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Completion of TVS-Cone Metric Spaces and Some Fixed Point Theorems

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ABSTRACT

In this paper a completion theorem for cone metric spaces and a completion theorem for cone normed space over a complete locally convex topological vector space E are proved. The completion spaces are defined by means of an equivalence relation obtained by convergence via the topology of the locally convex space E. Very recently some fixed point theorems obtained in cone Banach spaces are generalized to locally convex cone Banach spaces. These theorems can not be generalized or proved trivially by using the nonlinear scalarization function used very recently by Wei-Shih Du in " A note on cone metric fixed point theory and its equivalence, Nonlinear Analysis Theory Methods and Applications 72 (5):2259-2261 (2010)".

Key Words: *Cone metric space, tvs- cone metric.space, tvs- cone Banach space, fixed point, normal cone.*

1. INTRODUCTION

In 2007, Huang and Zhang [1] gave the definition of a cone metric space (CMS) by using the same idea, namely, by replacing real numbers with a ordering Banach space. In that paper,they proved the fixed point theorem of contractive mappings for cone metric spaces: Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$, has a unique fixed point. Lately, many results on fixed point theorems have been extended to cone metric spaces [1-15]. Motivated by that most of those results have been proved in complete cone metric spaces, the author in [2] proved completion theorems for cone metric spaces and cone normed spaces, where the cone of the real Banach space was assumed to be normal. Recently, Du [16] gave the definition of generalized cone metric space, namely topological vector space-cone metric space (TVS-CMS), and proved some fixed point theorem on that class. The author showed also that Banach contraction principles in usual metric spaces and in TVS-CMS are equivalent.

In this manuscript, motivated by all the above, we prove a completion theo- rem for TVS-cone metric spaces and a completion theorem for TVS-cone normed spaces. The topological space E under consideration will be assumed to be complete Hausdorff locally convex and its cone P will be normal. Also, we prove some fixed point theorems in TVS-cone Banach spaces. The proofs will be adap tation to those in [2] and [8]. However, this will be possible after proving some technical Lemmas about convergence in TVS-cone metric spaces.

Throughout this paper, E stands for real topological

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vector space (t.v.s.) with zero vector. A non-empty subset P of E is called cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. The cone P will be assumed to be closed with nonempty interior as well. For a given cone P, one can define a partial ordering (denoted by $\leq$: or $\leq_P$) with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x = y$ while $x \; y$ will show $y - x \in \mathrm{int}P$, where intP denotes the interior of P. Continuity of the algebric operations in a topological vector space and the properties of the cone imply the relations:

$$\mathrm{Int}P + \mathrm{Int}P \subseteq \mathrm{Int}P$$

and

$$\lambda \mathrm{Int}P \subseteq \mathrm{Int}P \;\; (\lambda > 0).$$

We appeal to these operations in the following.

Definition 1. (see [16-18]) *For $c \in IntP$, the nonlinear scalarization function φ_C: $E \to R$ is defined by*

$$\phi_c(y) = \inf\{t \in \mathbb{R} : y \in tc - P\}, \; \textit{for all } y \in E.$$

Lemma 2. (See [16-18]) *For each $t \in \mathbb{R}$ and $y \in E$, the following are satisfied:*

(i) $\varphi_C(y) \leq t \Leftrightarrow y \in tc - P$,

(ii) $\varphi_C(y) > t \Leftrightarrow y \notin tc - P$,

(iii) $\varphi_C(y) \geq t \Leftrightarrow y \notin tc - \mathrm{int}(P)$,

(vi) $\varphi_C(y) < t \Leftrightarrow y \in tc - \mathrm{int}(P)$,

(v) $\varphi_C(.)$ *is positively homogeneous and continuous on E*,

(vi) if $y_1 \in y_2 + P$, then $\varphi_C(y_2) \leq \varphi_C(y_1)$,

(vii) $\varphi_C(y_1 + y_2) \leq \varphi_C(y_1) + \varphi_C(y_2)$, for all y_1, $y_2 \in E$.

Definition 3. *[16] Let X be non-empty set. Suppose a vector-valued function*

$p: X \times X \to E$ satisfies:

(M 1) $0 \leq p(x, y)$ for all $x, y \in X$

(M 2) $p(x, y) = 0$ if and only if $x = y$,

(M 3) $p(x, y) = p(y, x)$ for all $x, y \in X$

(M 4) $p(x, y) \leq p(x, z) + p(z, y)$, for all $x, y, z \in X$

Then, p is called TVS-cone metric on X, and the pair (X, p) is called a TVS- cone metric space (in short, TVS-CMS).

