear equation system (LES):

$$C_j\cdot\sum_{k=0}^{n-\nu}\alpha_k t_j^k + D_j\cdot\sum_{k=-n}^{-1}\alpha_k t_j^k + +2\pi i\lambda\cdot\sum_{k=0}^{2(n-\nu)} k(t_j,t_k)\Lambda_{-1}^{(k)}\varphi_{n-\nu}(t_k) + \sum_{k=1}^{-\nu}\varepsilon_{k,n}h_k(t_j) = f_j, j=0,1,\cdots,2(n-\nu). \qquad (18)$$

Here, C_j, D_j and f_j are the respective values of the functions) $C(t)$, $D(t)$ and $f(t)$ at the points $t=t_j$. The points t_j. $j=0,1,\cdots,2(n-\nu)$ are points that are defined by (13).

In the space X, we can write equation (9) in the form of a linear operator equation:

$$\overline{K}_\varepsilon x \equiv \psi^- P\varphi + t^\nu\psi^+ Q\varphi + \lambda d\cdot k\varphi + +d\cdot\sum_{k=1}^{-\nu}\varepsilon_k h_k = g, \qquad (19)$$

where the functions $\psi^\pm$ are the functions that are defined in formulas (4) and

$$x=(\varphi,\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{-\nu})\in X, \qquad d=\psi^-/C,$$

$$g=d\cdot f.$$

Now we take the following theorem about the existence of the approximate solution of the equation (9) and the convergence of the quadrature method.

Theorem 3.1. *Let* $\gamma \in (A)$ *The functions* a(t), b(t), *f*(*t*) *and k*(t τ) *(for every two variable) are of the class* $H_\alpha^{(r)}(\gamma)$, $0<\alpha<1$ *(* $0 \le r$ *- is an integer). For every* $t\in\gamma$, $a^2(t)-b^2(t)\neq 0$ *an index* $\nu < 0$.

If the homogeneous Fredholm integral equation (11) *has only the zero solution,*

then, for every $n > n_0$,

$$\begin{pmatrix} d_5 + d_6 \ln(n-\nu) + \\ +d_7 \ln^2(n-\nu) \end{pmatrix} (n-\nu)^{\beta-\alpha-r} \cdot \left\| K_\varepsilon^{-1} \right\| < 1 \ \},$$

the LES (18) *has the unique solution* $\left(\alpha_{-n}^*,\cdots,\alpha_{n-\nu}^*,\varepsilon_{1,n}^*,\cdots,\varepsilon_{-\nu,n}^*\right)$.

The approximate solution $x_{n-\nu}^* = \left(\varphi_{n-\nu}^*,\varepsilon_{1,n}^*,\cdots,\varepsilon_{-\nu,n}^*\right)$ *of equation* (9) *converges to the unique solution* $x^* = \left(\varphi^*,\varepsilon_1^*,\cdots,\varepsilon_{-\nu}^*\right)$, *and the following estimate is correct:*

$$\left\| x_{n-\nu}^* - x^* \right\|_X \le \begin{pmatrix} d_8 + d_9 \cdot \ln(n-\nu) + \\ +d_{10} \cdot \ln^2(n-\nu) \end{pmatrix} \cdot (n-\nu)^{\beta-\alpha-r}, 0<\beta<\alpha<1 \tag{20}$$

where, $\varphi_{n-\nu}^*(t) = \sum_{k=-n}^{n-\nu} \alpha_k^* t^k$.

Proof. Let the index of the SIE (9) be $\nu < 0$ Let us take points $t_j \in \gamma$, $j = 0,1,\cdots,2(n-\nu)$ as the points that are defined by formula (13), and the operator $U_{n-\nu}$ is the operator that is defined by (15).

Let us denote

$$X_{n-\nu} = \left\{ x_{n-\nu} : x_{n-\nu} = \left(\varphi_{n-\nu},\varepsilon_{1,n},\varepsilon_{2,n},\cdots,\varepsilon_{-\nu,n}\right)\right\}$$

finite dimensional subspace of X .

Let us write the LES (18) in the space $X_{n-\nu}$ as a linear operator equation:

$$\begin{aligned} \bar{K}_{n,\varepsilon} x_{n-\varepsilon} &\equiv U_{n-\nu}\left(\psi^- P\varphi_{n-\nu} + t^\nu \psi^+ Q\varphi_{n-\nu}\right) + \\ &+\lambda \cdot U_{n-\nu}\left[d(t)\cdot \int_\gamma U_{n-\nu}(k(t,\tau)\varphi_{n-\nu}(\tau))d\tau \right] + \\ &+U_{n-\nu}\left(d(t)\cdot \sum_{k=1}^{-\nu} \varepsilon_{k,n} h_k(t)\right) = U_{n-\nu} g \equiv g_{n-\nu} . \end{aligned} \tag{21}$$

