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Isogonal Conjugates in Poincaré Upper Half Plane

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ABSTRACT

In this study, we give isogonal conjugates from major contributions of the modern synthetic geometry of the hyperbolic triangle in Poincaré upper half plane model of hyperbolic geometry.

Key Words: *Hyperbolic Ceva theorem and Hyperbolic sines theorem.*

1. INTRODUCTION

Hyperbolic geometry, which is the axiomatic geometry obtained by replacing the uniqueness stipulation of Playfair's postulate with the seemingly more relaxed alternative that through a given point outside a given line there exist more than one line parallel to the given line. There are several different ways of constructing hyperbolic geometry. These different constructions are called “models”. There are major five basic models in hyperbolic space and isometric passes between this models. The five basic models are Poincaré upper half plane, Poincaré disk model, Hyperboloid model, Klein model, half-sphere model [1]. In this study, we will discuss one particularly simple and convenient model of hyperbolic geometry, namely the Poincaré upper half plane model H.

Poincaré upper half plane was formulated by the French mathematician Henri Poincaré. Poincaré upper half plane H is the upper half plane of the Euclidean analytical plane $\Re^2$. Although the points in the H are the same as the points in the upper half plane of the Euclidean analytical plane $\Re^2$, the lines (geodesic segments) and the distance function between any two points are different [2]. The geodesic segments of the H are either arcs of Euclidean semicircles that are centered on the x-axis or segments of Euclidean straight lines that are perpendicular to the x-axis [1].

If $A=(x_1,y_1)$ and $B=(x_2,y_2)$ are any two points in the H then the Poincare distance between these points is given by [2].

$$d_H(A,B)=\begin{cases} \left|\ln\left(\dfrac{y_2}{y_1}\right)\right|, & \text{If} \quad x_1=x_2 \\ \left|\ln\left(\dfrac{y_2(x_1-c+r)}{y_1(x_2-c+r)}\right)\right|, & \text{If} \quad x_1\neq x_2 \end{cases}$$

In the H, a hyperbolic triangle ABC consists of three points A,B,C (vertices) that do not lie on a single geodesic segments or Euclidean semicircles (sides) that join each pair of vertices [1,3] (see Figure1.a,b).

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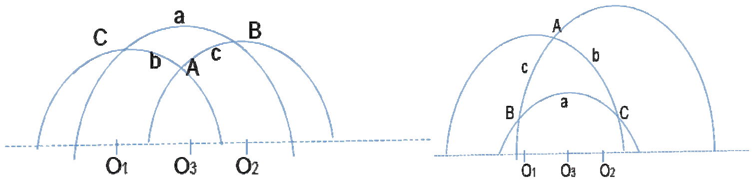

Figure 1.a Hyperbolic Triangles

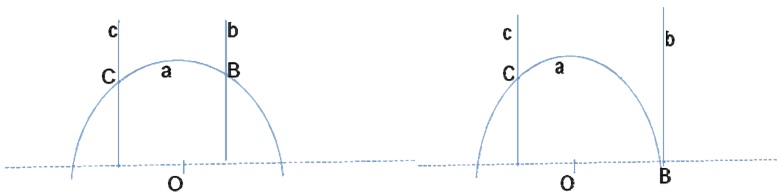

Figure 1.b Hyperbolic Triangles

In this study, we give isogonal conjugates from major contributions of the modern synthetic geometry of the hyperbolic triangles in Figure 1.a of Poincaré upper half plane model.

Lemma 1.1 (Hyperbolic Sines Theorem): In the hyperbolic triangle ABC let α,β,γ denote at A,B,C and a,b,c denote the hyperbolic lengths of the sides opposite A,B,C, respectively [1], then

$$\frac{\sin\alpha}{\mathrm{sha}}=\frac{\sin\beta}{\mathrm{shb}}=\frac{\sin\gamma}{\mathrm{shc}}$$

Below the analog of Ceva's theorem for a triangle in Klein model $H^2(-1)$ of hyperbolic plane is given [4].