Note that in [1], the authors considered *E* as a real Banach space in the definition of TVS-CMS. Thus, a cone metric space (in short, CMS) in the sense of Huang and Zhang [1] is a special case of TVS-CMS

Lemma 4. *(See [16]) Let (X, p) be a TVS-CMS. Then, $d_p : X \times X \to [0,\infty)$ defined by $d_p = \varphi_C(y) \; p$ is a metric.*

Remark 5. *Since a cone metric space (X, d) in the sense of Huang and Zhang [1], is a special case of TVS-CMS, then $d_p : X \times X \to [0,\infty)$ defined by $d_p = \varphi_C(y) \; d$ is also a metric*

Definition 6. (See [16]) Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X.

(i) $\{x_n\}_{n=1}^{\infty}$ *TVS-cone converges to $x \in X$ whenever for every $0 << c \in E$ there is a natural number M such that $p(x_n, x) << c$ for all $n \geq M$ and denoted by cone $-\lim_{n\to\infty} x_n = x$ (or $x_n \to x$ as $n \to \infty$)*

(ii) $\{x_n\}^{\infty}$ TVS-cone Cauchy sequence in (X, p) whenever for every $0 << c \in E$, there is a natural number M such that $p(x_n, x_m) << c$ for all $n, m \geq M$,

(iii) (X, p) *is TVS-cone complete if every TVS-cone Cauchy sequence in X is a TVS-cone convergent.*

Lemma 7 . (See [16]) Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X.

Set $d_p = \varphi_C(y) \circ p$. Then the following statements hold:

(i) If $\{x_n\}^{\infty}$ $n \to \infty$, converges to x in TVS-CMS (X, p), then $d_p(x_n, x) \to 0$ as

(ii) If $\{x_n\}^{\infty}$ is Cauchy sequence in TVS-CMS (X, p), then $\{x_n\}^{\infty}$ is a Cauchy sequence (in usual sense) in (X, d_p),

(iii) If (X,p) is complete TVS-CMS, then (X, d_p) is a complete metric space

Regarding (iii) of Lemma 7 above see [19].

Proposition 8. (See [16]) Let (X, p) be a complete TVS-CMS and T: $X \to X$ satisfy the contractive condition $p(Tx, Ty) \leq kp(x, y)$(1) for all $x, y \in X$ and $0 \leq k < 1$. Then, T has a unique fixed point in X.

Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to fixed point.

Definition 9. [20] A cone P of a topological vector space (X,τ) is said to be normal whenever τ has a base of zero consisting of P_ full sets. A subset A of an order vector space via a cone P is said to be P_full if for each x,y∈A we have $\{a \in X: x \leq a \leq y\} \subset A$.

Theorem 10. [20] (a) A cone P of a topological vector space (X,τ) is normal if and only if whenever $\{x_\alpha\}$ and $\{y_\alpha\}$, $\alpha \in \Delta$ are two nets in X with $0 \leq x_\alpha \leq y_\alpha$ for each $\alpha \in \Delta$ and $y_\alpha \to 0$, then $x_\alpha \to 0$.

(b) The cone of an ordered locally convex space (X,

τ) is normal if and only

if τ is generated by a family of monotone τ – continuous seminorms. Where a seminorm q on X is called monotone if $q(x) \leq q(y)$ for all x y $\in$ X with $0 \leq x \leq y$.

As a particular case of (b) above, the cone P of a real Banach space E is called normal if there is a number $K \geq 1$ such that for all x,y $\in$ E: $0 \leq x \leq y \Rightarrow \|x\| \leq K \|y\|$. The least positive integer K, satisfying this inequality, is called the normal constant of *P*.

The following lemma generalizes Lemma 1 and Lemma 4 in [1].

Lemma 11. *Let (X, d) be a cone metric space over a locally convex space (E, S), where S is the family of seminorms defining the locally convex topology. Let $\{x_n\}$ be a sequence in E. Then*

(i) $x_n \to x$ in (X, d) if and only if $d(x_n, x) \to 0$ in (E, S).

