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Omega invariant of the line graphs of tricyclic graphs

Hacer ÖZDEN AYNA*

Bursa Uludağ Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Bursa.

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Abstract

Graphs are probably one of the few fastest growing subjects due to their applications in many areas including Chemistry, Physics, Biology, Anthropology, Finance, Social Sciences, etc. One of the ways of classifying graphs is according to the number of faces. A graph having no cycle is called acyclic, and a graph having one, two, three, faces are respectively called unicyclic, bicyclic, tricyclic. Recently, a new graph invariant denoted by Ω(D) for a realizable degree sequence D is defined. Ω(D) gives a list of information on the realizability, number of faces, components, chords, multiple edges, loops, pendant edges, bridges, cyclicness, connectedness, etc. of the realizations of D and is shown to have several explicit applications in Graph Theory. Acyclic, unicyclic and bicyclic graphs have been studied already in relation with Ω invariant. In this paper, we study tricyclic graphs by means of Ω invariant.

Keywords: *Omega invariant, degree sequence, tricyclic graph.*

Üç yüzlü grafların doğru graflarının omega invaryantı

Özet

Kimya, Fizik, Biyoloji, Antropoloji, Finans, Sözel Bilimler vb alanlardaki uygulamaları nedeniyle graflar en hızlı gelişen alanlardan birisidir. Grafları sınıflandırma yollarından birisi grafların yüz sayılarıdır. Hiçbir yüzü olmayan grafa yüzü olmayan graf (acyclic), bir, iki, üç yüzü olan graflara sırasıyla bir yüzlü (unicyclic), iki yüzlü (bicyclic) ve üç yüzlü (tricyclic) graflar denir. Son zamanlarda çizilebilir bir derece dizisi için adına omega invaryantı denilen bir sayı tanımlanmıştır. Ω(D), çizilebilirlik, yüz sayısı, bileşen, kiriş, katlı kenar, döngü, sallanan kenar, köprü sayıları, döngüsellik ve bağlantılılık gibi D nin çizimlerinin sahip olduğu çeşitli özelliklerle ilgili bilgi vermektedir ve graf teorinin çeşitli uygulamalarında faydalıdır. Yüz bulundurmayan, bir

* Hacer ÖZDEN AYNA, hozden@uludag.edu.tr, http://orcid.org/0000-0003-1556-3511

ve iki yüze sahip graflar **Ω** *invaryantı ile bağlantılı olarak çalışılmıştır. Bu çalışmada üç yüze sahip grafları* **Ω** *invaryantı yardımıyla inceleyeceğiz.*

Anahtar Kelimeler: *Omega invaryant, derece dizisi, üç yüzlü graf.*

1. Introduction

Let $G=(V,E)$ be a graph having order *n* and size *m*. Let $v\in V(G)$ be a vertex of G. The degree of v is denoted by d_v. A vertex of degree one will be called a pendant vertex and an edge having a pendant vertex will be called a pendant edge. The largest vertex degree in a graph is denoted by Δ. If *u* and *v* are two adjacent vertices of G, then the edge *e* connecting these vertices will be denoted by $e=uv$ and also the vertices *u* and *v* are called adjacent vertices. *e* will be said to be incident with the vertices *u* and *v*. A graph is said to be connected if there is a path between every pair of vertices and disconnected if not.

In many occasions, we shall classify our graphs under consideration according to whether they have at least one cycle or not. Those graphs having no cycle will be called acyclic. For example, all trees are acyclic. The remaining graphs are called cyclic graphs. A grapgh having one, two, three cycles is called unicyclic, bicyclic and tricyclic, respectively. The relation between omega invariant and acyclic graphs is studied in [5], between omega invariant and caterpillar trees is studied in [6]. In this work, we study the tricyclic graphs in a similar manner.

An edge connecting a vertex to itself is called a loop, and at least two edges connecting two vertices will be called multiple edges. When there are no loops nor multiple edges, the graph will be called simple.

