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GENERALIZED PROJECTIVE CURVATURE TENSOR OF NEARLY COSYMPLECTIC MANIFOLD

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Abstract. In this paper, we concentrated our attention on geometry of generalized projective tensor of nearly cosymplectic manifold. In particular, we studied the flatness property of generalized projective tensor. This property helped us to find the necessary and sufficient condition that nearly cosymplectic manifold is a generalized Einstein manifold.

1. Introduction

One of the important curvature tensors is the projective tensor. According to this importance, many authors focused on its geometrical properties. Kirichenko [11] proved that nontrivial projective-recurrent K-space of maximal rank is 6-dimensional manifold of constant curvature tensor. Abood [3] studied the projective tensor of nearly Kähler manifold. Abood and Mohammed [4] proved that almost Kähler manifold is a Kähler manifold if it is a projective parakähler manifold. Shashikala and Venkatesha [20] studied the generalized pseudo-projective Φ-recurrent $N(k)$-contact metric manifold. Later on, Abood and Abd Ali [1] found the necessary condition that Viasman-Grey manifold has flat generalized projective tensor. Abood and Abd Ali [2] studied the projective-recurrent Viasman-Gray manifold. Finally, Atceken, Yildirim and Dirik [6], [7], [21], [22] studied certain curvature tensors including the pseudo-projective on some contact metric manifolds.

In this paper, we obtain some results on generalized projective tensor when it's act on nearly cosymplectic manifold. In particular, we found the necessary and sufficient conditions that nearly cosymplectic manifold is generalized Einstein manifold.

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2. Preliminaries

Let M be a smooth manifold of dimension $2n+1$ greater than 3, $X(M)$ be the module of smooth vector fields on M, $X^c(M)$ be the complexification of the module $X(M)$ and $T_p^c(M)$ be the complexification of tangent space $T_p(M)$ at the point $p \in M$.

An almost contact manifold (AC-manifold) is the set (M, η, ξ, Φ, g), where η is differential 1-form called a *contact form*, ξ is a vector field called a *characteristic*, Φ is endomorphism of $X(M)$ called a *structure endomorphisim* and $g = \langle .,. \rangle$ is the Riemannian metric on M. Moreover, the following conditions are fulfilled:

$$\eta(\xi) = 1, \Phi(\xi) = 0, \eta \circ \Phi = 0, \Phi^2 = -id + \eta \otimes \xi,$$

and $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y);\ X, Y \in X(M)$ [8].

In the module $X^c(M)$, define two endomorphisms σ and $\bar{\sigma}$ as follows: $\sigma = \frac{1}{2}(id - \sqrt{-1}\Phi)$ and $\bar{\sigma} = -\frac{1}{2}(id + \sqrt{-1}\Phi)$, then we can define two projections as follows:

$$\Pi = \sigma \circ \ell = -\frac{1}{2}(\Phi^2 - \sqrt{-1}\Phi) \ \textit{and}\ \bar{\Pi} = \bar{\sigma} \circ \ell = \frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi),$$

where $\sigma \circ \Phi = \Phi \circ \sigma = i\sigma$ and $\bar{\sigma} \circ \Phi = \Phi \circ \bar{\sigma} = -i\bar{\sigma}$. Therefore, If we denote $\mathrm{Im}\Pi = D_\Phi^{\sqrt{-1}}$ and $\mathrm{Im}\bar{\Pi} = D_\Phi^{-\sqrt{-1}}$, then

$$X^c(M) = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{-\sqrt{-1}} \oplus D_\Phi^0,$$

where $D_\Phi^{\sqrt{-1}}$, $D_\Phi^{-\sqrt{-1}}$ and D_Φ^0 are proper submodules of the endomorphism Φ with proper values $\sqrt{-1}, -\sqrt{-1}$ and 0 respectively [13].

At each point $p \in M$, we can construct a frame in $T_p^c(M)$ by the form $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$, where $\varepsilon_a = \sqrt{2}\sigma_p(e_p)$, $\varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}(e_p)$ and $\varepsilon_0 = \xi_p$. The frame $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$ is called A-frame [16].

The principle fiber of all A-frames with structure group $\{1\} \times U(n)$ is called an G-adjoined structure space.