(ii) x_n is Cauchy in (X, d) if and only if $\lim_{m,n\to\infty} d(x_n, x_m) = 0$ in (E, S)

Proof. (i) Suppose $\{x_n\}$ converges to x. Let $\in > 0$ and p $\in$ S be given, choose p (c) >> 0 such that $p(c) < \in$. This is possible by taking $c = \in c_0/2P(c_0)$, where c_0 is an interior point of *P*. Then there is n_0 such that $d(x_n, x) << c$, for all $n > n_0$. Then by normality of the cone P we have $p(d(x_n, x)) \leq p(c) <$ for all $n > n_0$. This means $d(x_n, x) \to 0$ in (E,S). Conversely, suppose that $d(x_n, x) \to 0$ in (E,S). For $c \in E$ with $c >> 0$ find $\delta > 0$ and $p \in S$ such that $p(b) < \delta$ implies $b << c$. For this δ and this p find n_0 such that $p(d(x_n, x)) < \delta$ for all $n > n_0$ and so $d(x_n, x) << c$ for all $n > n_0$. Therefore $x_n \to x$ in (X,d).

(ii) The proof is similar to that in (i)

Definition 12. A TVS-cone normed space is an ordered pair (X, $\|.\|_C$) where X is a vector space over R and $\|.\|_C : X \to (E, S)$ is a function satisfying:

C1) $\|x\|_C \geq 0$, for all $x \in X$.

C2) $\|x\|_C = 0$ if and only if $x = 0$.

C3) $\|x\|_C = |\alpha| \|x\|_C$, for each $x \in X$ and $\alpha \in R$.

C4) $\|x+y\|_C \leq \|x\|_C + \|y\|_C$ for all $x, y \in X$.

It is clear that each TVS- cone normed space is TVS-cone metric space. In fact, the cone metric is given by $d(x, y) = \|x - y\|_C$. Complete TVS-cone normed spaces are called TVS-cone Banach spaces.

According to the definition of convergence in TVS-cone metric spaces and Lemma 11, we see that $x_n \to x$ in (X, $\|.\|_C$) if and only if for all c >> 0 in *E* there exists n_0 such that $\|x_n - x\|_C << c$ for all $n \geq n_0$ and, if the cone is normal if and only if $\lim_{m,n\to\infty} q(\|x_n - x_m\|_C) = 0$, for all $q \in S$

Example 13. Let $X = R^2$, $P = \{(x,y): x \geq 0, y \geq 0\} \subset R^2$ and $\|(x, y)\|_C = (\alpha|x|, \beta|y|)$, $\alpha > 0$, $\beta > 0$. Then, (X, $\|\,\|_C$) *is a cone normed space over R^2*

Example 14. Let $X = \omega$: *the vector space of all real sequences, $E = \omega$ and $P = \{x = \{x_i\}: x_i \geq 0$, for all i$\}$. Then ω is a complete metrizable locally convex space (Fréchet space) when its topology is generated by the seminorms $\{q_k : k = 1, 2, 3, ..\}$ where $q_k(x) = \Sigma^k_{i=1} |x_i|$, $x = \{x_i\} \in \omega$. Clearly P is a normal cone in E. On X define $d(x, y) = \{|x_i - y_i|\}_i$, $x = \{x_i\}, y = \{y_i\} \in X$. Then (X, d) is a TVS-cone metric space over E.*

2. COMPLETION THEOREMS

Before proceeding to prove a TVS-completion theorem, we first give the meaning of isometries of cone metric spaces.

Definition 15. Let (X, d) and (Y, ρ) be TVS-cone metric spaces over the same TVS E. A mapping T of X into Y is said to be an isometry if it preserves cone distances, that is, if for all $x_1, x_2 \in X$,

$$\rho(Tx_1, Tx_2) = d(x_1, x_2) \tag{2}$$

Throughout, we shall say that the TVS- cone metric space X is isometric with the TVS-cone metric space Y if there exists a bijective isometry of X onto Y. In the sequel, one has to note that every cone isometry is one to one. The following two lemmas are essential to prove the completion Theorems.

Lemma 16. *Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in a cone metric space (X, d) over a normal cone of a complete locally convex space (E, S). Then, $\lim_{n\to\infty} d(x_n, y_n)$ exists in (E,S).*

Proof. Since (E,S) is complete, it will be enough to show that the sequence $\{d(x_n, y_n)\}$ is Cauchy in (E,S) To this end, let > 0 and $p \in S$ be given, then choose $c \in E$ with $c >> 0$ such that $p(c) < (\in/6)$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, there exists a natural number n_0 such that

$$d(x_i, x_j) << c \text{ and } d(y_i, y_j) << c \tag{3}$$

for all $i, j > n_0$. Then, we have,

$$d(x_i, y_i) \leq d(x_i, x_j) + d(x_j, y_j) + d(y_j, y_i) \leq d(x_j, y_j) + 2c \tag{4}$$

and

$$d(x_j, y_j) \leq d(x_j, x_i) + d(x_i, y_i) + d(y_i, y_j) \leq d(x_i, y_i) + 2c \tag{5}$$

and hence, (4) and (5) lead to

$$0 \leq d(x_j, y_j) + 2c - d(x_i, y_i) \leq d(x_i, y_i) + 2c + 2c - d(x_i, y_i) = 4c \tag{6}$$