A degree sequence written with multiplicities is given as $D=\left\{d_1^{(a_1)},d_2^{(a_2)},d_3^{(a_3)},\dots,\Delta^{(a_\Delta)}\right\}$
where a_i's are positive integers. It is also possible to state a degree sequence as

$$D=\left\{1^{(a_1)},2^{(a_2)},3^{(a_3)},\dots,\Delta^{(a_\Delta)}\right\}$$

where some of a_i's could be zero.

Let $D=\left\{d_1,d_2,d_3,\dots,\Delta\right\}$ be a set of non-decreasing non-negative integers. If the degree sequence of a graph G is equal to D, then D is said to be realizable and G is a realization of D.

For a realizable degree sequence, there is at least one graph with this degree sequence. Usually their number is quite big and there is no formula for it yet. The most well-known realizability test is known as Havel-Hakimi.

The degree sequence of some graphs are as follows: $D(\mathrm{P}_n)=\{1^{(2)},2^{(n-2)}\}$, $D(C_n)=\{2^{(n)}\}$, $D(S_n)=\{1^{(n-1)},(n-1)^{(1)}\}$, $D(K_n)=\{(n-1)^{(n)}\}$, $D(K_{r,s})=\{r^{(s)},s^{(r)}\}$ and $D(T_{r,s})=\{1^{(1)},2^{(r+s-2)},3^{(1)}\}$.

2. Ω invariant

In this section, for a realizable degree sequence D or for a given graph G, we recall the definition and some fundamental properties of the number **Ω***(D)* or **Ω***(G),* respectively, which are defined and studied in [2]. The number a_1 of pendant vertices of a tree T is given by

$$a_1 = 2 + a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta, \tag{1}$$

where a_i is the number of vertices of degree *i.* Note that Eqn. (1) can be restated as

$$a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta - a_1 = -2. \tag{2}$$

Generalizing this, Ω*(D)* is defined in [2] as follows:

Definition 1. Let $D=\{1^{(a_1)},2^{(a_2)},3^{(a_3)},\ldots,\Delta^{(a_\Delta)}\}$ be a set with a realization G. The **Ω***(G)* of the graph G is defined only in terms of the degree sequence as

$$\Omega(G) = a_3 + 2a_4 + 3a_5 + 4a_6 + \cdots + (\Delta - 2)a_\Delta - a_1 = \sum_{i=1}^{\Delta}(i-2)a_i.$$

Ω*(G)* of some graphs such as *T,* P_n ,C_n , S_n , K_n , $T_{r,s}$, $K_{r,s}$ where $n = r + s$ which respectively denote a tree, path, cycle, star, complete, tadpole and complete bipartite graphs with *n* vertices are

$\Omega(C_n) = 0$
$\Omega(P_n) = -2$
$\Omega(S_n) = -2$
$\Omega(T) = -2$
$\Omega(K_n) = n(n-3)$
$\Omega(K_{r,s}) = 2[rs - (r+s)]$
$\Omega(T_{r,s}) = 0.$

The most useful graph classes in the study of Ω are the path, cycle and tree graphs. Note that the Ω of a path, star or tree is equal to *-2.* This is true for all trees as stated in [2].

We now recall some basic properties of Ω.. In many cases, we study with disconnected graphs. The following useful result shows that Ω of *G* is additive on the components of *G:*

Theorem 1. [2] Let G be a disconnected graph with components $G_1, G_2, \ldots, G_c$. Then $\Omega(G) = \sum_{i=1}^{c} \Omega(G_i)$.

The following relation is useful in calculating $\Omega(G)$:

Theorem 2. [2] For a graph

$G\,\Omega(G) = 2(m - n)$.

That means, for any graph G, $\Omega(G)$ is even. Therefore if $\Omega(D)$ is odd for a set D of non-negative integers, then D is not realizable, which can be taken as a new realizability test.

The number r of non-overlapping cycles in a given connected planar graph G is known as the cyclomatic number of G. r is stated in terms of $\Omega(G)$:

Theorem 3. [2] Let $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \ldots, \Delta^{(a_\Delta)}\right\}$. If D is realizable as a connected planar graph G, then the number r of faces is given by

$$r = \frac{\Omega(G)}{2} + 1.$$

This result is useful in many applications. The following is a direct generalization of Theorem 3 to disconnected graphs:

Corollary 1. [2] Let $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \ldots, \Delta^{(a_\Delta)}\right\}$ be realizable as a graph G with c components. r of G is given by

$$r = \frac{\Omega(G)}{2} + c.$$

For the definitions of the fundamental notions in Graph Theory, see [1], [3], [4], [7], [8].