The matrices of the AC-structure Φ_p and Riemannian metric g_p in A-frame are given by the following forms:

$$(\Phi_j^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & o \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix},\ (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix} \qquad (2.1)$$

where I_n is the identity matrix of order n [14].

An almost contact manifold is called a nearly cosymplectic manifold (NC-manifold) if the equality $\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0;\ X, Y \in X(M)$ holds [9].

The following theorem explains the structure equations of NC-manifold in the G-adjoined structure space.

Theorem 2.1. [15] *In the G-adjoined structure space, the structure equations of NC-manifold are given by the following forms:*

(1) $d\omega^a = \omega^a_b \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c + \frac{3}{2}C^{ab}\omega_b \wedge \omega$;
(2) $d\omega_a = -\omega^b_a \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c + \frac{3}{2}C_{ab}\omega^b \wedge \omega$;
(3) $d\omega = C^{bc}\omega_b \wedge \omega_c + C_{bc}\omega^b \wedge \omega^c$;
(4) $d\omega^a_b = \omega^a_c \wedge \omega^c_b + [A^{ad}_{bc} - 2B^{adh}B_{hbc} + \frac{3}{2}C^{ad}C_{bc}]\omega^c \wedge \omega_d$,

where $B^{abc} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},\hat{c}}, C^{ab} = \sqrt{-1}\Phi^a_{0,\hat{b}}, C_{ab} = -\sqrt{-1}\Phi^{\hat{a}}_{b,0}$ *and* $B_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,c}$. *The tensors B, C and A are called the first, second and third structure tensors respectively.*

Definition 2.1. [17] *A Riemann-Christoffel tensor R of a smooth manifold M is a tensor of type* $(4,0)$ *which is defined by*

$$R(X,Y,Z,W) = g(R(Z,W)Y,X),$$

where $R(X,Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$, *and has the following properties:*

(1) $R(X,Y,Z,W) = -R(Y,X,Z,W)$;
(2) $R(X,Y,Z,W) = -R(X,Y,W,Z)$;
(3) $R(X,Y,Z,W) = R(Z,W,X,Y)$;
(4) $R(X,Y,Z,W) + R(X,Z,W,Y) + R(X,W,Y,Z) = 0$.

The components of Riemann-Christoffel tensor of NC-manifold are given in theorem below.

Lemma 2.1. [15] *In the G-adjoined structure space, the components of Riemann-Christoffel tensor of NC-manifold have the following forms:*

(1) $R_{\hat{a}bcd} = 0$;
(2) $R_{abcd} = -2B_{ab[cd]}$;
(3) $R_{\hat{a}\hat{b}cd} = -2B^{abh}B_{hcd}$;
(4) $R_{\hat{a}0b0} = C^{ac}C_{bc}$;
(5) $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}$.

The other components of Riemann-Christoffel tensor R can be obtained by the property of symmetry for R or equal to zero.

Definition 2.2. [10] *A generalized Riemannian curvature tensor G_R on NC-manifold M is a tensor of type* $(4,0)$ *which is defined as the following form:*
$G_R(X,Y,Z,W) = \frac{1}{16}\{3[R(X,Y,Z,W) + R(\Phi X,\Phi Y,Z,W) + R(X,Y,\Phi Z,\Phi W) + R(\Phi X,\Phi Y,\Phi Z,\Phi W)] - R(X,Z,\Phi W,\Phi Y) - R(\Phi X,\Phi Z,W,Y) - R(X,W,\Phi Y,\Phi Z) - R(\Phi X,\Phi W,Y,Z) + R(\Phi X,Z,\Phi W,Y) + R(X,\Phi Z,W,\Phi Y) + R(\Phi X,W,Y,\Phi Z) + R(X,\Phi W,\Phi Y,Z)\}$,
where $R(X,Y,Z)$ is the Riemann-Christoffel tensor, $X,Y,Z,W \in T_p(M)$ *and has the following properties:*

(1) $G_R(X,Y,Z,W) = -G_R(Y,X,Z,W) = -G_R(X,Y,W,Z)$;
(2) $G_R(X,Y,Z,W) = G_R(Z,W,X,Y)$;
(3) $G_R(X,Y,Z,W) + G_R(X,Z,W,Y) + G_R(X,W,Y,Z) = 0$;

(4) $G_R(X, \Phi X, \Phi X, X) = R(X, \Phi X, \Phi X, X)$.