Lemma 1.2 (Hyperbolic Ceva Theorem): Let (a,b,c) be the geodesic triangle on a hyperbolic plane and suppose that the three points $a'\in bc, b'\in ac, c'\in ab$ lie on the sides of this triangle. Then the three geodesics aa',bb',cc' intersect at a single point if and only if

$$\frac{\mathrm{sha'b}}{\mathrm{sha'c}}.\frac{\mathrm{shb'c}}{\mathrm{shb'a}}.\frac{\mathrm{shc'a}}{\mathrm{shc'b}}=1$$

Note that a trigonometric form for the theorem of hyperbolic Ceva is following by

$$\frac{\sin\angle baa'}{\sin\angle caa'}.\frac{\sin\angle cbb'}{\sin\angle abb'}.\frac{\sin\angle acc'}{\sin\angle bcc'}=1$$

Now, we remind the definitions isogonal conjugate and symmedian.

Definition 1.1: If $\overline{AE}$ is the bisector of angle A and if $\angle DAE\cong\angle FAE$, $\overline{AD}$ and $\overline{AF}$ are called *isogonal lines* and one is called the *isogonal conjugate* of the other. In other words, $\overline{AD}$ is the isogonal conjugate of a segment $\overline{AF}$ if $\angle DAE\cong\angle FAE$. The bisector of the angle is the bisector of the angle between two isogonal conjugates. According to this definition, the adjacent sides of the triangle are themselves isogonal conjugates.

Definition 1.2: A *symmedian* is the isogonal conjugate of a median [5].

Suppose, in Figure 2 that geodesic segment $\overline{AF}$ is a median in the H. Then geodesic segment $\overline{AD}$ is called symmedian.

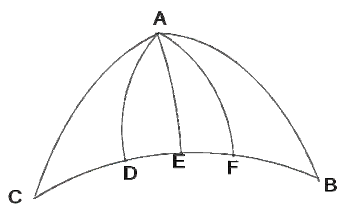

Figure 2 Symmedian and median

2. MAIN RESULTS

Theorem 2.1: The isogonal conjugates of a set of concurrent geodesic segments from the vertices to the opposite sides of a triangle are also concurrent in H.

Proof: Assume first that G is the point of concurrency of three geodesic segments and that $\overline{AD}, \overline{BE}$ and $\overline{CF}$ are the isogonal conjugates of these geodesic segments (see Figure 3).

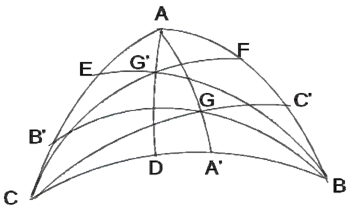

Figure 3 Isogonal conjugates

In this proof, it is not assumed that A', B', C' are midpoints. Consider hyperbolic triangles $AA'C, AA'B$. By the law of hyperbolic sines (see Lemma 1.1),

$$\frac{\mathrm{shCA'}}{\sin\angle \mathrm{CAA'}} = \frac{\mathrm{shCA}}{\sin\angle \mathrm{CA'A}}$$

$$\mathrm{shCA'} = \frac{\mathrm{shCA}\sin\angle \mathrm{CAA'}}{\sin\angle \mathrm{CA'A}}$$

Similarly,

$$\mathrm{shA'B} = \frac{\mathrm{shAB}\sin\angle \mathrm{BAA'}}{\sin\angle \mathrm{BA'A}}$$

Since the sines of the suplementary angles at A' are equal [1], we have

$$\frac{\mathrm{shCA'}}{shA'B} = \frac{\mathrm{shCA}\sin\angle \mathrm{CAA'}}{\mathrm{shAB}\sin\angle BAA'}$$

Expressing each ratio of division of the sides of hyperbolic triangle ABC in this form leads to a trigonometric form for the theorem of hyperbolic Ceva:

$$\frac{\sin\angle CAA'}{\sin\angle BAA'} \cdot \frac{\sin\angle BCC'}{\sin\angle ACC'} \cdot \frac{\sin\angle ABB'}{\sin\angle CBB'} = 1$$

Because $\overline{AD}, \overline{BE}$ and $\overline{CF}$ are isogonal conjugates of the three original geodesic segments, various angles can be substituted. For example, $\angle BAA' \cong \angle CAD$ and so on. When all of the appropriate substitutions are made,

$$\frac{\sin\angle BAD}{\sin\angle CAD} \cdot \frac{\sin\angle ACF}{\sin\angle BCF} \cdot \frac{\sin\angle CBE}{\sin\angle ABE} = 1$$

$\overline{AD}, \overline{BE}$ and $\overline{CF}$ are concurrent by hyperbolic Ceva's theorem.