3. Line graphs

Let G be a simple undirected graph. A graph obtained by associating a new vertex onto each edge of G and connecting two such new vertices with an edge iff the corresponding edges of G have a vertex in common is called the line graph of G and denoted by $L(G)$. $L(G)$ is one of the derived graphs with many useful properties. It is known that the order of the line graph is equal to the size of the graph, and the size of the line graph is given by the formula

$$m\big(L(G)\big) = \frac{1}{2}\sum_{u \in V(G)} d_u^2 - m(G).$$

In [19] this formula is restated as

$$m\big(L(G)\big) = \frac{1}{2}M_1(G) - m(G).$$

where $M_1(G)$ denotes the first Zagreb index of G. It is clear that $L(P_n) = P_{n-1}$, $L(C_n) = C_n$ and $L(S_n) = K_{n-1}$.

The following result was obtained for characterization of connected unicyclic graphs:

Lemma 1. [5] Let G be a connected graph. G is unicyclic iff $m(G) = n(G)$.

That is, the necessary and sufficient condition for a connected graph to be unicyclic is that the order and size are equal. As $n\big(L(G)\big) = m(G)$, Lemma 1 immediately gives the following result which characterises all the graphs G such that the orders of G and $L(G)$ are equal:

Theorem 4. [5] Let G be a connected graph. G is unicyclic iff $n(L(G)) = n(G)$.

By Theorem 4, the number of vertices are the same in G and $L(G)$ whenever the graph G is unicyclic, and vice-versa. Of course, the line graph is rarely unicyclic. First we characterize all graphs having the property that $n(L(G)) = n(G)$. Actually even if the graph is unicyclic or acyclic, its line graph may have a large number of faces. The structure of the line graph of a tree is given as follows:

Theorem 5. [5] Let G be a connected simple acyclic graph with degree sequence $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \ldots, \Delta^{(a_\Delta)}\right\}$. Then its line graph $L(G)$ consists of a_2 times $K_2's$, a_3 times $K_3's$, …, a_Δ times $K_\Delta's$ where K_r and K_s have a unique common vertex in $L(G)$ for $d_{v_i} = r$ and $d_{v_j} = s$ iff v_i and v_j are adjacent in G.

In the following result, for a graph G, the necessary and sufficient condition for $L(G)$ to have a pendant vertex was given:

Theorem 6. [5] Let G be any graph. G has a support vertex of degree 2 iff $L(G)$ has a pendant vertex.

4. Omega invariant of the line graphs of tricyclic graphs

In this section, we will determine all the integer values that can be attained by the omega of the line graphs of tricyclic graphs. In our results, we shall need two new notions:

Definition 1. Two graphs will be called adjacent if they have a common cutvertex. Two graphs will be called neighbor if they have an edge in common.

In Fig. 1, G and H are adjacent and H and I are neighbour.

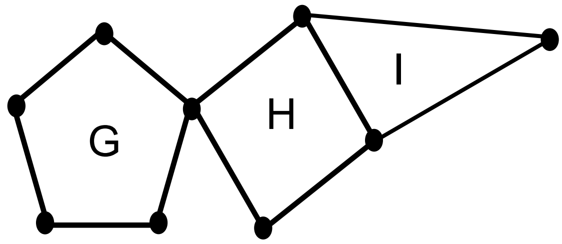

Figure 1. Adjacent and neighbor graphs.

Our aim is to find the lowest possible r $(and\ \Omega)$ for the line graph of a connected tricyclic graph.