Since the cone P is normal, then (6) implies that

$$p(d(x_j, y_j) + 2c - d(x_i, y_i)) \leq 4p(c) \quad (7)$$

Finally, by the triangle inequality of the seminorm p and (7) we have

$$p(d(x_j,y_j) - d(x_i,y_i)) \leq p(d(x_j,y_j) + 2c - d(x_i,y_i) + 2p(c) \leq 6p(c) < \in \quad (8)$$

Therefore, $\{d(x_i,y_i)\}$ is Cauchy in (E,S) and hence convergent

Lemma 17. Let $\{x_n\}$ $\{x'_n\}$; $\{y_n\}$ $\{y'_n\}$ be sequences of a cone metric space (X; d) over a normal cone P in a locally convex space (E;S). If $\lim_{n\to\infty} d(x_n; x'_n) = 0$ and $\lim_{n\to\infty} d(y_n; y'_n) = 0$ in (E;S), then

$$\lim_{n\to\infty} d(x_n; y_n) = \lim_{n\to\infty} d(x'_n; y'_n) \text{ in } (E; S) \quad (9)$$

Proof. Let $\in > 0$ and $p \in S$ be given, then choose $c \in E$ such that $c >> 0$ and $p(c) < (\in/6)$.

For this $c >> 0$ find $\delta > 0$ and $q \in S$ such that

$$q(b) < \delta \text{ implies } c - b \in Int(P) \quad (10)$$

By assumption, for the above $\delta > 0$ and q fin n_0 such that for all $n \geq n_0$ we have

$$q(d(x_n; x'_n)) < \delta \text{ and } q(d(y_n; y'_n)) < \delta \quad (11)$$

But then (10) and (11) imply that

$$d(x_n; x'_n) << c \text{ and } d(y_n; y'_n) << c \quad (12)$$

for all $n > n_0$. Now, by the triangle inequality and (12), for all $n \geq n_0$ we have

$$d(x_n; y_n) \leq d(x_n; x'_n) + d(x'_n; y'_n) + d(y'_n; y_n) \leq d(x'_n; y'_n) + 2c \quad (13)$$

and

$$d(x'_n; y'_n) \leq d(x_n; x'_n) + d(x_n; y_n) + d(y'_n; y_n) \leq d(x_n; y_n) + 2c \quad (14)$$

and hence, (13) and (14) lead to

$$0 \leq d(x'_n; y'_n) + 2c - d(x_n; y_n) \leq d(x_n; y_n) + 2c + 2c - d(x_n; y_n) = 4c \quad (15)$$

Since the cone is normal, then (15) together with the choice of $c >> 0$ imply that

$$p(d(x_n;y_n) - d(x'_n;y'_n)) \leq p(d(x'_n; y'_n)) + 2c - d(x_n; y_n)) + p(2c) < \in \quad (16)$$

for all $n > n_0$, which completes the proof.

Theorem 18. For a TVS-cone metric space (X, d) over a normal cone there exists a complete TVS-cone metric space (X_S, d_S) which has a subspace W that is isometric with X and dense in X_S. The space (X^S, d_S) is unique except for isometries, that is, if Z is any complete cone metric space having a dense subspace U isometric with X, then Z and X^S are isometric.

Proof. The proof will be divided into four steps. We construct:

(a) (X^S, d_S),

(b) an isometry T of X onto W, where W is dense in X^S.

Then, we prove

(c) completeness of X^S,

(d) uniqueness of X^S except for isometries

Here are the details of these steps:

(a) Let (x_n) and (x'_n) be Cauchy sequences in (X, d). Define (x_n) to be equivalent to (x'_n), written (x_n) (x'_n), if

$$\lim_{n\to\infty} d(x_n; x'_n) = 0 \text{ in } (E; S) \quad (17)$$

Let X^S be the set of all equivalence classes x^S, y^S, of Cauchy sequences. We write $\{x_n\} \in x^S$ to mean that $\{x_n\}$ is a member x^S (a representative of the class x^S). We now set

$$d_S(x^S, y^S) = \lim_{n\to\infty} d(x_n; y_n) \quad (18)$$

By Lemma 16, the limit in (18) exists and by Lemma 17 it is independent of the particular choice of the representatives. The rest of the proof is the same as in [2].

