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Converse Theorems for the Cesàro Summability of Improper Integrals

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Abstract

In this paper we prove converse theorems to obtain usual convergence of improper integrals from Cesàro summability.

Keywords: Converse theorem, Tauberian condition, Cesàro summability

1. INTRODUCTION

Given a complex-valued function $f:\mathbb{R}_+\to\mathbb{C}$, that is Lebesgue integrable over any finite interval $(0,t)$ for $0<t<\infty$, in symbol: $f\in L^1_{loc}(\mathbb{R}_+)$, we define

$$s(t)=\int_0^t f(y)dy \text{ and } \sigma(t)=\frac{1}{t}\int_0^t s(y)dy,\, t>0.$$

If the limit

$$\lim_{t\to\infty}\sigma(t)=\ell \tag{1}$$

exists, then the improper integral $\int_0^\infty f(t)dt$ is called Cesàro (or briefly $(C,1)$) summable to ℓ and we denote $s(t)\to\ell(C,1)$. It is obvious that

$$\lim_{t\to\infty}s(t)=\ell \tag{2}$$

implies (1). However, the converse of this implication is not always true. The purpose of this work is to determine Tauberian conditions for the Cesàro summability of improper integrals under which the converse implication holds.

For any $s(t)=\int_0^t f(y)dy$, we have the identity [1]

$$s(t)-\sigma(t)=v(t) \tag{3}$$

where

$$v(t)=\frac{1}{t}\int_0^t yf(y)dy.$$

For each integer $k\geq 1$, we introduce

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$$\sigma_k(t)=\frac{1}{t}\int_0^t \sigma_{k-1}(y)dy \text{ and } v_k(t)=\frac{1}{t}\int_0^t v_{k-1}(y)dy$$

where $\sigma_0(t)=s(t)$ and $v_0(t)=v(t)$.

The classical control modulo of $s(t)$ is defined by

$$\omega_0(t)=tf(t)=t\frac{d}{dt}s(t)$$

and the general control modulo of order $k\in\mathbb{N}$ of $s(t)$ is given by [2]

$$\omega_k(t)=\omega_{k-1}(t)-\sigma(\omega_{k-1}(t)).$$

A complex-valued function $s(t)$ defined on $[0,\infty)$ is called slowly oscillating [6] if

$$\lim_{\lambda\to 1^+}\limsup_{t\to\infty}\max_{t\le u\le \lambda t}|s(u)-s(t)|=0.$$

For the proofs of our results, we require the following lemmas.

Lemma 1. ([3]) *For $\lambda>1$, we have*

$$s(t)-\sigma(t)=\frac{\lambda}{\lambda-1}(\sigma(\lambda t)-\sigma(t))-\frac{1}{\lambda t-t}\int_t^{\lambda t}s(u)-s(t)du.$$

For any function $s(t)$, we denote

$$\left(t\frac{d}{dt}\right)_k s(t)=\left(t\frac{d}{dt}\right)_{k-1} t\frac{d}{dt}s(t) \text{ and}$$

$$\left(t\frac{d}{dt}\right)_0 s(t)=s(t), \left(t\frac{d}{dt}\right)_1 s(t)=t\frac{d}{dt}s(t).$$

The following identities are useful.

Lemma 2. ([2]) *Let k be a positive integer, then we have*

(*i*) $v_{k-1}(t)=t\frac{d}{dt}\sigma_k(t)$,

(*ii*) $\omega_k(t)=\left(t\frac{d}{dt}\right)_k v_{k-1}(t)$.

2. MAIN RESULTS

In this section, we determine new Tauberian conditions for the Cesàro summability of improper integrals.

Theorem 3. *If $\int_0^\infty f(t)dt$ is Cesàro summable to ℓ and*

$$\lim_{\lambda\to 1^+}\limsup_{t\to\infty}\int_t^{\lambda t}\frac{|\omega_0(y)|^p}{y}dy<\infty,\ p\in(1,\infty), \tag{4}$$

then $\int_0^\infty f(t)dt=\ell$.

Proof. From Lemma 1,

$$|s(t)-\sigma(t)|\le\frac{\lambda}{\lambda-1}|\sigma(\lambda t)-\sigma(t)|+\int_t^{\lambda t}|f(y)|dy.$$

Since $\sigma(t)$ is convergent, we obtain

$$\limsup_{t\to\infty}|s(t)-\sigma(t)|\le\limsup_{t\to\infty}\int_t^{\lambda t}|f(y)|dy. \tag{5}$$

Also, using the Hölder's inequality we get

$$\begin{aligned}\int_t^{\lambda t}|f(y)|dy&=\int_t^{\lambda t}\frac{|\omega_0(y)|}{y}dy\\ &\le\left(\int_t^{\lambda t}dy\right)^{\frac{1}{q}}\left(\int_t^{\lambda t}\frac{|\omega_0(y)|^p}{y^p}dy\right)^{\frac{1}{p}}\\ &\le(\lambda t-t)^{\frac{1}{q}}\left(\frac{1}{t^{p-1}}\int_t^{\lambda t}\frac{|\omega_0(y)|^p}{y}dy\right)^{\frac{1}{p}}\\ &=(\lambda-1)^{\frac{1}{q}}\left(\int_t^{\lambda t}\frac{|\omega_0(y)|^p}{y}dy\right)^{\frac{1}{p}}\end{aligned} \tag{6}$$

where $1/q+1/p=1$. Then, considering (5) and (6) we find

$$\limsup_{t\to\infty}\left|s(t)-\sigma(t)\right|$$

$$\le(\lambda-1)^{\frac{1}{q}}\limsup_{t\to\infty}\left(\int_t^{\lambda t}\frac{\left|\omega_0(y)\right|^p}{y}dy\right)^{\frac{1}{p}}. \tag{7}$$

Letting $\lambda\to1^+$ in (7), it follows

$$\limsup_{t\to\infty}\left|s(t)-\sigma(t)\right|\le0,$$

which implies $\int_0^\infty f(t)dt=\ell.$ □

The following corollary is a classical Hardy-type ([4], p.149) Tauberian theorem.