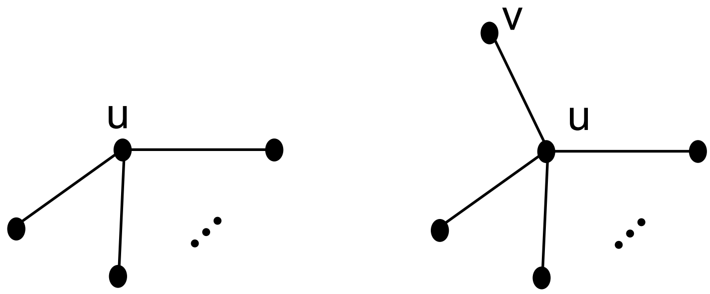

Figure 2. Adding a new pendant vertex v to G.

It is clear that pendant vertices do not change the number of faces of the graph G. Therefore Ω does not change as well. But a pendant vertex v increases the degree of its support vertex u resulting in an increase on the number of faces of the line graph $L(G)$, see Fig. 2. If the degree of u in G is $d_G u$, after adding a new pendant edge uv, it will become $d_G u + 1$ and the increase in the number of faces of $L(G)$ will be

$$
\begin{aligned}
&T_{d_G u+1-2} - \left(T_{d_G u-2}\right) \\
&= T_{d_G u-1} - T_{d_G u-2} \\
&= \frac{(d_G u - 1)(d_G u)}{2} - \frac{(d_G u - 2)(d_G u - 1)}{2} \\
&= \frac{d_G u - 1}{2}\left(d_G u - (d_G u - 2)\right) \\
&= d_G u - 1.
\end{aligned}
$$

Therefore to find the connected tricyclic graph G with lowest possible number $r(L(G))$ of faces and hence lowest Ω(L(G)), we need to look for the connected tricyclic graphs without pendant edges.

Next, we consider possible connected tricyclic graph structures. As we can omit pendant edges, we need only to consider those graphs with $\delta \geq 2$. By Theorem 5, for each vertex v of degree d_v in G, we have a complete graph $K_{d_G v}$ in $L(G)$. Because of this, the number of faces in a caterpillar tree was given in [6] by

$$r(L(G)) = \sum_{i=3}^{\Delta} a_i T_{i-2}.$$

We extend this result to tricyclic graphs as follows:

Theorem 7. Let G be a connected tricyclic graph. The number of faces of $L(G)$ is given by

$$r(L(G)) = 3 + \sum_{i=3}^{\Delta} a_i T_{i-2}.$$

Proof. For a vertex v of degree 2 in G lying between two edges e_1 and e_2, there is an edge (K_2) between e_1 and e_2 in $L(G)$. For a vertex v of degree 3 in G lying between three edges e_1, e_2 and e_3, there is a triangle (K_3) around v in $L(G)$. For a vertex v of degree $k \geq 4$ in G lying between k edges $e_1, e_2, \ldots, e_k$, there is a complete graph (K_k) around v in $L(G)$. As K_n has $\frac{(n-1)(n-2)}{2}$ faces, the contribution of each vertex v of degree $k \geq 2$ to the number of faces in $L(G)$ is $\frac{(k-1)(k-2)}{2}$. Also as G is tricyclic and for each cycle C_n, there is another cycle C_n in $L(G)$ with each vertex lying on an edge of G, we have three more faces each one is lying inside a cycle. So the result follows.

Next we shall find all the values of $r(L(G))$ for all connected tricyclic graphs:

Theorem 8. Let G be a connected tricyclic graph. $r(L(G))$ can take all integer values ≥ 7. That is, the line graph of a connected tricyclic graph can have at least 7 faces.

Proof. First we prove that for a connected tricyclic graph G, $r(L(G)) \geq 7$. Note that by Theorem 7, $r(L(G)) = 3 + \sum_{i=3}^{\Delta} a_i T_{i-2}$. Therefore for the minimum value of $r(L(G))$, we must find the minimum value of the sum

$$\sum_{i=3}^{\Delta} a_i T_{i-2} = a_3 + 3a_4 + 6a_5 + 10a_6 + 15a_7 + \cdots + a_\Delta T_{\Delta-2}.$$

To achieve this, we consider all possible connected tricyclic graph structures without pendant edges. We obtain the lowest number of faces in $L(G)$ which is 7 for the following graphs. Note that there are infinitely many other graphs with similar structures for which this minimum number is attained:

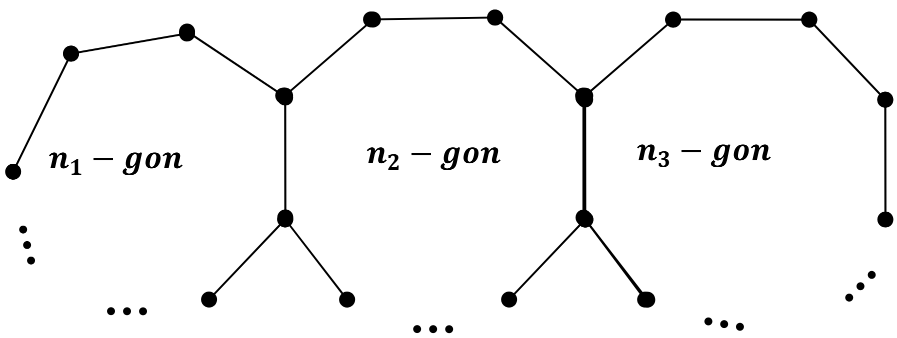

Figure 3. Connected tricyclic graphs with lowest number of faces in their line graphs.

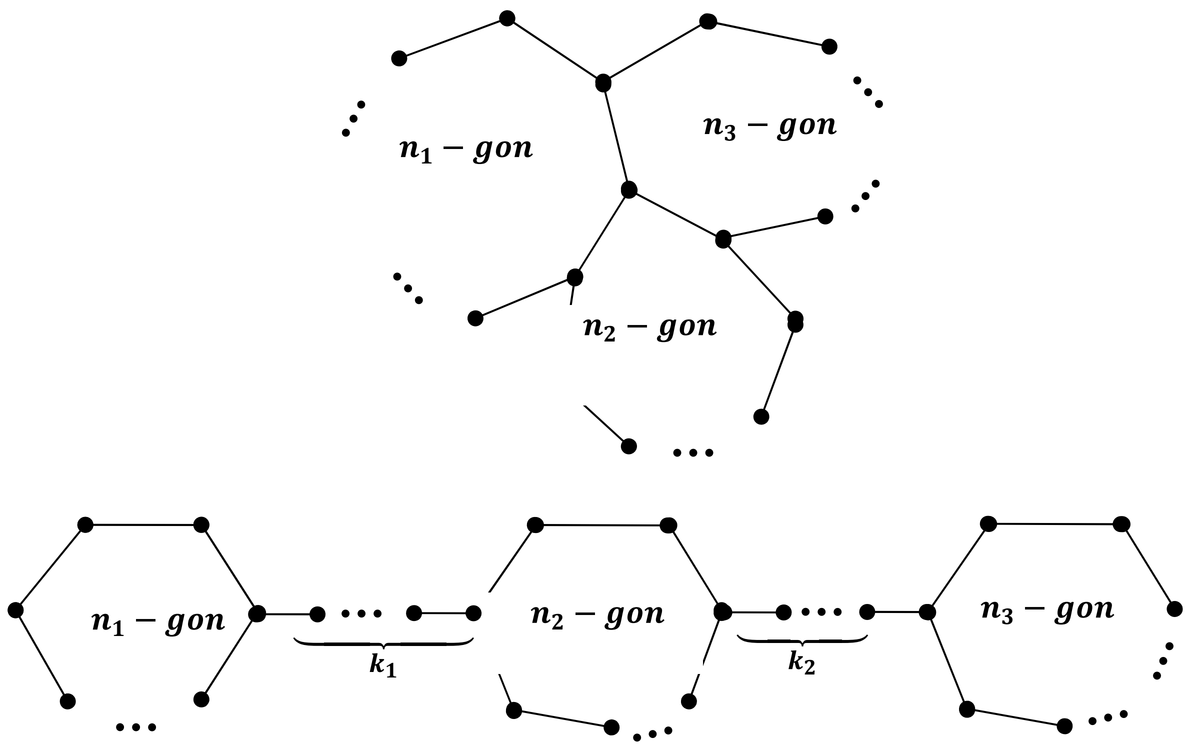

Figure 3. (continued).