Definition 2.3. [18] *A tensor G_r of type $(2,0)$ which is defined as $(G_r)_{ij} = (G_R)^k_{ijk}$ is called a generalized Ricci tensor.*

Remark 2.1. [18] *A generalized Ricci tensor is symmetric , this follows form the properties of symmetry of generalized Riemannian curvature tensor. This mean $(G_r)_{ij} = (G_r)_{ji}$.*

Definition 2.4. *A generalized projective tensor G_P is a tensor of type $(4,0)$ which is defined as the form:*

$$(G_P)_{ijkl} = (G_R)_{ijkl} - \frac{1}{2n}[(G_r)_{ik}g_{jl} - (G_r)_{jk}g_{il}].$$

Definition 2.5. [11] *Let M be an AC-manifold, an Φ-holomorphic sectional curvature (ΦHS-curvature) of a manifold M in the direction $X \in X(M), X \neq 0$ is a function $H(X)$ which is defined as:*

$$H(X) = \langle R(X, \Phi X)X, \Phi X\rangle \|X\|^{-4}$$

Definition 2.6. [11] *An AC-manifold is called a manifold of point constant ΦHS-curvature if*

$$\langle R(X, \Phi X)X, \Phi X\rangle = c\|X\|^4,$$

where $c \in C^\infty(M)$, for all $X \in X(M)$

Theorem 2.2. [11] *An AC-manifold is a manifold of point constant ΦHS-curvature C_0 if and only if, on the G-adjoined structure, the following equation holds:*

$$R^{(ad)}_{(bc)} = \frac{C_0}{2}\tilde{\delta}^{ad}_{bc}, \tag{2.2}$$

where $C_0 \in C^\infty(M)$ and $\tilde{\delta}^{ad}_{bc} = \delta^a_b\delta^d_c + \delta^a_c\delta^d_b$.

Definition 2.7. [19] *A Riemannian manifold is called an Einstein manifold, if the Ricci tensor satisfies the equation $r_{ij} = eg_{ij}$, where e is an cosmological constant.*

Similar to the above definition, we can introduce the following definition.

Definition 2.8. *A Riemannian manifold is called a generalized Einstein manifold, if the generalized Ricci tensor satisfies the equation $(G_r)_{ij} = (G_e)g_{ij}$, where G_e is a generalized cosmological constant.*

3. The main results

In this section, we calculated the components of the generalized Riemannian curvature tensor. Moreover, the necessary and sufficient condition that a nearly cosymplectic manifold is generalized Einstein manifold has been found.

Lemma 3.1. *In the G-adjoined structure space, the components of the generalized Riemannian curvature tensor of NC-manifold are given by the following forms:*

(1) $(G_R)_{\hat{a}b\hat{c}d} = -A^{ac}_{bd}$;
(2) $(G_R)_{\hat{a}bc\hat{d}} = -\frac{1}{2}[A^{ad}_{bc} - 3B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}]$.
And the others are conjugate to the above components or equal to zero.

Proof: By using the Lemma 2.1 and Definition 2.2, we compute the components of generalized projective tensor as the following:
1) Put $i = \hat{a}, j = b, k = \hat{c}$ and $l = d$, we have

$$(G_R)_{\hat{a}b\hat{c}d} = \frac{1}{16}\{3[R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d}] + R_{\hat{a}\hat{c}db} + R_{\hat{a}\hat{c}db} - R_{\hat{a}db\hat{c}} - R_{\hat{a}db\hat{c}} + R_{\hat{a}\hat{c}db} + R_{\hat{a}\hat{c}db} - R_{\hat{a}db\hat{c}} - R_{\hat{a}db\hat{c}}\}$$

Making use of the properties of G_R, we get

$$(G_R)_{\hat{a}b\hat{c}d} = -A^{ac}_{bd}.$$

2) Put $i = \hat{a}, j = b, k = \hat{c}$ and $l = d$, we obtain

$$(G_R)_{\hat{a}b\hat{c}d} = \frac{1}{16}\{3[R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d} + R_{\hat{a}b\hat{c}d}] + R_{\hat{a}\hat{c}db} + R_{\hat{a}\hat{c}db} - R_{\hat{a}db\hat{c}} - R_{\hat{a}db\hat{c}} + R_{\hat{a}\hat{c}db} + R_{\hat{a}\hat{c}db} - R_{\hat{a}db\hat{c}} - R_{\hat{a}db\hat{c}}\}$$

According the the Definition 2.2, consequently we deduce

$$(G_R)_{\hat{a}bc\hat{d}} = -\frac{1}{2}[A^{ad}_{bc} - 3B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}].$$

By the same manner, we can get the other components.