As every TVS-cone normed space is TVS-cone metric space and TVS-cone metric spaces can be completed, as we have done above, we can also complete TVS-cone normed spaces. Before stating and proving this result we define the meaning of isometry of TVS-cone normed spaces.

Definition 19. Two TVS-cone normed spaces (X, $\|.\|_{c1}$) ,(Y, $\|.\|_{c2}$), over the same TVS E are said to be isometric if there exists a bijective linear operator T: $X \to Y$ such that,

$$\|Tx\|_{c1} = \|x\|_{c2} \quad \text{for all } x \in X$$

Theorem 20*. Let (X, $\|.\|_c$) be a TVS-cone normed space over a normal cone. Then there is a TVS- cone Banach space (X^S $\|.\|_s$) and an isometry T from X onto a subspace W of X^S which is dense in X^S. The space X^S is unique, except for isometries.*

The proof is the same as in [2] except we make use of Theorem 18 above.

3. FIXED POINT THEOREMS

The following lemma will be useful in proving the fixed point theorems in this section and elsewhere.

Lemma 21. *Let (X, d) be a TVS-cone metric space over a normal cone of a locally convex space (E, S), where is the family of seminorms defining the locally convex topology. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x$, $y_n \to y$. Then $d(x_n; y_n) \to d(x; y)$ in (E; S)*

Proof. Let $\in > 0$ and $p \in S$ be given. Choose $c \in E$ with $c >> 0$ such that $p(c) < (\in/6)$. From $x_n \to x$ and $y_n \to y$, find n_0 such that for all $n > n_0$, $d(x_n, x) << c$

and $d(y_n, y) << c$. Then for all $n > n_0$ we have

$$d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n) \le d(x, y) + 2c$$

and

$$d(x, y) \le d(x, x_n) + d(x_n, y_n) + d(y_n, y) \le d(x_n, y_n) + 2c$$

Hence

$$0 \le d(x, y) + 2c - d(x_n, y_n) \le 4c$$

and so by the normality of P we obtain

$$p(d(x_n, y_n) - d(x, y)) \le p(d(x, y) + 2c - d(x_n, y_n)) + p(2c) \le 6p(c) < \epsilon$$

Therefore $d(x_n, y_n) \to d(x, y)$ in (E, S)

Theorem 22. Let $a \in R$ with $a > 1$ and (X, d) be a complete TVS-complete cone metric space (the cone not necessary normal) and T: $X \to X$ an onto mapping which satisfies the condition

$$d(Tx, Ty) \ge ad(x, y)$$

Then, T has a unique fixed point

Proof. The proof is the same as in Theorem 13 [8]. However, in place we make use of Theorem 2.3 in [16].

From now on, throughout this section $X = (X, \|.\|_C)$ will be a TVS-cone Banach space, P a normal cone in a locally convex Hausdorff topological vector space (E, S) and T a self-mapping defined on a subset C of X. Also d will be the TVS-cone metric induced by the TVS-cone norm $\|.\|_C$. The proof of the main results of this section will be generally adaptation to the proof of the main results in [8] and by making use of Lemma 21 which generalizes Lemma 5 in [1].

Theorem 23. Let C be a closed and convex subset of a TVS-cone Banach space X with the TVS-cone norm $d(x, y) = \|x - y\|_C$ and $T : C \to C$ a mapping which satisfies the condition

$$d(x, Tx) + d(y, Ty) \le qd(x, y) \quad (19)$$

for all $x, y \in C$, where $2 \le q < 4$. Then, T has at least one fixed point.

Proof. The proof is the same as in Theorem 16 in [8] except we make use of Lemma 21.

Analogously as in [8] we can also state Theorem 24. Let C be a closed and convex subset of a TVS-cone Banach space X with the TVS-cone norm $d(x, y) = \|x - y\|_C$ and $T : C \to C$ a mapping which satisfies the *condition*

$$d(x, Tx) + d(y, Ty) \le qd(x, y) \quad (20)$$

for all $x, y \in C$, where $0 \le r < 2$. Then, T has at least one fixed point.

Theorem 25. Let C be a closed and convex subset of a TVS-cone Banach space X with the TVS-cone norm $d(x, y) = \|x - y\|_C$ and $T : C \to C$ a mapping which satisfies the condition

$$d(Tx, Ty) + d(y, Ty) + d(x, Tx) \le rd(x,y) \square \quad (21)$$

for all $x, y \in C$, where $2 \le r < 5$. Then, T has at least one fixed point.

Proof The proof is the same as in Theorem 16 in [8] except we make use of Lemma 21.

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