From (19) and (21), we can write:

$$\begin{aligned} &\bar{K}_{n,\varepsilon} x_{n-\nu} - \bar{K}_\varepsilon x_{n-\nu} = \\ &= \left(I - U_{n-\nu}\right) \begin{bmatrix} \left(\psi_{n-\nu}^- - \psi^-\right) P\varphi_{n-\nu} + \\ +t^\nu \cdot \left(\psi_{n-\nu}^+ - \psi^+\right) Q\varphi_{n-\nu} \end{bmatrix} - \\ &-\lambda \cdot \begin{Bmatrix} U_{n-\nu}\left[d(t)\cdot \int_\gamma U_{n-\nu}(k(t,\tau)\varphi_{n-\nu}(\tau))d\tau \right] \\ -d(t)\cdot \int_\gamma k(t,\tau)\varphi_{n-\nu}(\tau)d\tau \end{Bmatrix} \\ &+\sum_{k=1}^{-\nu} \varepsilon_{k,n}\left[U_{n-\nu}\left(d(t)\cdot h_k(t)\right) - d(t)\cdot h_k(t)\right]. \end{aligned} \tag{22}$$

Here,

$$\psi_{n-\nu} = \psi_{n-\nu}^+ - \psi_{n-\nu}^- \qquad \left(\psi_{n-\nu}^+(t) = \sum_{k=0}^{n-\nu} \beta_k t^k, \ \psi_{n-\nu}^-(t) = \sum_{-n+\nu}^{-1} \beta_k t^k \ \right)$$

is the best approximation to the function $\psi = \psi^+ - \psi^-$ with rational polynomials whose degree do not exceed $n-\nu$.

From the bounded operators *P* and *Q* in Hölder space [1-3], and the following estimates (see [11, Corollary 1.1.5])

$$\left\|\psi_{n-\nu}^+ - \psi^+\right\|_\beta \le d_{11}\cdot(n-\nu)^{\beta-\alpha-r},$$

$$\left\|\psi_{n-\nu}^- - \psi^-\right\|_\beta \le d_{12}\cdot(n-\nu)^{\beta-\alpha-r}$$

and (see [29, Lemma 2.1]).

$$\left\|U_{n-\nu}\right\|_\beta \le d_{14} + d_{15}\cdot\ln(n-\nu) \tag{23}$$

it can be seen that the following evaluation is correct:

$$\begin{aligned} &\left\| \left(I - U_{n-\nu}\right) \begin{bmatrix} \left(\psi_{n-\nu}^- - \psi^-\right) P\varphi_{n-\nu} + \\ +t^\nu \left(\psi_{n-\nu}^+ - \psi^+\right) Q\varphi_{n-\nu} \end{bmatrix} \right\|_\beta \le \\ &\le \left(d_{16} + d_{17}\cdot\ln(n-\nu)\right)(n-\nu)^{\beta-\alpha-r} \cdot \left\|\varphi_{n-\nu}\right\|_\beta . \end{aligned} \tag{24}$$

From the following

$$U_{n-\nu}\left[d(t)\cdot\int_\gamma U_{n-\nu}\left(k(t,\tau)\varphi_{n-\nu}(\tau)\right)d\tau\right]-$$
$$-d(t)\cdot\int_\gamma k(t,\tau)\varphi_{n-\nu}(\tau)d\tau=$$
$$U_{n-\nu}\left\{d(t)\cdot\int_\gamma\begin{bmatrix}U_{n-\nu}\left(k(t,\tau)\varphi_{n-\nu}(\tau)\right)-\\-k(t,\tau)\varphi_{n-\nu}(\tau)\end{bmatrix}d\tau\right\}+$$
$$+U_{n-\nu}\left[d(t)\cdot\int_\gamma k(t,\tau)\varphi_{n-\nu}(\tau)d\tau\right]-$$
$$-d(t)\cdot\int_\gamma k(t,\tau)\varphi_{n-\nu}(\tau)d\tau$$

equality, and from Lemma 2.1 and inequality (23) we have

$$\left\|\begin{matrix}U_{n-\nu}\left[d(t)\cdot\int_\gamma U_{n-\nu}\left(k(t,\tau)\varphi_{n-\nu}(\tau)d\tau\right)\right]-\\-d(t)\cdot\int_\gamma k(t,\tau)\varphi_{n-\nu}(\tau)d\tau\end{matrix}\right\|_\beta$$
$$\le\left(d_{18}+d_{19}\cdot\ln(n-\nu)+d_{20}\cdot\ln^2(n-\nu)\right)\cdot$$
$$(n-\nu)^{\beta-\alpha-r}\cdot\|\varphi_{n-\nu}\|_\beta. \tag{25}$$

Furthermore, the following inequality is obvious

$$\left\|\sum_{k=1}^{-\nu}\varepsilon_{k,n}\left[U_{n-\nu}(d\cdot h_k)-d\cdot h_k\right]\right\|_\beta\le$$
$$\le\sum_{k=11}^{-\nu}\left|\varepsilon_{k,n}\right|\left\|U_{n-\nu}(d\cdot h_k)-d\cdot h_k\right\|_\beta. \tag{26}$$