Lemma 3.2. *In the G-adjoined structure space, the components of the generalized Ricci tensor of NC-manifold are given as the following form:*

$$(G_r)_{\hat{a}b} = \frac{1}{2}A^{ac}_{cb} + 3B^{ach}B_{hcb} + \frac{5}{3}C^{ac}C_{cb}.$$

And the others are conjugate to the above component or equal to zero.

Proof: By using the Lemma 2.1 and Definition 2.3, directly we obtain the above components.

Lemma 3.3. *In the G-adjoined space, the components of the generalized projective tensor of NC-manifold take the following forms:*

(1) $(G_P)_{\hat{a}\hat{b}cd} = -\frac{1}{2n}((\frac{1}{2}A^{af}_{fc} + 3B^{afh}B_{hfc} + \frac{5}{3}C^{af}C_{fc})\delta^b_d) - (\frac{1}{2}A^{bf}_{fc} - 3B^{bfh}B_{hfc} - \frac{5}{3}C^{bf}C_{fc})\delta^a_d$;
(2) $(G_P)_{\hat{a}b\hat{c}d} = -A^{ac}_{bd} + \frac{1}{2n}(\frac{1}{2}A^{cf}_{fb} + 3B^{cfh}B_{hfb} + \frac{5}{3}C^{cf}C_{fb})\delta^a_d$;
(3) $(G_P)_{\hat{a}bc\hat{d}} = -\frac{1}{2n}(\frac{1}{2}A^{ad}_{bc} + 3B^{adh}B_{hbc} + \frac{5}{3}C^{ad}C_{bc}) - \frac{1}{2}(A^{af}_{fc} + 3B^{afh}B_{hfc} + \frac{5}{3}C^{af}C_{fc})\delta^d_b$.

The remaining components are obtained by taking the conjugated operation to the above components or are identical equal to zero.

Proof:
1. Put $i=\hat{a},\ j=\hat{b},\ c$ and $l=d$.
According to the Definition 2.4, we obtain

$$(G_P)_{\hat{a}\hat{b}cd}=(G_{\Re})_{\hat{a}\hat{b}cd}-\tfrac{1}{2n}[(G_r)_{\hat{a}c}g_{\hat{b}d}-(G_r)_{\hat{b}c}g_{\hat{a}d}]$$

By using the Lemmas 3.1, 3.2 and the matrices (2.1), we have

$$(G_P)_{\hat{a}\hat{b}cd}=$$
$$-\tfrac{1}{2n}((\tfrac{1}{2}A^{af}_{fc}+3B^{afh}B_{hfc}+\tfrac{5}{3}C^{af}C_{fc})\delta^b_d)-(\tfrac{1}{2}A^{bf}_{fc}-3B^{bfh}B_{hfc}-\tfrac{5}{3}C^{bf}C_{fc})\delta^a_d.$$

2. Put $i=\hat{a},\ j=b,\ k=\hat{c}$ and $l=d$.
Harmonize to the Definition 2.4, we get

$$(G_P)_{\hat{a}b\hat{c}d}=(G_{\Re})_{\hat{a}b\hat{c}d}-\tfrac{1}{2n}[(G_r)_{\hat{a}\hat{c}}g_{bd}-(G_r)_{b\hat{c}}g_{\hat{a}d}]$$

Taking into account the Lemmas 3.1, 3.2 and the matrices (2.1), we obtain

$$(G_P)_{\hat{a}b\hat{c}d}=-A^{ac}_{bd}+\tfrac{1}{2n}(\tfrac{1}{2}A^{cf}_{fb}+3B^{cfh}B_{hfb}+\tfrac{5}{3}C^{cf}C_{fb})\delta^a_d.$$

By the same technique, we can compute the other components.