To prove that $r(L(G))$ can take all integer values ≥ 7, we define an operation as in Fig. 4 where we add a new pendant vertex to G.

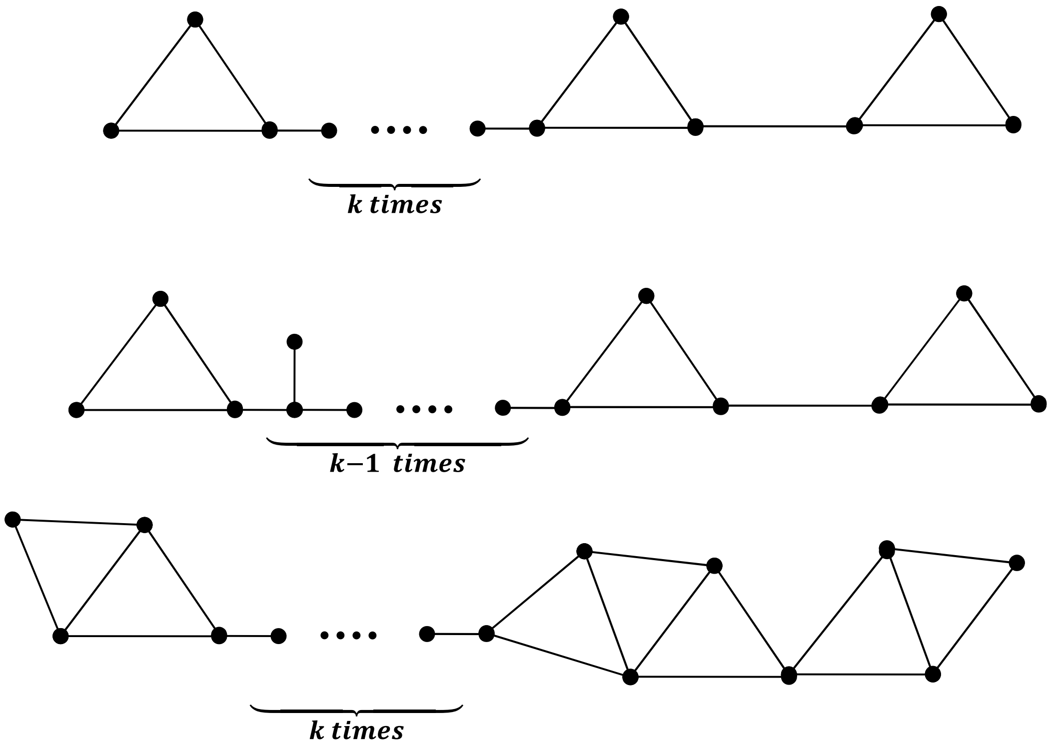

Figure 4. Effect of adding a new pendant vertex.

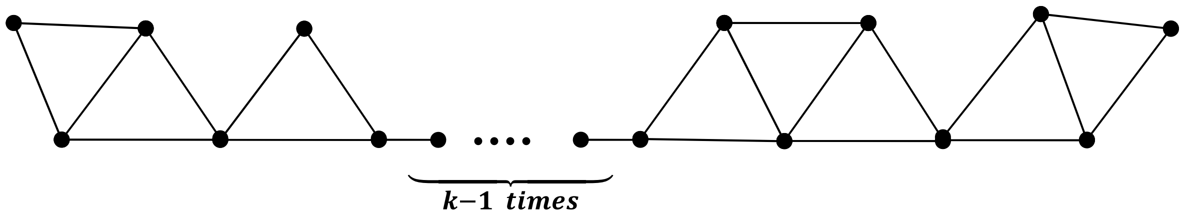

Figure 4. (continued).

This operation increases $r(L(G))$ by 1 when the pendant vertex is added to be adjacent to a vertex of degree 2. Recall that if the added pendant vertex is adjacent to a vertex of degree k, then by the proof of Theorem 4, we know that $r(L(G))$ increases by *k-1.* As this can be repeated many times by adding a new pendant vertex to another vertex of degree 2. The result then follows.

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