Corollary 4. *Let* $\int_0^\infty f(t)dt$ *be Cesàro summable to* ℓ. *If*

$$tf(t)=O(1),\ t\to\infty, \tag{8}$$

then $\int_0^\infty f(t)dt=\ell.$

Proof. Let (8) holds, that is $|tf(t)|\le M$ for some $M>0$. Hence

$$\int_t^{\lambda t}\frac{\left|\omega_0(y)\right|^p}{y}dy\le M^p\int_t^{\lambda t}\frac{dy}{y}$$
$$=M^p\log\lambda\to0\ \text{ as }\ \lambda\to1^+.$$

Since all hypotheses of Theorem 3 are satisfied, the proof follows. □

For a different proof of Corollary 4, see Laforgia [5].

As in the following theorem, in place of recovering the usual convergence of $\int_0^\infty f(t)dt$, we may get more general information about the behaviour of $\int_0^\infty f(t)dt$ if we replace Cesàro summability of $\int_0^\infty f(t)dt$ with Cesàro summability of $v(t)$.

Theorem 5. *If* $v(t)$ *is Cesàro summable to* ℓ *and*

$$\lim_{\lambda\to1^+}\limsup_{t\to\infty}\int_t^{\lambda t}\frac{|\omega_1(y)|^p}{y}dy<\infty,\ p\in(1,\infty), \tag{9}$$

then $s(t)$ is slowly oscillating.

Proof. Taking Lemma 1 into account for $v(t)$, we obtain

$$\left|v(t)-v_1(t)\right|\le\frac{\lambda}{\lambda-1}\left|v_1(\lambda t)-v_1(t)\right|+\int_t^{\lambda t}\left|\frac{d}{dy}v(y)\right|dy.$$

Since $v(t)\to\ell(C,1)$, we get

$$\limsup_{t\to\infty}\left|v(t)-v_1(t)\right|\le\limsup_{t\to\infty}\int_t^{\lambda t}\frac{|\omega_1(y)|}{y}dy$$

by using Lemma 2. Now, from the Hölder's inequality

$$\limsup_{t\to\infty}\left|v(t)-v_1(t)\right|$$

$$\le(\lambda-1)^{\frac{1}{q}}\limsup_{t\to\infty}\left(\int_t^{\lambda t}\frac{|\omega_1(y)|^p}{y}dy\right)^{\frac{1}{p}}, \tag{10}$$

where $1/q+1/p=1$. Now, taking the limit of both sides of (10) as $\lambda\to1^+$ gives

$$\limsup_{t\to\infty}\left|v(t)-v_1(t)\right|\le0.$$

This necessiate that $\limsup_{t\to\infty}v(t)=\ell$. Moreover, by Lemma 2

$$|\sigma(u)-\sigma(t)|=\left|\int_t^u\frac{d}{dy}\sigma(y)dy\right|$$
$$\le\int_t^u\frac{|v(y)|}{y}dy.$$

From the boundedness of $v(t)$, we also have

$$\max_{t\le u\le\lambda t}|\sigma(u)-\sigma(t)|\le M\int_t^{\lambda t}\frac{dy}{y}$$
$$=M\log\lambda,$$

whenever $M>0$. Then, we conclude

$$\lim_{\lambda\to 1^+}\limsup_{t\to\infty}\max_{t\le u\le \lambda t}|\sigma(u)-\sigma(t)|=0.$$

This indicates that $\sigma(t)$ is slowly oscillating. Therefore, it follows from (3) that, $s(t)$ is also slowly oscillating. □

Corollary 6 *Let* $\int_0^\infty f(t)dt$ *be Cesàro summable to* ℓ *. If* (9) *is satisfied, then* $\int_0^\infty f(t)dt=\ell$ *.*

Proof. The proof easily follows from Theorem 1 of Çanak and Totur [1]. □

3. CONCLUSION

In this work, we present new Tauberian conditions for Cesàro summable improper integrals. We emphasise that, our main results may be extended to the weighted mean summability method given by Móricz [7].

4. REFERENCES

[1] İ. Çanak, Ü. Totur, "A Tauberian theorem for Cesàro summability of integrals," Appl. Math. Lett., vol. 24 no. 3, pp. 391–395, 2011.

[2] İ. Çanak, Ü. Totur, "Tauberian conditions for Cesàro summability of integrals," Appl. Math. Lett., vol. 24 no. 6, pp. 891–896, 2011.

[3] İ. Çanak, Ü. Totur, "Tauberian conditions for the (C,α) integrability of functions," Positivity, vol. 21 no. 1, pp. 73–83, 2017.

[4] G. H. Hardy, "Divergent Series," Clarendon Press, Oxford, 1949.

[5] A. Laforgia, "A theory of divergent integrals," Appl. Math. Lett., vol. 22 no. 6, pp. 834–840, 2009.

[6] F. Móricz, Z. Németh, "Tauberian conditions under which convergence of integrals follows from summability $(C,1)$ over $\mathbb{R}_+$," Anal. Math., vol. 26, no. 1, pp. 53–61, 2000.

[7] F. Móricz, "Necessary and sufficient Tauberian conditions in the case of weighted mean summable integrals over $\mathbb{R}_+$," Math. Inequal. Appl., vol. 7, no. 1, pp. 87–93, 2004.