In the H, if $\overline{AA'}, \overline{BB'}, \overline{CC'}$ are medians, then their isogonal conjugates are symmedians; thus the symmedians of a hyperbolic triangle are concurrent.

Theorem 2.2: The symmedian from one vertex of a hyperbolic triangle sometimes bisects geodesic segments parallel to the opposite side of the hyperbolic triangle in H.

Proof: i) Let $\overline{DE}$ be geodesic segment which share the same center with $\overline{BC}$. In this case $\overline{DE} // \overline{BC}$. Because of the two geodesic segments $\overline{DE}$ and $\overline{BC}$ have the same hyperbolic lenght [1], $\overline{DF} = \overline{FE}$ and F is the midpoint $\overline{DE}$.

ii) Let $\overline{DE}$ not be geodesic segment which share the same center with $\overline{BC}$. Also, let $\overline{DE}$ is a side of the orthic triangle [2]. Hence (see Figure 4),

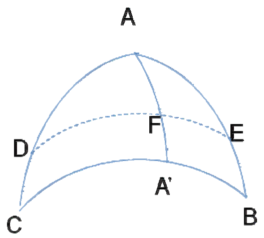

Figure 4. Figure showing the relationship between parallel segments and symmedian

$$\mathrm{shAD.shAC} = \mathrm{shAE.shAB}$$

By the theorem 2.1,

$$\frac{\mathrm{shCA'}}{shA'B} = \frac{\mathrm{shCA}\sin\angle \mathrm{CAA'}}{\mathrm{shAB}\sin\angle BAA'}$$

If A' is the midpoint of $\overline{BC}$, then

$$\frac{\sin\angle \mathrm{CAA'}}{\sin\angle BAA'} = \frac{shAB}{shAC}$$

so that

$$\frac{\text{shDF}}{shFE}=\frac{\text{shADsin}\angle \text{D}AF}{\text{shEAsin}\angle EAF}$$

$$=\frac{\text{shAB}}{\text{shAC}}.\frac{\text{shAB}}{\text{shAC}}\neq 1$$

and F isn’t the midpoint $\overline{DE}$.

iii) Let $\overline{DE}$ not be geodesic segment which share the same center with $\overline{BC}$. Also, let $\overline{DE}$ isn’t a side of the orthic triangle. Let $shAD=shEA.$.

$$\frac{\text{shDF}}{shFE}=\frac{\text{shADsin}\angle \text{D}AF}{\text{shEAsin}\angle EAF}=\frac{\text{shAB}}{\text{shAC}}$$

If ABC is a isosceles hyperbolic triangle then above ratio is equal to 1 and F is the midpoint $\overline{DE}$. In another cases, F isn't the midpoint $\overline{DE}$.

Theorem 2.3: The ratio of hyperbolic sines of the distances from a point on the symmedian to the adjacent sides of the hyperbolic triangle equals the ratio of hyperbolic sines of the lengths of those sides.

Proof: Let $\overline{AD}$ be the symmedian and $\overline{AA'}$ is the median (see Figure 5). Let $shAF=shAG.$

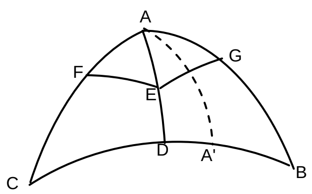

Figure 5. The relationship between a point on the symmedian and the adjacent sides of the hyperbolic triangle

From the law of hyperbolic sines,

$$\frac{\text{shEF}}{shEG}=\frac{\sin\angle CAD}{\sin\angle BAD}=\frac{\sin\angle BAA'}{\sin\angle CAA'}$$

$$=\frac{\text{shAC}}{\text{shAB}}$$

since A' is the midpoint of $\overline{CB}$.

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