Thus, according to inequalities (24), (25), (26), Lemma 2.1 and the equality (22) we obtain the following evaluation:

$$\left\|\bar{K}_{n,\varepsilon}x_{n-\nu}-\bar{K}_\varepsilon x_{n-\nu}\right\|_\beta\le$$
$$\le\left(d_{21}+d_{22}\cdot\ln(n-\nu)+d_{23}\cdot\ln^2(n-\nu)\right)\cdot$$
$$(n-\nu)^{\beta-\alpha-r}\cdot\|x_{n-\nu}\|_X. \tag{27}$$

Conversely, according to Lemma 2.1, the following is true:

$$\|g_{n-\nu}-g\|_\beta\le$$
$$\le\left(d_{24}+d_{25}\cdot\ln(n-\nu)\right)(n-\nu)^{\beta-\alpha-r}. \tag{28}$$

Thus, from inequalities (27) and (28) (according to Theorem 7 [5], there exist a unique solution $x^*_{n-\nu}\in X_{n-\nu}$ for operator equation (21) that satisfies the following condition for every $n\in\mathbb{N}$:

$$n>n_0=\min\{n\in N:$$
$$\delta_{n,\nu}\equiv\left(d_{26}+d_{27}\cdot\ln(n-\nu)+d_{28}\cdot\ln^2(n-\nu)\right)\cdot$$
$$(n-\nu)^{\beta-\alpha-r}\cdot\left\|K_\varepsilon^{-1}\right\|<1\}.$$

In this case, for the $x^*\in X$ to be the solution of operator equation (19), then following estimate is correct:

$$\left\|x^*_{n-\nu}-x^*\right\|_X\le\begin{pmatrix}d_{29}+d_{30}\cdot\ln(n-\nu)+\\+d_{31}\cdot\ln^2(n-\nu)\end{pmatrix} \tag{29}$$
$$(n-\nu)^{\beta-\alpha-r}.$$

Thus, the LES (18) has a unique solution. Therefore, equation (9) can be solved approximately. Furthermore, the solution of equation (21) is the approximate solution of equation (9). Besides, the efinite solution of equation (19) is the definite solution of equation (9).

The theorem has been proved now.

Now let us propose the following theorem in order to discuss the approximate solution of the SIE (1). This theorem can easily prove.

Theorem 3.2. *Let us suppose that the parameter* λ *is not an eigenvalue of the homogeneous Fredholm integral equation* (11) *and let* $\nu<0$. *Let the definite solution of equation* (9) *be* $x^*=\left(\varphi^*,\varepsilon_1^*,\varepsilon_2^*,\cdots,\varepsilon_{-\nu}^*\right)$ *and the approximate solution be* $x^*_{n-\nu}=\left(\varphi^*_{n-\nu},\varepsilon^*_{1,n},\varepsilon^*_{2,n},\cdots,\varepsilon^*_{-\nu,n}\right)$. *The necessary and sufficient condition for the function* φ^* *to be the unique solution of the SIE* (1) *is* $\lim_{n\to\infty}\varepsilon^*_{k,n}=0$ *for every* 1,2, , *k=1,2,...,* - ν.

4. CONCLUSIONS

Essentially, in this study we apply the quadrature method to the "regularization" equation (9), not to the equation (1). We proposed the quadrature method in the context of the SIE (9) and we derived sufficient conditions for the convergence of this method. As evidenced by Theorem 3.2, if the vector $x^*=\left(\varphi^*,\varepsilon_1^*,\varepsilon_2^*,\cdots,\varepsilon_{-\nu}^*\right)$ is the definite solution of equation (9) and the vector $x^*_{n-\nu}=\left(\varphi^*_{n-\nu},\varepsilon^*_{1,n},\varepsilon^*_{2,n},\cdots,\varepsilon^*_{-\nu,n}\right)$ is the approximate solution, and if condition (7) is satisfied for the function φ^* then $\varepsilon_1^*=\varepsilon_2^*=\cdots=\varepsilon_{-\nu}^*=0$. Therefore, the function φ^* is the unique solution of SIE (1). In this case, for sufficiently large values of the natural number *n,* we can take the rational polynomial $\varphi^*_{n-\nu}(t)=\sum_{k=-n}^{n-\nu}\alpha_k^*t^k$ as the approximate solution of the SIE (1).

Furthermore, according to the Theorem 3.1, the following estimate is true:

$$\left\|\varphi^*_{n-\nu}-\varphi^*\right\|_\beta \le (n-\nu)^{\beta-\alpha-r}\times$$
$$\times\left(d_{32}+d_{33}\ln(n-\nu)+d_{34}\ln^2(n-\nu)\right).$$

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