Theorem 3.1. *If M is vanishing generalized projectively NC-manifold, then the necessary and sufficient condition for M to be vanishing generalized Ricci tensor is the holomorphic tensor vanishes.*

Proof: Let M be vanishing generalized projectively NC-manifold.
According to the Lemma 3.3, we have

$$(G_{\Re})_{\hat{a}b\hat{c}d}+\frac{1}{2n}[(G_r)_{b\hat{c}}g_{\hat{a}d}]=0 \tag{3.1}$$

If M is vanishing generalized Ricci tensor, then directly we get

$$A^{ac}_{bd}=0.$$

Conversely, if M is vanishing holomorphic tensor, then we have

$$\frac{1}{2n}[(G_r)_{b\hat{c}}\delta^a_d]=0 \tag{3.2}$$

Contracting (3.2) by the indices (b,a) , it follows that

$$(G_r)_{d\hat{c}}=0.$$

Theorem 3.2. *An NC-manifold has vanishing holomorphic tensor if and only if, M is a manifold of vanishing generalized projective tensor.*

Proof: Let M be NC-manifold with vanishing generalized projective tensor. Making use of the Lemma 3.3, we obtain

$$-A^{ac}_{bd}+\frac{1}{2n}(\frac{1}{2}A^{cf}_{fb}+3B^{cfh}B_{hfb}+\frac{5}{3}C^{cf}C_{fb})\delta^a_d=0 \tag{3.3}$$

Symmetrizing and then antisymmetrizing the equation (3.3) by the indices (c, f), we get

$$A^{ac}_{bd} = 0.$$

Conversely, let M be NC-manifold with vanishing holomorphic tensor, then the equation (3.3) takes the following formula:

$$(G_P)_{\hat{a}b\hat{c}d} = \frac{1}{2n}(3B^{cfh}B_{hfb} + \frac{5}{3}C^{cf}C_{fb})\delta^a_d \tag{3.4}$$

Symmetrizing and then antisymmetrizing the equation (3.4) by the indices (c, f), we deduce

$$(G_p)_{\hat{a}b\hat{c}d} = 0.$$

Lemma 3.4. *An NC-manifold has Φ-invariant generalized Ricci tensor if and only if,*

$$\Phi \circ G_r = G_r \circ \Phi.$$

Theorem 3.3. *Let M be NC-manifold. Then M has Φ-invariant generalized Ricci tensor if and only if, $(G_r)^{\hat{a}}_b = 0$ hold in the G-adjoined structure space.*

Proof: Suppose that M is Φ- invariant generalized Ricci tensor.
According to the Lemma 3.4, we have

$$\Phi \circ G_r = G_r \circ \Phi$$

By the G-adjoined structure space, the above equation becomes

$$(\Phi \circ G_r)^i_j = (G_r \circ \Phi)^i_j$$

This means

$$\Phi^i_k (G_r)^k_j = (G_r)^i_k \Phi^k_j \tag{3.5}$$

Put $i = \hat{a}$ and $j = b$, then the equation (3.5) becomes

$$\Phi^{\hat{a}}_c (G_r)^c_b + \Phi^{\hat{a}}_{\hat{c}} (G_r)^{\hat{c}}_b + \Phi^{\hat{a}}_0 (G_r)^0_b = (G_r)^{\hat{a}}_c \Phi^c_b + (G_r)^{\hat{a}}_{\hat{c}} \Phi^{\hat{c}}_b + (G_r)^{\hat{a}}_0 \Phi^0_b$$

By using (2.1), we have

$$(G_r)^{\hat{a}}_b = 0.$$

Theorem 3.4. *Suppose that M is NC-manifold with vanishing generalize projective tensor and Φ-invariant generalized Ricci tensor. Then the necessary and sufficient condition for M to be generalized Einstein manifold is $A^{bc}_{bd} = \frac{Ge}{2n}\delta^c_d$, where Ge is a generalized Cosmological constant.*

Proof: Let M be NC-manifold with vanishing generalized projective tensor.
According to the Lemma 3.3, we have

$$-A^{ac}_{bd} + \frac{1}{2n}[(G_r)_{b\hat{c}}g_{\hat{a}d}] = 0 \tag{3.6}$$

Making use of the Definition 2.8, the equation (3.6) becomes

$$A^{ac}_{bd} = \frac{Ge}{2n}\delta^c_b\delta^a_d \tag{3.7}$$

Contracting the equation (3.7) by the indices (b, a), it follows that

$$A^{bc}_{bd} = \frac{Ge}{2n}\delta^c_d \tag{3.8}$$

Conversely,
Contracting the equation (3.6) by the indices (b, a), we have

$$-A^{bc}_{bd} + \frac{1}{2n}[(G_r)_{d\hat{c}}] = 0 \tag{3.9}$$

Combining the equations (3.8) and (3.9), we conclude

$$(G_r)^c_d = Ge\delta^c_d$$

Therefore, by the Definition 2.8 and Theorem 3.3, M is generalized Einstein manifold.

Theorem 3.5. *Suppose that M is NC-manifold with vanishing generalized Riemannian curvature tensor and Φ-invariant generalized Ricci tensor. If M is a generalized Einstein manifold then $Ge = \frac{10}{3}C^{bc}C_{bd}$.*

Proof: Let M be NC-manifold with vanishing generalized Riemannian curvature tensor. Then from Lemma 3.1 we have

$$-\frac{1}{2}[A^{ad}_{bc} - 3B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}] = 0 \tag{3.10}$$

By symmetrization and antisymmetrizatin the equation (3.10) by the induces (d, h) we get

$$A^{ad}_{bc} - \frac{5}{3}C^{ad}C_{bc} = 0 \tag{3.11}$$

Contracting the equation (3.11)by the induces $(b, a), (d, c)$ and (c, d), we deduce

$$A^{bc}_{bd} - \frac{5}{3}C^{bc}C_{bd} = 0 \tag{3.12}$$

Since M is generalized Einstein manifold, then from the Theorem 3.4, the equation (3.12) becomes

$$\frac{Ge}{2n}\delta^c_d - \frac{5}{3}C^{bc}C_{bd} = 0 \tag{3.13}$$

Contracting the equation (3.13)by the induces (d, a), implies

$$Ge = \frac{10}{3}C^{bc}C_{bd}.$$

Theorem 3.6. [5] *Suppose that M is NC-manifold. Then the necessary and sufficient condition that M is a manifold of point constant ΦHS-curvature C_0 is*

$$A_{bc}^{ad} = B^{adh}B_{hbc} + \frac{5}{3}C^{ad}C_{bc} + \frac{C_0}{2}\tilde{\delta}_{bc}^{ad}.$$

Theorem 3.7. *Suppose that M is NC-manifold of point constant ΦHS-curvature C_0 and vanishing generalized projective tensor with Φ-invariant generalized Ricci tensor, then $C^{af}C_{fc} = -\frac{C_0(n+1)}{10}\delta_c^a$.*

Proof: Let M be NC-manifold of ΦHS-curvature tensor and vanishing generalized projective tensor.
According to the Lemma 3.3, we have

$$-\frac{1}{2n}((\frac{1}{2}A_{fc}^{af} + 3B^{afh}B_{hfc} + \frac{5}{3}C^{af}C_{fc})\delta_d^b) - (\frac{1}{2}A_{fc}^{bf} - 3B^{bfh}B_{hfc} - \\ -\frac{5}{3}C^{bf}C_{fc})\delta_d^a \quad (3.14)$$

By using Theorem 3.6, the equation (3.14) becomes

$$-\frac{1}{2n}(\frac{1}{2}(B^{afh}B_{hfc} + \frac{5}{3}C^{af}C_{fc} + \frac{c_0}{2}\tilde{\delta}_{fc}^{af}) + 3B^{afh}B_{hfc} + \frac{5}{3}C^{af}C_{fc})\delta_d^b - \\ -\frac{1}{2}(-2B^{bfh}B_{hfc} + \frac{c_0}{2}\tilde{\delta}_{fc}^{bf})\delta_d^a = 0 \quad (3.15)$$

Symmetrizing and then antisymmetrizing the equation (3.15) by the indices (b, f) and (f, h), we conclude that

$$C^{af}C_{fc} = -\frac{C_0(n+1)}{10}\delta_c^a.$$

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